

A METHOD OF IMPROVING THE STABILITY OF LIAO'S HIGHER-ORDER ABSORBING BOUNDARY CONDITION

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Abstract—Liao's absorbing boundary condition (ABC) is a classic ABC algorithm. It has the advantages of better absorption effect, easy programming and needless to split field. However its numerical stability is poor, especially for the higher-order ones, which greatly limits the scope of its application. To solve this problem, a weighting method for improving the stability of Liao's higher-order ABC is presented in this paper. This method is simple and effective, and it can be implemented easily compared with other improvement methods before. It improves the stability of Liao's higher-order ABC remarkably, and extends its application range.

1. INTRODUCTION

Due to the limitation of the computer capacity, appropriate boundary condition should be assigned at the boundaries of the domain when we solve the electromagnetics problems with the finite difference time domain (FDTD) method. To solve this problem, many absorbing boundary condition (ABC) algorithms were proposed [1–5]. The presentation of Brenger's perfectly matched layer (PML) ABC [6–8] promoted the development of FDTD a big step. Because of its perfect absorption effect, it is widely used in numerical solution of EM scattering problems [9–15]. Afterward, scholars proposed many modified PML ABCs, such as uniaxial PML (UPML) [16, 17], convolutional PML (CPML) [18]. These algorithms all have better accuracy. Although there are so many better ABCs, Liao's ABC

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still is used in some numerical calculations which don't need higher requirement in accuracy, because of its simple form and lower requirement in computer performance. So studying the stability of Liao's ABC has theoretical and practical significance.

Liao's ABC was acquired by extrapolating wave function in time and space domain based on the Newton's backward differential polynomial. It owns advantages of simple form, easy implementation and better absorption capability, which can be used near the corners of a computational grid and meet the requirements of more engineering numerical applications. However, lack of stability has restricted its application range, especially for the higher-order ones. Then W. C. Chew and other scholars analyzed the reason of the instability of Liao's ABC [19–23], and they found that the theoretical value of the pole of reflection coefficient is at the unit circle of the complex plane. However, by the limitations of computer precision, this pole may be shifted outside the unit circle, producing instability in the FDTD scheme. They pointed out that the higher stability can be realized when the poles of the reflection coefficient are within the unit circle of the complex plane, and also stabilized Liao's ABC by adding small damping factor. Nevertheless, this method is not easy to achieve, especially in Liao's higher-order ABC, which is sensitive to the extrapolation coefficient, a small change may produce catastrophic instability.

Based on this, the stability of Liao's ABC lies on its reflection coefficient pole stability, if we can enhance the stability of the pole of the reflection coefficient, we can make the Liao's higher-order ABC stable. With the calculation accuracy improved, the two dimensional (2D) Liao's second-order, third-order ABC and three dimensional (3D) Liao's second-order ABC are very stable. The Liao's lower-order ABC is very stable, its pole is stable and not easy to be shifted by the calculation error. However the Liao's higher-order ABC pole is sensitive to the calculation error, and easy to be shifted outside the unit circle. Thus we can improve the stability of the pole of Liao's higher-order ABC by using of the advantage of Liao's lower-order ABC in stability. In this paper, we propose to improve the stability of Liao's higher-order ABC through weighting unstable Liao's higher-order ABC with stable Liao's lower-order ABC. After weighted with Liao's lower-order ABC, the stability of the pole of Liao's higher-order ABC is enhanced, thus its stability is improved greatly, especially in 3D. Further, this method will not increase any additional storage and CPU time. Numerical experiments demonstrate that the method is accurate and effective.

2. THEORY

The FDTD method is well known for the solution of the wave equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \phi(x, y; t) = 0 \tag{1}$$

where ϕ is scalar field, which is computed on a rectangular finite-difference grid with grid spacing Δs and time step Δt : $\phi(m\Delta s, n\Delta s; l\Delta t) = \phi_{m,n}^l$. Using central differencing in time and space domain, the wave equation is approximated as:

$$\frac{\phi_{m+1,n}^l - 2\phi_{m,n}^l + \phi_{m-1,n}^l}{\Delta s^2} + \frac{\phi_{m,n+1}^l - 2\phi_{m,n}^l + \phi_{m,n-1}^l}{\Delta s^2} - \frac{1}{c^2} \frac{\phi_{m,n}^{l+1} - 2\phi_{m,n}^l + \phi_{m,n}^{l-1}}{\Delta t^2} = 0 \tag{2}$$

At the boundary of the finite computational domain, an absorbing boundary condition must be used to model radiation in a free space.

For a “right” boundary at $x = x_{\max}$, Liao’s ABC gives the updated boundary field $\phi(x_{\max}, y_j, t + \Delta t)$ in terms of field values at previous times lying along a straight line perpendicular to the boundary:

$$\phi(x_{\max}, y_j, t + \Delta t) = \sum_{i=1}^N (-1)^{i+1} C_i^N \phi(x_{\max} - i\alpha c\Delta t, y_j, t - (i-1)\Delta t) \tag{3}$$

where C_i^N is the binomial coefficient $N!/[i!(N-i)!]$, N the order of the boundary condition, i the space step, and $\alpha c\Delta t$ the space sample interval. If we set $\alpha c\Delta t$ equal to Δs , the sample interval just coincides with the grid, so the form of Liao’s ABC is very simple, the updated boundary field of Liao’s second-order ABC is given as:

$$\phi(x_{\max}, y_j, t + \Delta t) = 2\phi(x_{\max-1}, y_j, t) - \phi(x_{\max-2}, y_j, t - \Delta t) \tag{4}$$

And the Liao’s fourth-order ABC is given as:

$$\phi(x_{\max}, y_j, t + \Delta t) = 4\phi(x_{\max-1}, y_j, t) - 6\phi(x_{\max-2}, y_j, t - \Delta t) + 4\phi(x_{\max-3}, y_j, t - 2\Delta t) - \phi(x_{\max-4}, y_j, t - 3\Delta t) \tag{5}$$

The improving algorithm is implemented by weighting Liao’s higher-order ABC with stable Liao’s lower-order ABC. The Liao’s higher-order ABC and Liao’s lower-order ABC are multiplied with factor a and $(1 - a)$ respectively, the range of a is from 0 to 1. The common weighting form of Liao’s ABC can be given as:

$$\phi(x_{\max}, y_j, t + \Delta t) = (1 - a) \sum_{i=1}^N (-1)^{i+1} C_i^N \phi(x_{\max-i}, y_j, t - (i-1)\Delta t) + a \sum_{i=1}^M (-1)^{i+1} C_i^M \phi(x_{\max-i}, y_j, t - (i-1)\Delta t) \tag{6}$$

where a is the weighting factor, and M, N are different order. In the latter numerical experiments, M is set as the higher order.

For example, when improving the stability of Liao's fourth-order ABC by weighting method, we can weight Liao's fourth-order ABC with Liao's second-order ABC. The improved updated boundary field can be given as:

$$\begin{aligned} & \phi(x_{\max}, y_j, t + \Delta t) \\ = & (4 * 0.8 + 2 * 0.2) \phi(x_{\max - 1}, y_j, t) - (6 * 0.8 + 1 * 0.2) \phi(x_{\max - 2}, y_j, t - \Delta t) \\ & + (4 * 0.8) \phi(x_{\max - 3}, y_j, t - 2\Delta t) - (1 * 0.8) \phi(x_{\max - 4}, y_j, t - 3\Delta t) \quad (7) \end{aligned}$$

where a is set as 0.8.

The above is improving method in 2D. The improving method in 3D is same as the one in 2D.

3. NUMERICAL VALIDATION

In order to validate the availability of the weighting method in improving the stability of Liao's ABC, we conducted numerical experiments which implemented the weighted Liao's ABC in 2D and 3D grids. In order to analyze the reflection error, we computed the global error [19], and compared them with six-cell-thick PML ABC ones and second-order Mur ABC ones. Cases discussed here include: 1) 2D TE grid, calculation region is $100 * 50$ cells; and 2) 3D full vector lattice, calculation region is $50 * 50 * 30$ cells.

Figure 1 shows the model of the calculation domain in the numerical experiments, Fig. 1(a) is the 2D model, its region is $100 * 50$ cells. Its four boundaries are set as the same ABC. Fig. 1(b) is the 3D model, its region is $50 * 50 * 30$ cells. When validating the stability, its six boundaries are set as the same ABC. While validating the precision, its five boundaries are set as PML ABC, and the last one is set as PML ABC, Liao's ABC or Mur ABC.

A sinusoidal point source (operating at $1 \text{ e } 8 \text{ Hz}$) was set at the center of the computational region, to validate the stability of the weighted Liao's ABC. The observation point was set five cells away from the source in every coordinate direction. After calculating forty thousand time steps, the wave data of the observation point were compared with the ones which using PML ABC. The former results agree well with the latter ones.

Figure 2 describes the electric field of observation point in the 2D TE grid. It can be clearly seen that the weighted Liao's fourth-order and fifth-order ABC are still stable after calculating forty thousand time steps. Compared with the original Liao's fourth-order and fifth-order ABC, its stability is improved notably. Because the original

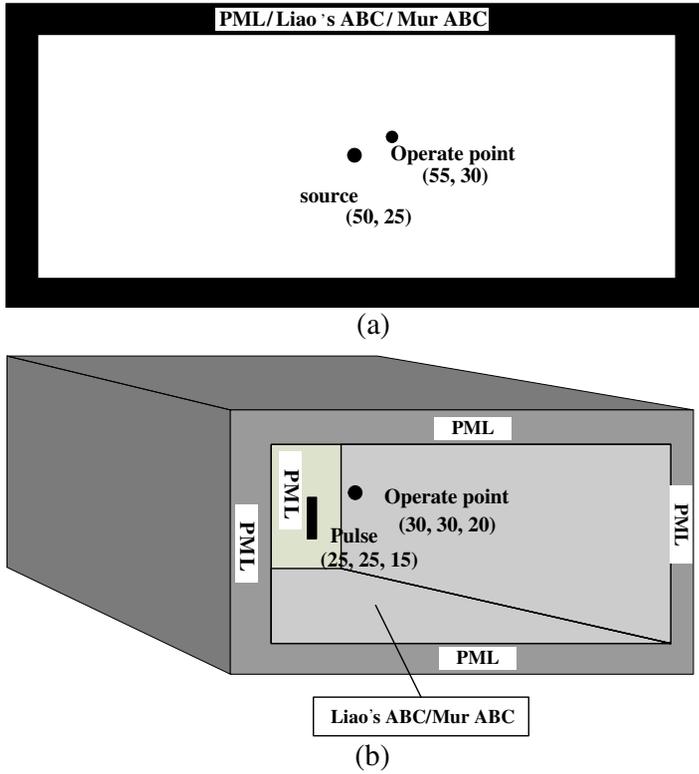


Figure 1. The model of the calculation domain. (a) The TE model in 2D grids (100 * 50). (b) The model in 3D grids (50 * 50 * 30).

Liao's fourth-order ABC can be only calculated four thousand time steps stably, and the original Liao's fifth-order ABC can be only calculated one thousand time steps stably in the same condition. We studied the range of the weighting factor in different conditions. When weighting Liao's fourth-order ABC with Liao's second or third-order ABC, and weighting Liao's fifth-order ABC with Liao's third-order ABC in 2D grids the weighting factor ranges from 0 to 0.8, while weighting Liao's fifth-order ABC with Liao second-order ABC, the weighting factor ranges from 0 to 0.5.

Figure 3 shows the electric field near forty thousand time steps. It can be clearly seen that, compared with the original Liao's ABC, the 3D weighted Liao's third-order and fourth-order ABC show excellent stability. It can be calculated at least forty thousand time steps stably. However, the original Liao's third-order ABC can only be calculated

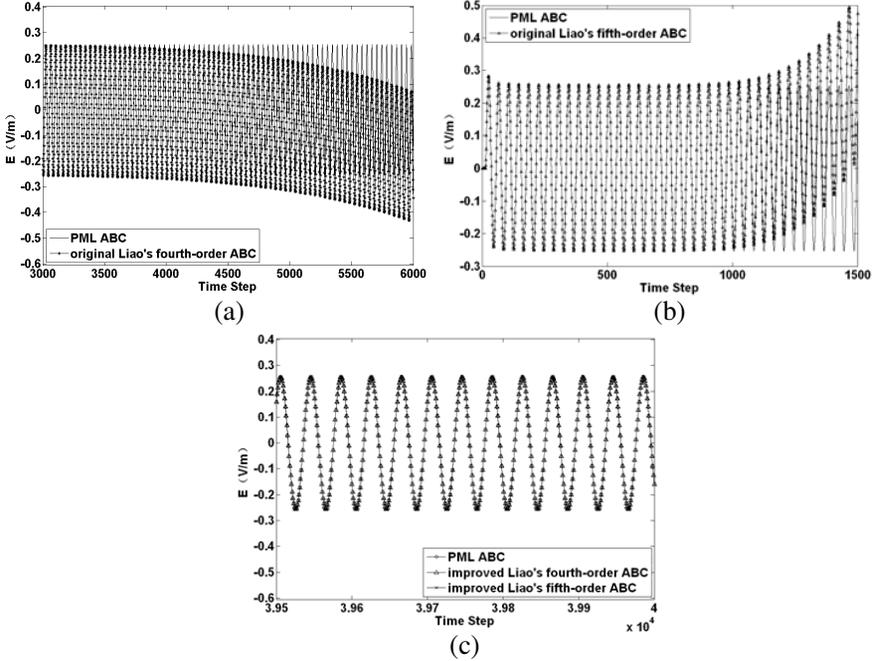


Figure 2. The electric field at the point (55, 30) within the 100×50 cells 2D TE FDTD grid for different order Liao’s ABC and 6-cell-thick PML ABC, plotted as a function of time step number. (a) Original Liao’s fourth-order ABC and PML ABC in 2D. (b) Original Liao’s fifth-order ABC and PML ABC in 2D. (c) Improved Liao’s fourth-order, fifth-order ABC and PML ABC in 2D.

four thousand steps, and the original Liao’s fourth-order ABC can only be calculated six hundred steps. When weighting different order Liao’s ABC, the weighting factor can be ranged from 0 to 0.8. In 3D grids we find that weighting Liao’s ABC and Mur ABC can also stabilize the higher-order Liao’s ABC.

When studying the numerical reflection errors, a “smooth compact pulse” source was excited in the calculation region. The pulse has an extremely smooth transition to zeros (its first five derivatives vanishing) [24]. In the 2D calculation region, the pulse was set at the center of the calculation region; a standard electric field was obtained by running a large mesh, Ω_T (having zero ABC artifact), centered upon and registered with Ω_B (having an outer boundary). The error due to numerical reflections caused by ABC was obtained by subtracting the field at any point inside Ω_T from the field at the corresponding point

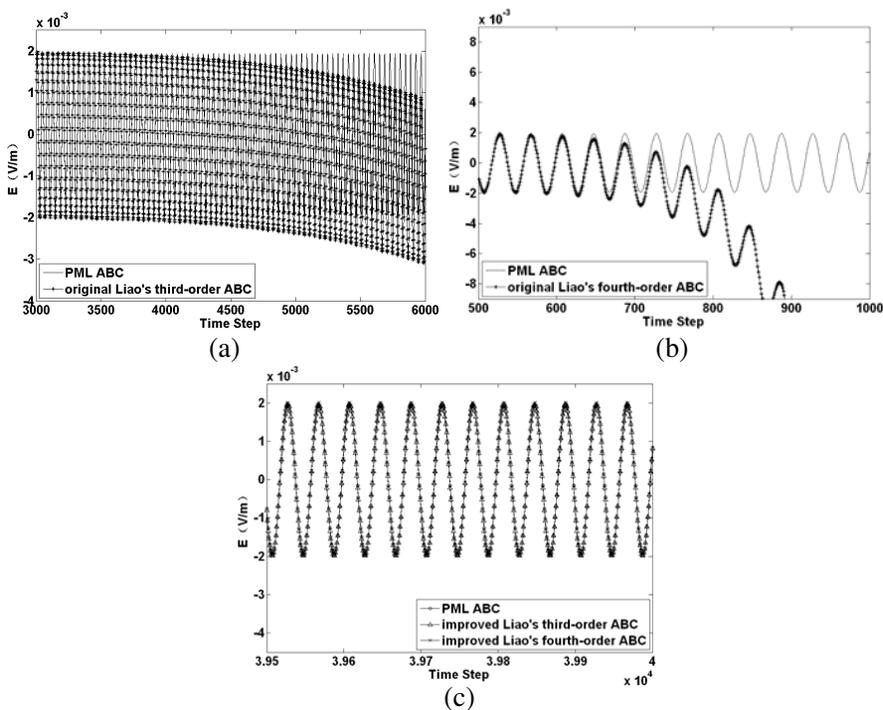


Figure 3. The electric field at the point (30, 30, 20) within the $50 \times 50 \times 30$ cells 3D FDTD grid for different order Liao’s ABC and 6-cell-thick PML ABC, plotted as a function of time step number. (a) Original Liao’s third-order ABC and PML ABC in 3D. (b) Original Liao’s fourth-order ABC and PML ABC in 3D. (c) Improved Liao’s third-order, fourth-order ABC and PML ABC in 3D.

in Ω_B , and the global error can be measured by summing the squares of the error at each time step. While in 3D calculation region, confined by the computer capacity, the standard electric field ($50 \times 50 \times 30 \times 500$) can’t be obtained as the 2D one. So an alternative method was used. Five boundaries of the 3D calculation region were set as 6-cell-thick PML ABC, and the last one was set as PML ABC, Mur ABC or Liao’s ABC (showing in Fig. 1(b)). The standard electric field was obtained by setting the last boundary as large mesh along the x -direction. The global error was obtained as the 2D ones.

Figure 4 shows the impact of the weighting factor on the global error in five hundred time steps. It can be seen that the global error of Liao’s higher-order ABC increases after weighting with lower-order one, but it is still superior to the second-order Mur ABC. And the

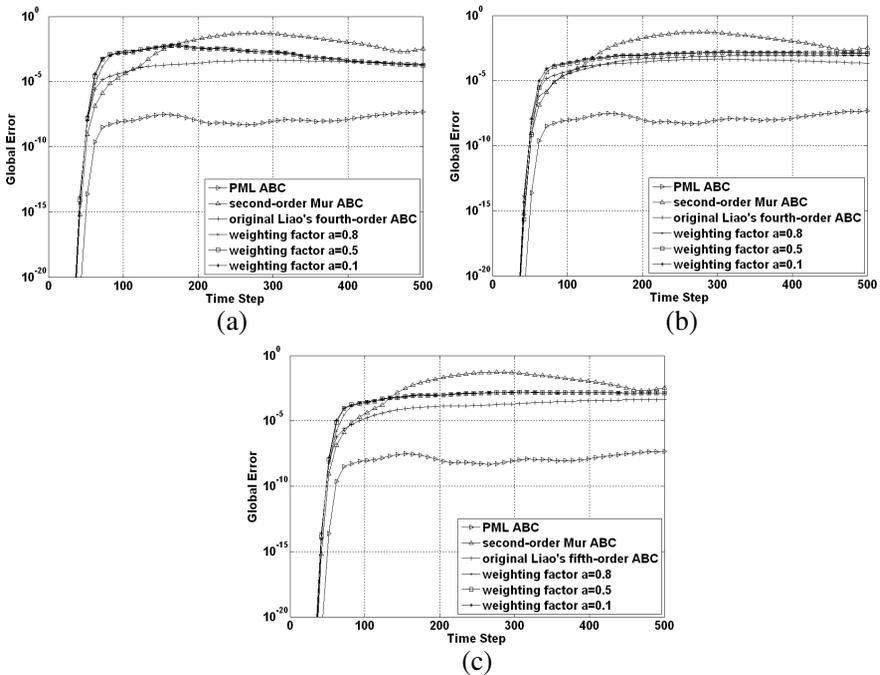


Figure 4. Global error energy (square of the electric field error at each grid cell summed throughout the entire grid) within the 100×50 cells 2D TE FDTD grid for different weighting factor ($a = 0.8, 0.5, 0.1$) Liao's ABC, second-order Mur ABC and 6-cell-thick PML ABC, plotted as a function of time step number on a logarithmic vertical scale. (a) Weighting Liao's fourth-order ABC with Liao's second-order ABC in 2D grids. (b) Weighting Liao's fourth-order ABC with Liao's third-order ABC in 2D grids. (c) Weighting Liao's fifth-order ABC with Liao's third-order ABC in 2D grids.

adding error of weighting Liao's fourth-order ABC with Liao's third-order ABC is the smallest in three weighting forms. The cause of global error increasing is that the global error of Liao's lower-order ABC is larger than the higher-order ones. With the weighting factor increasing, the global error changes unclearly, namely, the impact of the weighting factor on the global error was unclear in 2D grids.

Figure 5 shows the impact of the weighting factor on the global error in five hundred time steps in 3D. Fig. 5(a) shows that the global error of Liao's third-order ABC has minor increase when weighting with the second-order one, and with the weighting factor increasing, the global error is close to the original Liao's third-order ABC one.

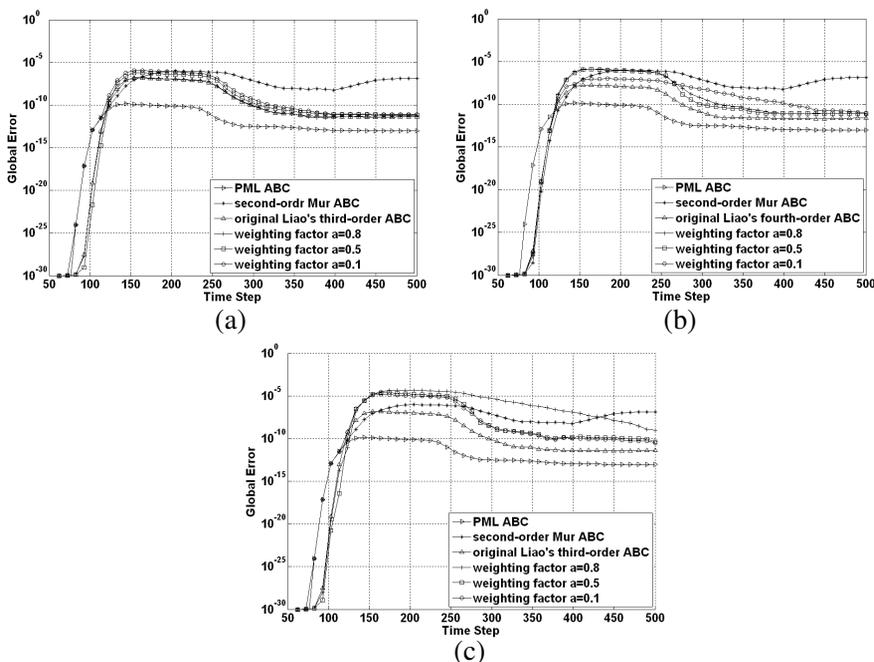


Figure 5. Global error energy (square of the electric field error at each grid cell summed throughout the entire grid) within the $50 \times 50 \times 30$ cell 3D grids for different weighting factor ($a = 0.8, 0.5, 0.1$) Liao's ABC, second-order Mur ABC and 6-cell-thick PML ABC, plotted as a function of time step number on a logarithmic vertical scale. (a) Weighting Liao's third-order ABC with Liao's second-order ABC in 3D grids. (b) Weighting Liao's fourth-order ABC with Liao's second-order ABC in 3D grids. (c) Weighting Liao's third-order ABC with first-order Mur ABC in 3D grids.

The weighted Liao's ABC is superior to the second-order Mur ABC in the term of reflection error. Fig. 5(b) shows that the global error of Liao's fourth-order ABC increases after weighting the second-order one, but it is still superior to the second-order Mur ABC. The global error decreases with the weighting factor increasing. Fig. 5(c) shows that the global error of Liao's third-order ABC increase clearly after weighting with the first-order Mur ABC, and it increases with the weighting factor increasing, which is different from the former. This weighting form can stabilize the Liao's third-order ABC, but it has no superiority compared with the standard second-order Mur ABC.

The above graphs show the impact of the weighting factor and weighting form on the global error, and compared with 6-cell-thick

PML ABC and second-order Mur ABC. It can be seen that the impact of weighting factor on the global error is significant in 3D grids than in 2D grids, and the greater the weighting factor, the smaller the global error in 3D, except weighting with Mur ABC. The error between improved Liao's ABC and original Liao's ABC can be decreased by selecting the closer stable order near the higher order, i.e., selecting Liao's third-order ABC instead of the second-order one weighting with Liao's fourth-order ABC in 2D.

4. CONCLUSION

A method of improving Liao's ABC stability is proposed in this paper. The method is implemented by weighting Liao's higher-order ABC with stable lower-order one or other ABC. Compared with others method, it is easy to implement, and effective to stabilize the Liao's higher-order ABC in 2D and 3D. It also can be used to stabilize more higher order Liao's ABC. The numerical experiments validate that the weighted Liao's ABC notably improves its stability, which is superior to the Mur ABC. With the help of this method, the stable Liao's higher-order ABC can be widely applied in engineering calculation. This method can be used to solve similar problems in other fields.

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