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The Fractional Fourier Transform of Hypergeometric-Gauss Beams through the Hard Edge Aperture

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Abstract—Based on the Collins integral formula and Lohmann optical system, we expand the hard edge aperture into complex Gauss function and derive an approximate analytic expression of intensity distribution theoretically for Hypergeometric-Gauss beams through the fractional Fourier transform (FRT) optical systems with hard edge aperture. The influences of FRT order, aperture size and other optical parameters on the light intensity distribution of Hypergeometric-Gauss beams passing through the FRT optical systems are discussed in detail. The results show that the FRT is an excellent beam-shaping method.

1. INTRODUCTION

Hollow beams attract wide interest and research due to their important applications in optical communication [1], laser optics [2], particle trapping [3], medical science [4], field atom and binary optics [5]. Sodha et al. have studied the transmission characteristics of focusing and self-focused hollow beam in different dielectrics [6]. There have been lots of research on its extensive applications since Kotlyar et al. proposed a new paraxial hypergeometric beam [7], and especially they obtained it by using another kind of solution of scalar Helmholtz equation, which has limited energy and can be produced in the experiments [8].

On the other hand, the FRT is an extension of the conventional Fourier transform. Since 1993, Ozaktas et al. have introduced the FRT into optics for the first time [9], which has been widely researched in optics and shows that the FRT has important applications for image encryption [10, 11], beam shaping [12], signal processing [13], etc.

This paper mainly studies the transmission properties of Hypergeometric-Gauss beams passing through the FRT optical systems. Based on the Collins integral formula and Lohmann optical system, the electric field analytic expressions of Hypergeometric-Gauss beams passing through the FRT optical systems are derived theoretically. The influences of the FRT order, aperture size and other optical parameters on the normalized intensity distribution are discussed in detail. For comparison, the normalized intensity distributions of Hypergeometric-Gauss beams through the FRT optical systems with hard edge aperture are also shown. The method used in this paper can be applied to the study on transmission characteristics of other beams through the FRT optical systems with hard edge aperture.

2. THE FRACTIONAL FOURIER TRANSFORM OF HYPERGEOMETRIC-GAUSS BEAMS

Lohmann [10] points out that the optical systems shown in Fig. 1 of the Lohmann I (as shown in Fig. 1(a)) and Lohmann II (as shown in Fig. 1(b)) can be used to realize the FRT for beams, where f

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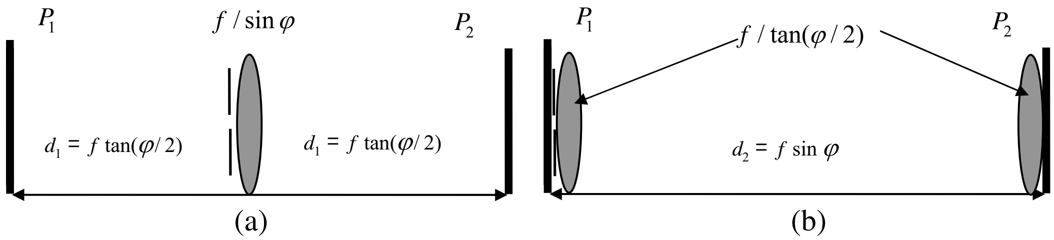


Figure 1. Two kinds of optical systems of FRT.

is the standard focal length. The lens focal lengths in Figs. 1(a) and 1(b) are $f/\sin\varphi$ and $f/\tan(\varphi/2)$, respectively. Here $\varphi = p\pi/2$ and p is the order of FRT. P_1 and P_2 are the input and output planes. Other parameters are shown in Fig. 1.

The FRT transmission matrices for Lohmann I and Lohmann II optical systems can be expressed as [14, 15]

$$M_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & d_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\sin\varphi/f & 1 \end{pmatrix} \begin{pmatrix} 1 & d_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\varphi & f\sin\varphi \\ -\sin\varphi/f & \cos\varphi \end{pmatrix} \tag{1}$$

$$M_2 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\tan(\varphi/2)/f & 1 \end{pmatrix} \begin{pmatrix} 1 & d_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\tan(\varphi/2)/f & 1 \end{pmatrix} = \begin{pmatrix} \cos\varphi & f\sin\varphi \\ -\sin\varphi/f & \cos\varphi \end{pmatrix} \tag{2}$$

One can see from the results of Eqs. (1) and (2) that Lohmann I and Lohmann II optical systems are identical in nature. Supposing an electric field $E(x_1, y_1)$ of a bunch of fixed quasi monochromatic laser beam, the expression of electric field from the lens to the output plane can be obtained according to the Collins integral formula [16]

$$\begin{aligned} \mathrm{E}_p(x_2, y_2) &= \frac{1}{i\lambda f\sin\varphi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(x_1, y_1) \exp\left[-\frac{i\pi\left(x_1^2 + y_1^2 + x_2^2 + y_2^2\right)}{\lambda f\tan\varphi}\right] \\ &\quad \times \exp\left[\frac{2\pi i(x_1x_2 + y_1y_2)}{\lambda f\sin\varphi}\right] dx_1dy_1 \end{aligned} \tag{3}$$

Here x_1, y_1 and x_2, y_2 indicate the Cartesian coordinates of input plane and output plane, respectively. λ is the wavelength. When $p = 4k + 1$, k is an arbitrary integer, and the FRT optical systems reduces to the conventional Fourier transform optical systems. By setting $x_1 = r_1\cos\theta_1$, $y_1 = r_1\sin\theta_1$, $x_2 = r_2\cos\theta_2$, $y_2 = r_2\sin\theta_2$ in Eq. (3), we can get the propagation equation for beams through the FRT optical systems in the cylindrical coordinate system as follows:

$$\begin{aligned} E_p(r_2, \theta_2) &= \frac{1}{i\lambda f\sin\varphi} \int_0^{2\pi}\int_0^{\infty} E(r_1, \theta_1) \exp\left[-\frac{i\pi\left(r_1^2 + r_2^2\right)}{\lambda f\tan\varphi}\right] \\ &\quad \times \exp\left[\frac{2\pi i r_1 r_2}{\lambda f\sin\varphi}\cos(\theta_1 - \theta_2)\right] r_1dr_1d\theta_1 \end{aligned} \tag{4}$$

Here r_1, θ_1 and r_2, θ_2 denote the radial and azimuth angle coordinates in the input and FRT planes, respectively. The analytical expression of Hypergeometric-Gauss beams $z = 0$ in the cylindrical coordinate system takes the form as [7]

$$E_n(r_1) = \left(\frac{r_1}{w}\right)^{n+i\gamma} \exp\left(-\frac{r_1^2}{w^2}\right) \tag{5}$$

where n, w represent the beam order and the beam waist radius, and γ is a real parameter. Eq. (5) shows the hollow Gauss beam when $\gamma = 0$. The area of the dark region increases with the increase of the beam order. By substituting Eq. (5) into Eq. (4) and using the integral formula [17]

$$\int_0^{2\pi} \exp\left[ix\cos\left(\theta_1-\theta_2\right)\right]d\theta_1 = 2\pi J_0\left(x\right) \tag{6}$$

$$\begin{aligned}\int_0^{\infty} x^u \exp\left(-\alpha x^2\right) J_v\left(\beta x\right)dx &= \beta^v \alpha^{-(u+v+1)/2} 2^{-v-1} \Gamma\left(\frac{u+v+1}{2}\right)\\ &\times \Gamma^{-1}\left(v+1\right) {}_1F_1\left(\frac{u+v+1}{2}, v+1, -\frac{\beta^2}{4\alpha}\right)\end{aligned} \tag{7}$$

Here $J_v(x)$ denotes the first kind of Bessel function of order v, $\Gamma(x)$ the gamma function, and ${}_1F_1(a,b,x)$ the confluent hypergeometric function.

The analytic expressions of output field $E_p\left(r_2,\theta_2\right)$ can be attained through the integral operation

$$E_p\left(r_2\right) = \frac{\pi w^{-(n+i\gamma)}}{i\lambda f\sin\varphi}\alpha^{-(n+i\gamma+2)/2}\exp\left(-\frac{i\pi r_2^2}{\lambda f\tan\varphi}\right)\Gamma\left(\frac{n+i\gamma+2}{2}\right){}_1F_1\left(\frac{n+i\gamma+2}{2}, 1, -\frac{\beta^2}{4\alpha}\right) \tag{8}$$

where

$$\alpha = \frac{1}{w^2} + \frac{i\pi}{\lambda f\tan\varphi}; \quad \beta = \frac{2\pi r_2}{\lambda f\sin\varphi} \tag{9}$$

Normally, the lens in the FRT optical systems is limited. Therefore, the condition of aperture should be considered. For the FRT optical systems of Lohmann I, beam propagation is divided into two processes. The first process is in the free space transmission from P_1 to the lens, the second process is from lens to P_2, where a circular aperture is placed in the front of the lens. Consequently, the expression of the electric field of beams through the FRT optical systems can be written as [16]

$$\begin{aligned}E_p\left(r_2,\theta_2\right) &= \frac{1}{i\lambda f\sin\varphi}\int_0^{2\pi}\int_0^{a} E\left(r_1,\theta_1\right)\exp\left[-\frac{i\pi\left(r_1^2+r_2^2\right)}{\lambda f\tan\varphi}\right]\\ &\times\exp\left[\frac{2\pi i r_1 r_2}{\lambda f\sin\varphi}\cos\left(\theta_1-\theta_2\right)\right]r_1 dr_1 d\theta_1\end{aligned} \tag{10}$$

Here a is the aperture radius. The window function of the hard edge aperture can be represented by a two-dimensional matrix function

$$A_p\left(r\right) = \begin{cases} 1, & r\le a\\ 0, & r>a\end{cases} \tag{11}$$

Hence, Eq. (10) can be converted into

$$\begin{aligned}E_p\left(r_2,\theta_2\right) &= \frac{1}{i\lambda f\sin\varphi}\int_0^{2\pi}\int_0^{\infty} A_p\left(r_1\right)E\left(r_1,\theta_1\right)\exp\left[-\frac{i\pi\left(r_1^2+r_2^2\right)}{\lambda f\tan\varphi}\right]\\ &\times\exp\left[\frac{2\pi i r_1 r_2}{\lambda f\sin\varphi}\cos\left(\theta_1-\theta_2\right)\right]r_1 dr_1 d\theta_1\end{aligned} \tag{12}$$

The aperture function can be expanded as the sum of complex Gaussian functions with finite numbers [18]

$$A_p\left(r\right) = \sum_{h=1}^{N} A_h \exp\left(-\frac{B_h}{a^2}r^2\right) \tag{13}$$

A_h and B_h are the expansion coefficient and Gauss coefficient [18, 19]. It is worth noting that Eq. (14) is only an approximate expression of the function of hard edge aperture. This expansion method has been proved reliable and efficient. The simulation accuracy improves as the expansion order N increases. For a circular hard aperture, $N = 10$ assures a very good description of the diffracted beam [20].

By substituting Eq. (5) and Eq. (13) into Eq. (12) and using the integral formula Eq. (6) and Eq. (7), the approximate analytic expression of the electric field for Hypergeometric-Gauss beams, which passes through the FRT optical systems with hard edge aperture, can be derived by the integral operation

$$\begin{aligned} E_p(r_2) &= \frac{\pi}{i\lambda f \sin\varphi} \exp\left(-\frac{i\pi r_2^2}{\lambda f \tan\varphi}\right) \sum_{h=1}^{N} A_h \left(\frac{1}{w}\right)^{n+i\gamma} \alpha'^{-(n+i\gamma+2)/2} \\ &\quad \times \Gamma\left(\frac{n+i\gamma+2}{2}\right) {}_1F_1\left(\frac{n+i\gamma+2}{2}, 1, -\frac{\pi^2 r_2^2}{\alpha' \lambda^2 f^2 \sin^2\varphi}\right) \qquad (14) \\ \alpha' &= \frac{B_h}{\delta^2 w^2} + \frac{1}{w^2} + \frac{i\pi}{\lambda f \tan\varphi}, \quad \delta = a/w \qquad (15) \\ I_p(r_2) &= E_p(r_2) \times E_p^*(r_2) \qquad (16) \end{aligned}$$

Here δ denotes the truncation parameter. The intensity distribution on axis can be obtained in the condition of $r_2 = 0$ in Eq. (16). The electric field analytic expressions of Hypergeometric-Gauss beams, passing through the FRT optical systems without and with hard edge aperture, are given by Eq. (8) and Eq. (14), respectively, which provide a convenient and important method to study the FRT of Hypergeometric-Gauss beams.

3. NUMERICAL RESULTS

Based on the analytic expressions we have numerically investigated the transmission properties of the Hypergeometric-Gauss beams through the FRT optical systems with the hard edge aperture. The parameters are $w = 1\,\text{mm}$, $\lambda = 632.8\,\text{nm}$, $f = 400\,\text{mm}$. In Fig. 2, the influences of FRT order p and the beam order n on the intensity distribution of Hypergeometric-Gauss beams are studied.

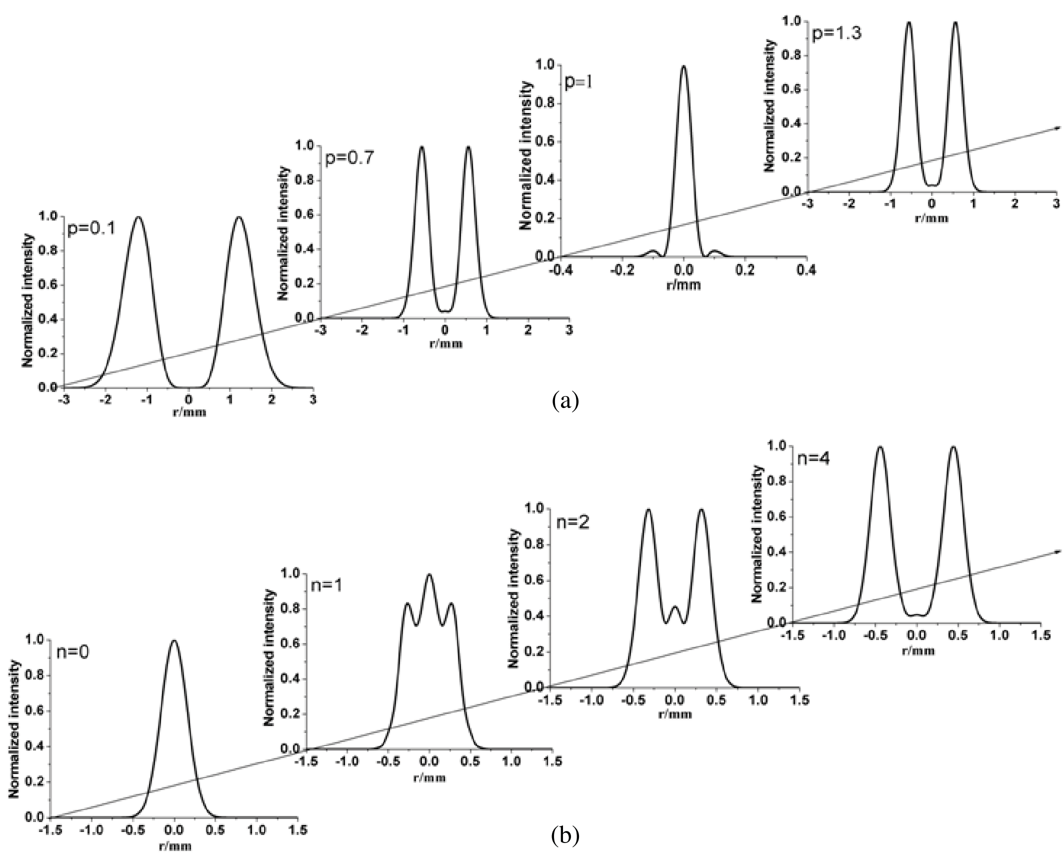


Figure 2. The normalized intensity distribution of Hypergeometric-Gauss beams through the FRT optical systems. $w = 1\,\text{mm}$, $n = 3$, $\lambda = 632.8\,\text{nm}$, $f = 400\,\text{mm}$, (a) $n = 3$; (b) $p = 0.8$.

In Fig. 2, we set $\gamma = 0$, and the normalized intensity distribution is greatly affected by the FRT order and beam order. From Fig. 2(a), it can be found that when $0 < p < 1$, the hollow size between the symmetrical double-peaks decreases gradually with an increasing order p of FRT, and eventually becomes the central peak. Meanwhile, the maximum light intensity position will change from axial external to axis; however, as $1 < p < 2$, the central peak gradually becomes symmetrical double-peaks, and the maximum light intensity position changes from axis to axial external. From Fig. 2(b), it is easy to see that the intensity distribution of Hypergeometric-Gauss beams is quasi-Gauss distribution as $n = 0$. The central peak gradually becomes symmetrical double-peaks with an increasing beam order n, and the hollow size increases with an increasing beam order. There is an excellent agreement between the above results and the transmission characteristics of FRT for the hollow Gauss beams [21].

The intensity distribution on axis can be obtained by making $r_2 = 0$ in Eq. (16). The change situation of intensity distribution on axis of Hypergeometric-Gauss beams, along with the FRT order p in the condition of different hard edge aperture truncation parameters δ, is shown in Fig. 3. The values of other parameters are the same as in Fig. 2.

The change situation of the intensity on axis of Hypergeometric-Gauss beams through the FRT optical systems with hard edge aperture is periodical as shown in Fig. 3, and the period is 2. The light intensity has a maximum when $p = 2n + 1$ and has a minimum as $p = 2n$. In addition, the minimum is uniform for different δ.

The change situation of the intensity distribution on axis of Hypergeometric-Gauss beams through the FRT optical systems with hard edge aperture along with the beam order n, the real variable γ in the condition of ideal condition, and truncation parameter $\delta = 1$ is shown in Fig. 4. The values of other parameters are the same as in Fig. 3.

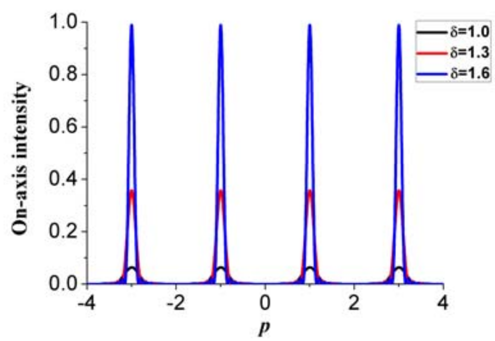


Figure 3. The intensity distributions on axis of Hypergeometric-Gauss beams through the FRT optical systems with different sizes of hard edge aperture with $w = 1\,\text{mm}$, $n = 3$, $\lambda = 632.8\,\text{nm}$, $f = 400\,\text{mm}$, $\gamma = 0$.

The change situation of the intensity on axis of Hypergeometric-Gauss beams through the FRT optical systems with hard edge aperture is periodical for different beam orders and real variables as shown in Fig. 4, and the period is 2. From Figs. 4(a)–4(b), the light intensity increases with the increase of beam order in the ideal condition, and the intensity attenuates obviously and decreases with the increase of beam order for the diffraction and limit of hard edge aperture as $\delta = 1$. In Figs. 4(d) and 4(c), the light intensity increases with increasing γ when $\delta = 1$, and the intensity decreases with increasing γ under the ideal condition. On the other hand, the light intensity has a maximum when $p = 2n + 1$ and has a minimum as $p = 2n$ in Figs. 4(a) and 4(b). In addition, the minimum is uniform for different beam orders n. Nevertheless, the maximum and minimum intensities on axis shift to the left owing to the effect of real variable γ in Figs. 4(c) and 4(d).

In Fig. 5, by using Eqs. (14)–(16), we numerically calculate the normalized intensity distribution of Hypergeometric-Gauss beams through the FRT optical systems with hard edge aperture (the blue solid lines shown in Fig. 5). Here, the truncation parameter $\delta = 0.5$, and other parameters are the same as in Fig. 2. Moreover, the red dotted lines in Fig. 5 are the results of numerical calculation of the integral formula (11) used by MATLAB.

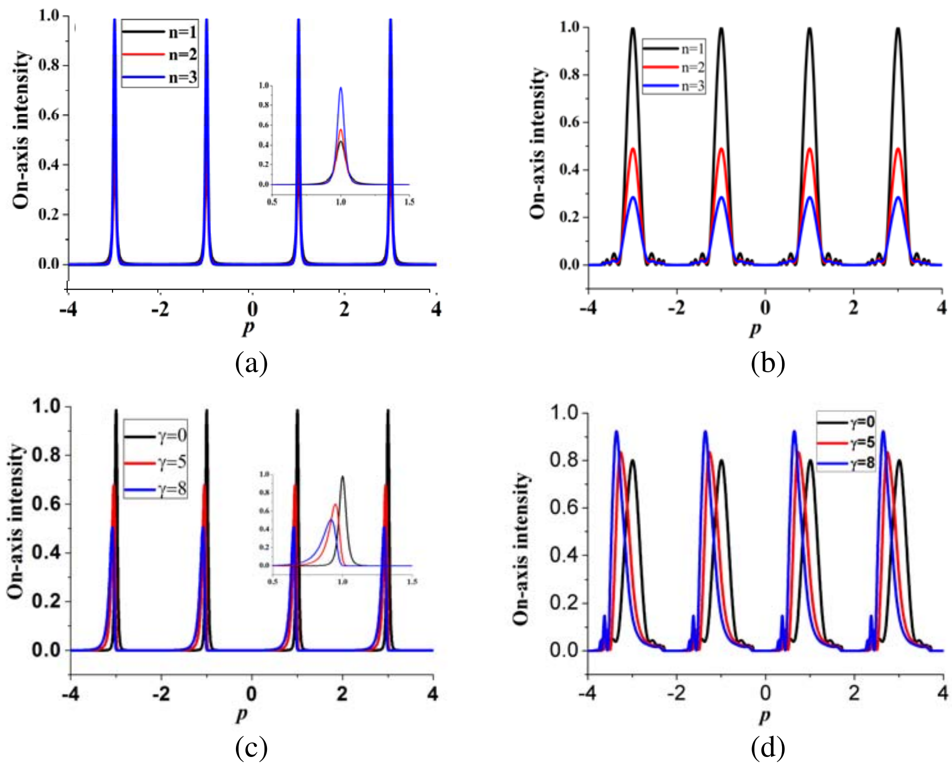
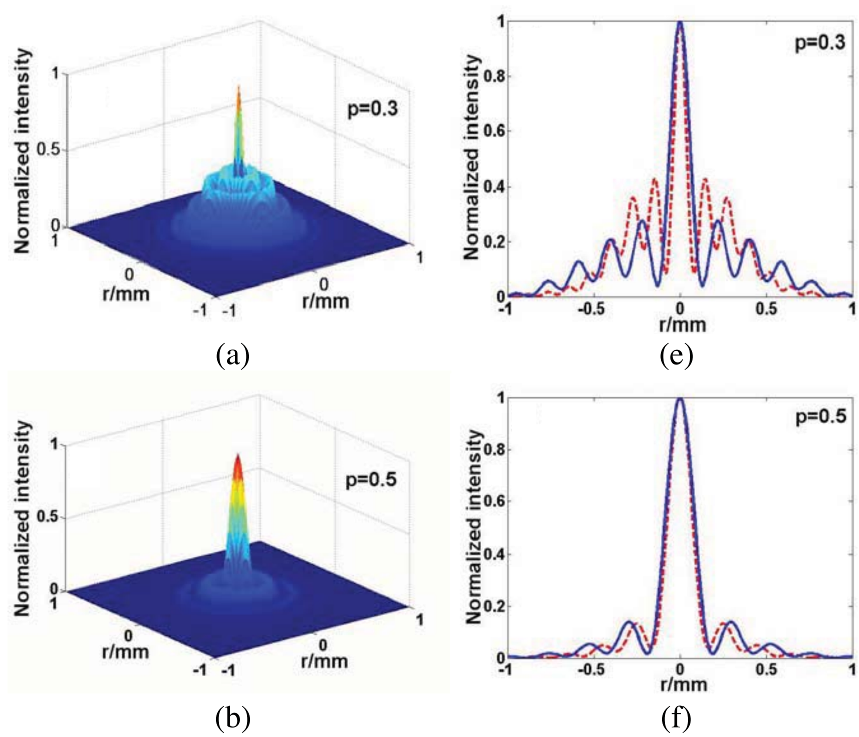


Figure 4. The intensity distribution on optical axis of Hypergeometric-Gauss beams (with different beam orders and real variables) through the FRT optical systems with hard edge aperture with $w = 1\,\text{mm}$, $n = 3$, $\lambda = 632.8\,\text{nm}$, $f = 400\,\text{mm}$. (a), (c) $\delta = \infty$; (b), (d) $\delta = 1$.



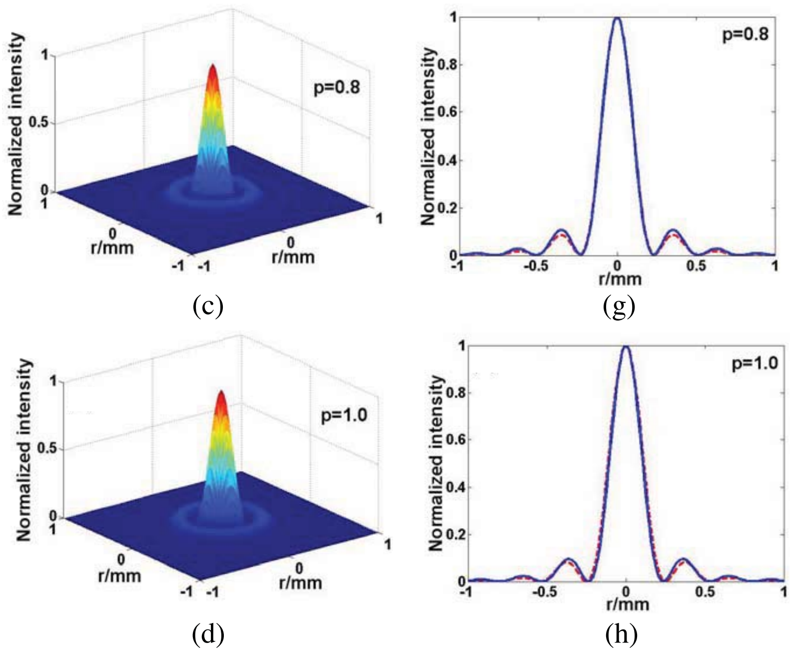


Figure 5. The normalized intensity distribution of Hypergeometric-Gauss beams through the FRT optical systems with hard edge aperture with $w = 1\,\text{mm}$, $n = 3$, $\lambda = 632.8\,\text{nm}$, $f = 400\,\text{mm}$, $\gamma = 0$, $\delta = 0.5$.

One can see from Fig. 5 that the normalized intensity of Hypergeometric-Gauss beams is greatly affected by the FRT order. Figs. 5(e)–(h) show that the approximate analytic Eq. (14) and direct integral formula (10) can obtain almost coincident calculation results, in addition to some deviations as smaller FRT order ($p < 0.3$). Consequently, the approximate solution method used in this paper can also be effective to research transmission characteristics of Hypergeometric-Gauss beams through the FRT optical systems with hard edge aperture.

4. CONCLUSION

Based on the Collins integral formula and Lohmann optical system, the analytic expressions of electric field and normalized intensity distribution of Hypergeometric-Gauss beams through the FRT optical systems with and without hard edge aperture are derived theoretically. Numerical results show that the intensity distribution of Hypergeometric-Gauss beams is related to not only the FRT order p, but also the beam order n, real variables γ and the aperture truncation parameter δ. The normalized intensity distribution obtained by approximate analytical method and numerical integral method are also given. The results have a certain reference value for the study on the transmission characteristics of beam through the FRT optical system with hard edge aperture, beam shaping and optical information transmission.

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