Two Finite-Difference Time-Domain Methods Incorporated with Memristor

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Abstract—Two finite-difference time-domain (FDTD) methods incorporated with memristor are presented. The update equations are derived based on Maxwell's equations, and the physical model is given by Hewlett-Packard (HP) lab. The first method is derived by calculating the memristance directly while the second method is derived by the relationship between electric charge and flux. Numerical results are given to discuss the accuracy, efficiency and stability of both proposed methods.

1. INTRODUCTION

Memristor is regarded as the fourth basic circuit element along with the widely used lumped elements including resistor, capacitor and inductor. It is characterized by a relationship between the electric charge and flux. The existence of memristor was predicted by Chua in 1971 [1] and it was not found until Strukov et al. in Hewlett-Packard (HP) lab successfully fabricated a memristor in 2008 [2]. They used a thin film of titanium dioxide to fabricate a nanoscale memristor and a coupled variable-resistor physical model is given to describe its characteristic. Potential applications based on memristor attract extensive attention worldwide such as reconfigurable electromagnetic switching devices [3], non-volatile random access memory [4] etc.

Subsequently, circuit analysis and simulation using memristor's SPICE model [5, 6] are developed due to the attractive potential applications in microwave circuits [7]. However, most circuits incorporated with memristor are simulated based on its SPICE model. To the best of the authors' knowledge, there are no full-wave electromagnetic update equations derived for memristor based on Maxwell's equations so far. Meanwhile, full-wave finite-difference time-domain (FDTD) method incorporated with resistor, capacitor and inductor have been derived and widely utilized in circuit simulation [8].

In this paper, two FDTD methods incorporated with memristor are proposed. The update equations are derived based on Maxwell's equations and the physical model given by HP lab. The first method is derived by calculating the memristance directly while the second method is derived by the relationship between electric charge and flux. Numerical results are given to discuss the accuracy, efficiency and stability of both proposed methods.

2. FORMULATIONS OF MEMRISTOR

The memristor links electric charge $q(t)$ and flux $\varphi(t)$. The memristance is expressed as

$$M[q(t)] = \frac{d\varphi[q(t)]}{dq(t)}. \tag{1}$$

According to the coupled variable-resistor physical model proposed by HP lab in [2], for a linear ionic drift in a uniform field with average ion mobility μ_v, the dopant width w (state variable) changes with the electric charge $q(t)$ as follows:

$$\frac{dw(t)}{dt} = \mu_v \frac{R_{ON}}{d} i(t) \tag{2}$$

$$w(t) = w(t = 0) + \mu_v \frac{R_{ON}}{d} q(t), \tag{3}$$

where R_{ON} is the lower bound of the memristance and d the thickness of the thin semiconductor film. The dopant width w determines the memristance $M[q(t)]$, which can be expressed as

$$M[q(t)] = R_{ON} \frac{w(t)}{d} + R_{OFF} \left[1 - \frac{w(t)}{d} \right]. \tag{4}$$

The initial memristance $M[q(t = 0)]$ can be calculated according to

$$M[q(t = 0)] = R_{ON} \frac{w(t = 0)}{d} + R_{OFF} \left[1 - \frac{w(t = 0)}{d} \right], \tag{5}$$

where R_{OFF} is the upper bound of the memristance. The memristance at a any time t can be derived as

$$M[q(t)] = M[q(t = 0)] - (R_{OFF} - R_{ON}) \mu_v \frac{R_{ON}}{d^2} q(t). \tag{6}$$

The above mathematical formulations are all based on the definition of memristance and the physical model in [2]. It can be seen that the memristance M is varied when the dopant width w or the electric charge q is changed.

3. FULL-WAVE UPDATE EQUATIONS INCORPORATED WITH MEMRISTOR

In this section, two memristor-incorporated FDTD methods will be presented. The update equations for these two different methods will be derived accordingly.

3.1. Memristor-Incorporated FDTD Method 1

The first method is derived by calculating the memristance directly according to the physical model in [2]. Assuming a z-directed memristor located along the field component $E_z|_{i-\frac{1}{2},j+\frac{1}{2},k+1}$, the corresponding Maxwell's differential equation can be discretized as

$$E_z|_{i-\frac{1}{2},j+\frac{1}{2},k+1}^{n+\frac{1}{2}} = E_z|_{i-\frac{1}{2},j+\frac{1}{2},k+1}^{n-\frac{1}{2}} + \frac{\Delta t}{\varepsilon \Delta x} \left(H_y|_{i,j+\frac{1}{2},k+1}^{n} - H_y|_{i-1,j+\frac{1}{2},k+1}^{n} \right)$$
$$- \frac{\Delta t}{\varepsilon \Delta y} \left(H_x|_{i-\frac{1}{2},j+1,k+1}^{n} - H_x|_{i-\frac{1}{2},j,k+1}^{n} \right) - \frac{\Delta t}{\varepsilon} J_{mem}|_{i-\frac{1}{2},j+\frac{1}{2},k+1}^{n}. \tag{7}$$

J_{mem} is the electric current density across the memristor and it can be expressed in terms of the electric fields (in a semi-implicit manner) and the memristance M_{mem} as

$$J_{mem}|_{i-\frac{1}{2},j+\frac{1}{2},k+1}^{n} = \frac{I_{mem}|_{i-\frac{1}{2},j+\frac{1}{2},k+1}^{n}}{\Delta x \Delta y} = \frac{V|_{i-\frac{1}{2},j+\frac{1}{2},k+1}^{n}}{M_{mem}|^{n} \Delta x \Delta y}$$
$$= \frac{1}{2} \frac{\Delta z \left(E_z|_{i-\frac{1}{2},j+\frac{1}{2},k+1}^{n+\frac{1}{2}} + E_z|_{i-\frac{1}{2},j+\frac{1}{2},k+1}^{n-\frac{1}{2}} \right)}{\Delta x \Delta y M_{mem}|^{n}} \tag{8}$$

After substituting (8) into (7), we obtain the FDTD update equation:

$$
\begin{aligned}
E_z|_{i-\frac{1}{2},j+\frac{1}{2},k+1}^{n+\frac{1}{2}} = {} & \frac{\left(1 - \dfrac{\Delta t \Delta z}{2\varepsilon \Delta x \Delta y M_{mem}|^n}\right)}{\left(1 + \dfrac{\Delta t \Delta z}{2\varepsilon \Delta x \Delta y M_{mem}|^n}\right)} E_z|_{i-\frac{1}{2},j+\frac{1}{2},k+1}^{n-\frac{1}{2}} \\
& + \frac{\dfrac{\Delta t}{\varepsilon \Delta x}\left(H_y|_{i,j+\frac{1}{2},k+1}^{n} - H_y|_{i-1,j+\frac{1}{2},k+1}^{n}\right)}{\left(1 + \dfrac{\Delta t \Delta z}{2\varepsilon \Delta x \Delta y M_{mem}|^n}\right)} - \frac{\dfrac{\Delta t}{\varepsilon \Delta y}\left(H_x|_{i-\frac{1}{2},j+1,k+1}^{n} - H_x|_{i-\frac{1}{2},j,k+1}^{n}\right)}{\left(1 + \dfrac{\Delta t \Delta z}{2\varepsilon \Delta x \Delta y M_{mem}|^n}\right)}. \quad (9)
\end{aligned}
$$

When the electric field E of the memristor is changed, the memristance will also be changed due to the different dopant width w. According to (2), the state of dopant width w at time step $n+1$ can be calculated using the following differential equation:

$$
w^{n+1} = w^n + \Delta t \mu_v \frac{R_{ON}}{d} i^{n+\frac{1}{2}} = w^n + \Delta t \mu_v \frac{R_{ON}}{d} \frac{\Delta z E_z|_{i-\frac{1}{2},j+\frac{1}{2},k+1}^{n+\frac{1}{2}}}{M_{mem}|^n} \quad (10)
$$

Next, the memristance can be calculated according to the dopant width w in (10).

$$
M_{mem}|^{n+1} = R_{ON}\frac{w^{n+1}}{d} + R_{OFF}\left[1 - \frac{w^{n+1}}{d}\right] \quad (11)
$$

Based on the above formulations, the time-stepping update equations for the proposed memristor-incorporated FDTD method 1 are summarized as:

- Update the $E_z|_{i-\frac{1}{2},j+\frac{1}{2},k+1}^{n+\frac{1}{2}}$ field component according to (9).

- Update the dopant width w^{n+1} using (10).

- Update the memristance $M_{mem}|^{n+1}$ according to (11).

3.2. Memristor-Incorporated FDTD Method 2

The second method is derived by using the relationship between electric charge $q(t)$ and flux $\varphi(t)$ based on the definition of the memristance and the physical model in [2].

When the memristance M in (1) is replaced by (6), the relationship between the electric charge $q(t)$ and flux $\varphi(t)$ subject to the initial condition $\varphi[q(0)] = 0$ can be expressed as

$$
d\varphi(t) = d\left\{q(t)R_0 - \frac{R_0}{2C_M}q(t)^2\right\}, \quad (12)
$$

$$
\varphi(t) = R_0\left[q(t) - \frac{1}{2C_M}q(t)^2\right] \quad (13)
$$

where

$$
C_M = \frac{R_0 d^2}{\mu_v R_{ON}(R_{OFF} - R_{ON})} \quad (14)
$$

and R_0 is the initial memristance $M[q(t=0)]$ which can be calculated from (5).

The electric charge $q(t)$ can be expressed in terms of flux $\varphi(t)$ based on (13) as

$$
q(t) = C_M\left[1 - \sqrt{1 - \frac{2}{C_M R_0}\varphi(t)}\right]. \quad (15)
$$

Thus, the current density across the memristor J_{mem} can be written as

$$J_{mem}|^n_{i-\frac{1}{2},j+\frac{1}{2},k+1} = \frac{1}{\Delta x \Delta y} \frac{dq|^n_{i-\frac{1}{2},j+\frac{1}{2},k+1}}{dt}$$

$$= \frac{\Delta z}{2R_0 \Delta x \Delta y} \frac{\left(E_z|^{n+\frac{1}{2}}_{i-\frac{1}{2},j+\frac{1}{2},k+1} + E_z|^{n-\frac{1}{2}}_{i-\frac{1}{2},j+\frac{1}{2},k+1} \right)}{\sqrt{1 - \frac{2}{C_M R_0} \varphi|^n_{i-\frac{1}{2},j+\frac{1}{2},k+1}}}. \quad (16)$$

After substituting the above equation into (7), the time-stepping update equations for the proposed memristor-incorporated FDTD method 2 are summarized as the following two procedures:

$$\left(1 + \frac{\Delta t}{\varepsilon} \frac{\Delta z}{2R_0 \Delta x \Delta y} \frac{1}{\sqrt{1 - \frac{2\varphi|^n_{i-\frac{1}{2},j+\frac{1}{2},k+1}}{C_M R_0}}} \right) E_z|^{n+\frac{1}{2}}_{i-\frac{1}{2},j+\frac{1}{2},k+1}$$

$$= \left(1 - \frac{\Delta t}{\varepsilon} \frac{\Delta z}{2R_0 \Delta x \Delta y} \frac{1}{\sqrt{1 - \frac{2\varphi|^n_{i-\frac{1}{2},j+\frac{1}{2},k+1}}{C_M R_0}}} \right) E_z|^{n-\frac{1}{2}}_{i-\frac{1}{2},j+\frac{1}{2},k+1}$$

$$+ \frac{\Delta t}{\varepsilon \Delta x} \left(H_y|^n_{i,j+\frac{1}{2},k+1} - H_y|^n_{i-1,j+\frac{1}{2},k+1} \right) - \frac{\Delta t}{\varepsilon \Delta y} \left(H_x|^n_{i-\frac{1}{2},j+1,k+1} - H_x|^n_{i-\frac{1}{2},j,k+1} \right) \varphi|^{n+1}_{i-\frac{1}{2},j+\frac{1}{2},k+1}$$

$$= \varphi|^n_{i-\frac{1}{2},j+\frac{1}{2},k+1} + \Delta z \Delta t E_z|^{n+\frac{1}{2}}_{i-\frac{1}{2},j+\frac{1}{2},k+1}. \quad (17)$$

Different from the previous memristor-incorporated FDTD method, the memristance in the second method is not involved in the FDTD update scheme. If the memristance is required, it still could be calculated separately according to the definition of memristance in (1) as well as the relationship between electric charge $q(t)$ and flux $\varphi(t)$ in (12). The equation for memristance is derived as

$$M_{mem}|^n_{i-\frac{1}{2},j+\frac{1}{2},k+1} = R_0 - \frac{R_0}{C_M} q|^n_{i-\frac{1}{2},j+\frac{1}{2},k+1}, \quad (18)$$

where the electric charge can be calculated according to (15).

Based on the above formulations derived directly based on the relationship between electric charge and flux, the time-stepping update equations for the proposed memristor-incorporated FDTD method 2 are summarized as:

- Update the $E_z|^{n+\frac{1}{2}}_{i-\frac{1}{2},j+\frac{1}{2},k+1}$ field component according to (17).
- Update the flux $\varphi(n+1)$ using (18).

4. NUMERICAL RESULTS

As shown in Fig. 1, a microstrip line terminated with a memristor is simulated to verify the accuracy of the proposed two methods. The relative dielectric constant ε_r of the circuit substrate is 9.4 and the thickness is 0.8 mm. A sinusoidal voltage source along the z-direction: $V_s = \sin(2\pi 1000)$ V is used to excite the electromagnetic field at the beginning of the microstrip circuit. The time step is set to the Courant-Friedrichs-Lewy (CFL) condition:

$$\Delta t = \Delta t_{CFL} = \frac{1}{c_{\max} \sqrt{\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} + \frac{1}{(\Delta z)^2}}} \quad (19)$$
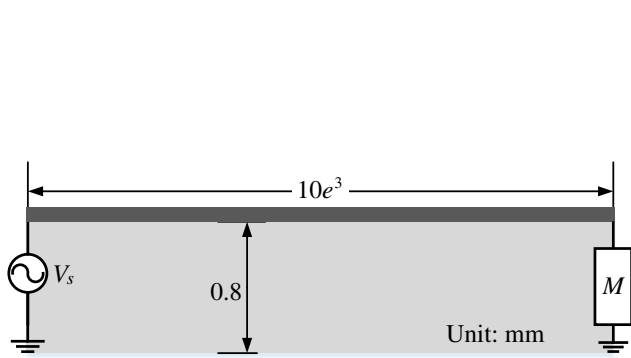
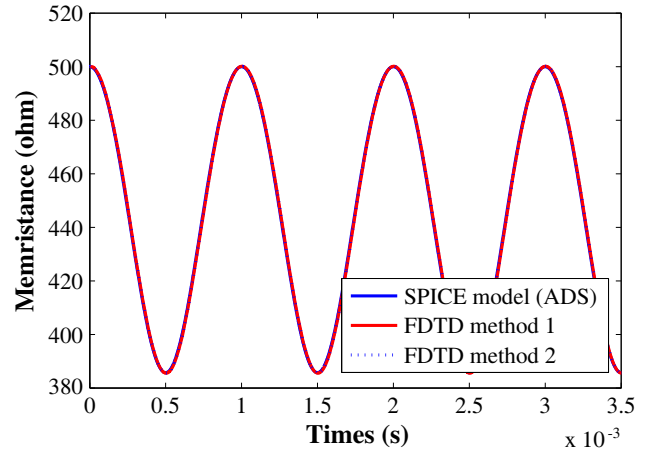**Figure 1.** Configuration of the microstrip circuit.

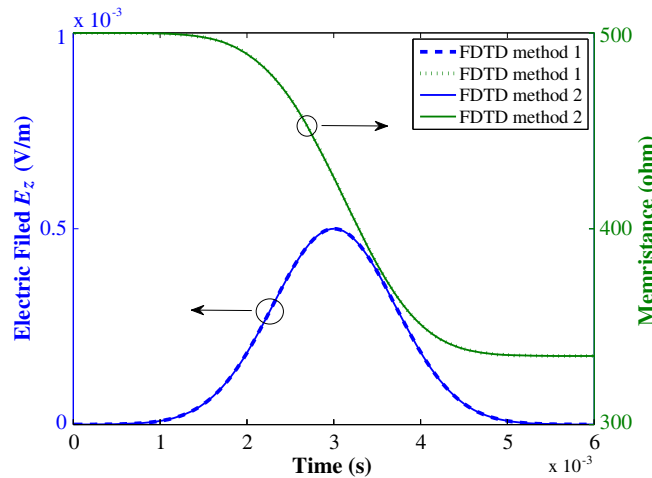**Figure 2.** Simulated results of the microstrip circuit with memristor.

Figure 3. Electric field and memristance of the memristor excited by Gaussian pulse.

According to the physical model of memristor [2], the average ion mobility is $\mu_v = 1e^{-14}$ m/s and the thickness of the thin semiconductor film is $d = 1e^{-8}$ m. The same mesh is utilized for both two proposed memristor-incorporated FDTD methods. The initial memristance is 500 Ohm and the range of the memristance $M \in \{R_{ON} = 100, R_{OFF} = 16e^3\}$ Ohm.

The simulated results are shown in terms of the memristance, as in Fig. 2. The memristance changes with time as the sinusoidal voltage source provides a varying flux across the memristor. The same microstrip circuit is simulated by a commercial software Advanced Design System (ADS) with the SPICE model in [7] to verify the accuracy of the two proposed methods. In Fig. 2, FDTD method 1 is the first proposed method by calculating the memristance and FDTD method 2 is the second method using the relationship between electric charge $q(t)$ and flux $\varphi(t)$. It is observed that both proposed simulation methods agree well with the simulated result using SPICE model in ADS.

Compared with the simulated memristance result of SPICE model in ADS, the norm-2 errors of the proposed two methods are 0.3759% and 0.2334%, respectively. Both two proposed FDTD methods are very close to the simulated result of SPICE model in ADS, but the second method is more accurate than the first method. We further investigate the efficiency of both methods. We run the programs using Matlab on Windows 7 platform with 3.1 GHz Intel i5 processor and 4 GB RAM. After 50000 main iterations by both FDTD methods, the CPU computational time for the first method is 1.82 s, and 3.55 s for the second method. More computational time is taken for the second FDTD method, although the

second method could achieve a more accurate result.

Since memristor is a non-linear component involving many parameters according to the physical model proposed by HP, commonly used stability analysis method such as Von-Neumann is not applicable for memristor-incorporated FDTD method. Nevertheless, the stability of FDTD methods incorporated with memristor can be verified numerically.

We have executed both FDTD methods for $1e^6$ iterations and both results of the electric fields across the memristors show that the algorithms are stable when the time step is set to the CFL limitation in (20).

To show the robustness of the FDTD method, another example with Gaussian excitation is simulated. The simulated model and the configuration are the same as the previous example, but the source is changed to a Gaussian pulse: $\frac{1}{2}\exp\left(-\frac{t-3\tau}{\tau}\right)^2$ mV/m, $\tau = 1$ ms. The electric field E_z of the memristor and its corresponding memristance are shown in Fig. 3. It could be observed that the memristance remains a stable value after the electric field E_z becomes zero.

5. CONCLUSION

Two FDTD methods incorporated with memristor have been presented. The update equations are derived based on Maxwell's equations and the physical model given by HP lab. The first method is derived by calculating the memristance directly while the second method is derived according to the relationship between electric charge and flux. A microstrip circuit with memristor has been simulated to verify the accuracy of both proposed methods by comparing with the result of SPICE model incorporated circuit simulation. Both simulated results agree well with the result using SPICE model in ADS. The accuracy, efficiency and stability of both methods have been discussed.

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