Development of Fundamental Theory of Thin Impedance Vibrators

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Abstract—In the paper, we prove two theorems relating to the theory of thin impedance vibrator radiators excited by a lumped voltage generator under rather general conditions. The first theorem proves that influence of external electrodynamic media on the vibrator current distribution is limited and can be estimated using a small natural parameter. The second theorem ascertains that there exists principal possibility to compensate influence of spatial boundaries upon current distributions on a perfectly conductive vibrator by applying to its surface complex impedance with predetermined variation along the vibrator length. Several corollaries disclose a range of the theorems application and their fundamental importance.

1. INTRODUCTION

The theory of thin vibrators is now considered as classic both for perfectly conducting [1, 2] and impedance vibrators [3–10]. The theory was outlined in a large number of well-known articles and monographs (see, e.g., references in [3]). However, this problem is still of great interest, since the vibrator structures are widely used in various devices and systems to provide the required mode of excitation. Since the problem is multivariable, an experimental optimization of devices is almost impossible, and physically adequate mathematical models are needed for compound boundary value problems, non-coordinate border of spatial domains, presence of scattering irregularities, medium inhomogeneity, etc. In any case, a key stage of modeling consists in a search of current distributions on a vibrator surface. The problem solution can be greatly simplified by selection of the basic current distribution. This choice should be done taking into account the vibrator surrounding which cannot always be done relying only on analysis of available publications. Therefore, the generalization of the theoretical results concerning the influence of surrounding media upon the current distribution on the thin impedance vibrator is an actual problem.

One approach to such generalization is based on an analytical solution of an integral equation for vibrator current using small natural parameter [7]. For example, in the monograph [11], attention was drawn to the fact that the functional effect of walls of a hollow rectangular waveguide upon the current on the linear scattering vibrator located inside the waveguide contains the proportionality factor equal to the natural small parameter of the problem. An analog situation has arisen during the analytical determination of the current on a radial impedance monopole allocated on the perfectly conducting sphere [6,9], and on the impedance vibrator over the perfectly conducting screen of finite size [10]. The boundary problem solution for these two cases requires that the total field should be represented by waves of electric and magnetic types. This article is aimed at the generalization of the influence on a boundary value problem with arbitrary boundaries that do not possess a property of mutual transformation of electric and magnetic fields. The second theorem assets that there exists principal possibility to compensate influence of spatial boundaries upon current distributions on a perfectly

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Lemma. Let a thin radiating impedance vibrator, excited by a point source is placed in an infinite homogeneous medium with material parameters \((\varepsilon_1, \mu_1)\). The vibrator is a segment of a circular cylinder, whose radius \(r\) and the length \(2L\) are such that inequalities \([r/(2L)] \ll 1\) and \(r\sqrt{\varepsilon_1\mu_1}/\lambda \ll 1\) (\(\lambda\) is the wavelength in free space) hold. Then the electric current on the vibrator can be represented by a power series \(J(s) = \alpha J_1(s) + \alpha^2 J_2(s) + \ldots\) in the small parameter \(\alpha = \frac{1}{2\ln[r/(2L)]}, |\alpha| \ll 1\), and \(J_n(s)\) is the current approximation of the \(n\)-th order \((n = 1, 2\ldots)\).

Proof. The proof is based upon the well-known solution of the integral equation for the vibrator current obtained using a small natural parameter \([7]\). Consider the following equation \([3]\)

\[
\frac{1}{i\omega\varepsilon_1} (\text{grad}\text{div} + k_1^2) \int_S \hat{G}^e(\vec{r}, \vec{r}') \vec{J}(\vec{r}') d\vec{r}' = -\vec{E}_0(\vec{r}) + z_i(\vec{r}) \vec{J}(\vec{r}),
\]

where \(z_i(\vec{r})\) is the linear intrinsic impedance ([Ohm/m]) of the vibrator, \(\vec{E}_0(\vec{r})\) is the field of extraneous sources, \(\hat{G}^e(\vec{r}, \vec{r}')\) is the tensor Green’s function of the spatial domain for the electric vector potential, \(k_1 = k\sqrt{\varepsilon_1\mu_1}, k = \omega/c = 2\pi/\lambda\) is wave number, \(c \approx 2.998 \times 10^{10} \text{ cm/s}\) is the speed of light in vacuum. Equation (1) was obtained using boundary conditions on the vibrator surface \(S\) if time \(t\) dependence is \(e^{i\omega t}\) and \(\omega\) is a circular frequency of a monochromatic process.

In a thin wire approximation, the electric current induced on the vibrator surface can be represented as

\[
\vec{J}(\vec{r}) = \vec{e}_s J(s) \psi(\rho, \varphi),
\]

where \(\vec{e}_s\) is the unit vector directed along the vibrator axis; \(s\) is the local coordinate along the vibrator axis; \(\psi(\rho, \varphi)\) is the function of transverse \((\perp)\) polar coordinates, satisfying the normalization condition \(\int \psi(\rho, \varphi) \rho d\rho d\varphi = 1\). If the relations

\[
\int_S \hat{G}^e(\vec{r}, \vec{r}') \vec{J}(\vec{r}') d\vec{r}' = \int_{-L}^{L} J(s') \int_{-\pi}^{\pi} \frac{e^{-ik_1 \sqrt{(s-s')^2 + [2r\sin(\varphi/2)]^2}}}{\sqrt{(s-s')^2 + [2r\sin(\varphi/2)]^2}} \psi(r, \varphi) r d\varphi ds' \\
\approx \int_{-L}^{L} J(s') \frac{e^{-ik_1 \sqrt{(s-s')^2 + r^2}}}{\sqrt{(s-s')^2 + r^2}} ds' = \int_{-L}^{L} J(s') \frac{e^{-ik_1 R(s, s')}}{R(s, s')} ds' = \int_{-L}^{L} J(s') \frac{e^{-ik_1 R(s, s')}}{R(s, s')} ds,
\]

are valid \([1, 3]\) and \(z_i(\vec{r}) = z_i(s) \equiv \text{const}\). The surface Equation (1) can be converted to an integral equation with a quasi-one-dimensional kernel

\[
\left(\frac{d^2}{ds^2} + k_1^2\right) \int_{-L}^{L} J(s') \frac{e^{-ik_1 R(s, s')}}{R(s, s')} ds' = -i\omega\varepsilon_1 E_{0s}(s) + i\omega\varepsilon_1 z_i J(s),
\]

where \(E_{0s}(s)\) is projection of the extraneous source field on the vibrator axis.

Let us isolate a logarithmic singularity in Equation (4) core using the method described in \([1, 7]\)

\[
\int_{-L}^{L} J(s') \frac{e^{-ik_1 R(s, s')}}{R(s, s')} ds' = \Omega(s) J(s) + \int_{-L}^{L} \frac{J(s') e^{-ik_1 R(s, s')} - J(s)}{R(s, s')} ds'.
\]
where

\[ \Omega(s) = \int_{-L}^{L} \frac{ds'}{\sqrt{(s-s')^2 + r^2}} = \Omega + \gamma(s). \quad (6) \]

\[ \gamma(s) = \ln \frac{[L+s] + \sqrt{(L+s)^2 + r^2}}{[L-s] + \sqrt{(L-s)^2 + r^2}} \]

is a function vanishing at the vibrator center and reaching maxima at the vibrator ends, where the current is zero as required by the boundary conditions \( J(\pm L) = 0 \) \([1, 3]\). \( \Omega = 2 \ln \frac{2L}{\Delta} \) is a large parameter. Then, taking into account Equation (5), Equation (4) can be converted to the following integral-differential equation with a small parameter

\[ \frac{d^2 J(s)}{ds^2} + k^2 J(s) = \alpha \{ i\omega \mathcal{E}_0(s) + F[s, J(s)] - i\omega \mathcal{E}_1 z_i J(s) \}. \quad (7) \]

Here \( \alpha = -\frac{1}{\Omega} = \frac{1}{2\pi \mu r / (2L)} \) is the small natural parameter \((|\alpha| \ll 1)\), and functional

\[ F[s, J(s)] = -\frac{dJ(s')}{ds'} R(s, s') \bigg|_{-L}^{L} + \left[ \frac{d^2 J(s)}{ds^2} + k^2 J(s) \right] \gamma(s) \]

\[ + \int_{-L}^{L} \frac{[d^2 J(s')}{ds^2} + k^2 J(s')] e^{-ik_1 R(s, s')} - \left[ \frac{d^2 J(s)}{ds^2} + k^2 J(s) \right] ds' \]

presents the vibrator eigenfield.

If we denote \( k = k_1 \sqrt{1 + i\omega \mathcal{E}_1 z_i / k_1} = k_1 \sqrt{1 + i2\alpha Z_S / (\mu_1 kr)} \) \((Z_S = Z_S / Z_0)\) is distributed surface impedance normalized to the wave resistance \( Z_0 = 120\pi \) \([\text{Ohm}]\), Equation (7) can be written as

\[ \frac{d^2 J(s)}{ds^2} + \tilde{k}^2 J(s) = \alpha \{ i\omega \mathcal{E}_0(s) + F[s, J(s)] \}. \quad (9) \]

Since Equation (9) is proportional to the small parameter \( \alpha \), its solution can be obtained by a successive approximation technique using the following algorithm

\[
\begin{align*}
\frac{d^2 J_1(s)}{ds^2} + \tilde{k}^2 J_1(s) &= i\omega \mathcal{E}_0(s), \\
\frac{d^2 J_2(s)}{ds^2} + \tilde{k}^2 J_2(s) &= F[s, J_1(s)], \\
&\vdots \\
\frac{d^2 J_n(s)}{ds^2} + \tilde{k}^2 J_n(s) &= F[s, J_{n-1}(s)].
\end{align*}
\]

The solution of each differential equation can be obtained using the boundary conditions for the current \( J_1(\pm L) = J_2(\pm L) = \ldots = J_n(\pm L) = 0 \). Thus, we obtain the current decomposition as power series in small parameter \( \alpha \), i.e., \( J(s) = \alpha J_1(s) + \alpha^2 J_2(s) + \ldots \), which was to be proved.

The zero approximation for the current \( J_0 \) was not included into the equation system (10), since its solution is \( J_0(s) = C_1 \cos \tilde{k}s + C_2 \sin \tilde{k}s \). Taking into account losses in the medium and/or on the vibrator surface, the trigonometric functions in the solution are complex and cannot be zero for any arguments. Therefore, to satisfy the boundary conditions \( J_0(\pm L) = 0 \), the constants \( C_1 \) and \( C_2 \) should be zero and identities \( J_0 \equiv 0 \), \( F[s, J_0(s)] \equiv 0 \) become valid for any vibrator length.

The first approximation of the vibrator current obtained from (10) as sum of the general and partial solutions is

\[ J(s) \approx \alpha J_1(s) = -\frac{i\omega \mathcal{E}_1 \tilde{k}}{\sin 2kL} \times \left\{ \begin{array}{l}
\sin \tilde{k} (L-s) \int_{-L}^{s} E_{0s}(s') \sin \tilde{k} (L+s') ds' \\
+ \sin \tilde{k} (L+s) \int_{s}^{L} E_{0s}(s') \sin \tilde{k} (L-s') ds',
\end{array} \right\} \quad (11) \]
and it does not depend on the function $F[s, J(s)]$ (8).

3. FORMULATION AND PROOF OF THE FIRST THEOREM

**Theorem 1.** Let a thin radiating impedance vibrator, exited by a point source, be placed in an electrodynamic volume filled by homogeneous medium with material parameters $(\varepsilon_1, \mu_1)$. The volume boundary is an arbitrary Lyapunov surface $S$ [12], not passing through sources of extraneous currents. The vibrator is a segment of a circular cylinder, whose radius $r$ and the length $2L$ are such that inequalities $\left| \frac{r}{4L} \right| \ll 1$ and $r \sqrt{\varepsilon_1 \mu_1} / \lambda \ll 1$ ($\lambda$ is the wavelength in free space) hold. Then the influence of the volume boundaries upon the current distribution on the vibrator surface does not exceed an amount proportional to the small natural parameter $\alpha = \frac{1}{2(4L)}$.

**Proof.** The kernel $\hat{G}^e(\vec{r}, \vec{r}')$ of integral Equation (1) is the electric tensor Green’s function of the closed volume. In a system of orthogonal curvilinear coordinates $(q_1, q_2, q_3)$, this function, according to the general properties [12], satisfies the inhomogeneous Helmholtz equation

$$\Delta \hat{G}^e (\vec{q}, \vec{q}') + k_1^2 \hat{G}^e (\vec{q}, \vec{q}') = -4\pi \delta (q_1 - q'_1) \delta (q_2 - q'_2) \delta (q_3 - q'_3),$$

where $I$ is unit tensor; $(q'_1, q'_2, q'_3)$ are source coordinates; $\delta (q - q')$ is Dirac delta function; $h_n$ are Lame coefficients. Laplacian $\Delta$ applies to all tensor components. Then, the solution for the vector Hertz potential can be written as

$$\hat{\Pi}^e (\vec{q}) = \frac{1}{i\omega \varepsilon_1} \int_V \hat{J}(\vec{q}') \hat{G}^e (\vec{q}, \vec{q}') \, dv + \oint_S \left\{ \text{div} \hat{\Pi}^e (\vec{q}') \hat{G}^e (\vec{q}, \vec{q}') \vec{n} - \text{div} \hat{G}^e (\vec{q}, \vec{q}') \hat{\Pi}^e (\vec{q}') \vec{n} \right\} ds',$$

where $\vec{n}$ is the unit vector of the external normal to the surface $S$. The volume integral is taken over the entire volume $V$ ($dv$ is volume element), and the surface integral is taken over the entire surface ($ds'$ is the area element in the primed coordinates). The expression in the curly brackets is a vector, and differentiation is performed over the primed coordinates.

Thus, the solution of the inhomogeneous Helmholtz equation is the sum of the volume and surface integrals. The surface integrals can be eliminated by building the Green’s function in a special way. If the components of the Green’s function $G^e(\vec{q}, \vec{q}')$ and components of the vector potentials $\hat{\Pi}^e(\vec{q})$ satisfy the boundary conditions on the surface $S$, the surface integrals vanish, since integrand of surface integrals in (13) vanish. Otherwise, the solution will have the general form (13). Thus, the expression (13) allows us to use alternative forms of the Green’s function.

If the Green’s tensor for unbounded domain, which is the solution of Equation (12) satisfying the boundary conditions at infinity is

$$\hat{G}^e(\vec{r}, \vec{r}') = I e^{-ik_1 |\vec{r} - \vec{r}'|} = I \hat{G}_0^e(\vec{r}, \vec{r}'),$$

the vector Hertz potential (13) can be presented as

$$\hat{\Pi}^e (\vec{r}) = \frac{1}{i\omega \varepsilon_1} \int_V \hat{J}(\vec{r}') \hat{G}^e (\vec{r}, \vec{r}') \, dv + \oint_S \left\{ \text{div} \hat{\Pi}^e (\vec{r}') \left( I \hat{G}_0^e(\vec{r}, \vec{r}') \right) \vec{n} - \text{div} \left( I \hat{G}_0^e(\vec{r}, \vec{r}') \right) \hat{\Pi}^e (\vec{r}') \vec{n} \right\} ds'.$$

Considering that in the general case $\hat{E}(\vec{r}) = \text{grad} \text{div} \hat{\Pi}^e(\vec{r}) + k_1^2 \varepsilon_1 \mu_1 \hat{\Pi}^e(\vec{r})$, substituting Equation (15)
field functionals
The influence is taken into account by successive approximations. Thus, the influence of the volume vibrator can be represented by a series in the small parameter $\epsilon$.
Corollary 1.4. One of the Theorem 1 conditions are the following requirements on the surface: 1) it must be a Lyapunov surface; 2) it should not cross current sources; 3) in a more general sense, it has to be passive, i.e., without generation of extraneous fields, and 4) it does not possess the properties of mutual transformation of spatial harmonics of the electric and magnetic fields. Since the type of electrodynamic boundary surfaces is not defined and do not exclude the possibility of its composite presentation, the theorem is valid both for different types of surfaces such as perfectly conducting, impedance, partially impedance, etc. and for the different scattering inhomogeneities in the spatial domain, whose surfaces can be interpreted as parts of the total surface.

Commentary. Parts of the boundary surface can be presented by impedance surfaces of coupling holes between coupling volumes. Separate areas of piecewise inhomogeneous of the magneto-dielectric filling of the electrodynamic volume can be thought as scattering irregularities.

The approach used to the proof of Theorem 1 allows us to formulate the second theorem. It concerns the fundamental possibility to compensate the influence of the spatial boundaries upon the current distribution on a perfectly conducting vibrator using “application” of the distributed impedance on the surface.

4. FORMULATION AND PROOF OF THE SECOND THEOREM

In the proof of the Lemma and Theorem 1, linear impedance of the vibrator was assumed to be constant $z_i(\vec{r}) = z_i(s) \equiv \text{const}$. Now, let us assume that the impedance can be distributed along the vibrator axis $z_i(\vec{r}) = z_i(s)$, or be concentrated at some points on the vibrator axis, or be superposition of these two options.

Theorem 2. Let a thin radiating impedance vibrator, exited by a point source, be placed in an electrodynamic volume filled by homogeneous medium with material parameters $(\varepsilon_1, \mu_1)$. The volume boundary is an arbitrary Lyapunov surface $S$ [12], not passing through sources of extraneous currents. The vibrator is a segment of a circular cylinder, whose radius $r$ and the length $2L$ are such that

\[ \text{inequalities } |r/(2L)| \ll 1 \quad \text{and} \quad r \sqrt{\varepsilon_1/\mu_1}/\lambda \ll 1 \quad (\lambda \text{ is the wavelength in free space}) \text{ hold. Then, the influence of the volume boundaries upon the current distribution on the vibrator surface can be compensated by coating its surface with the complex impedance, varying along the vibrator axis, } z_i(s) = \frac{F_S(s, J(s))}{J(s)} \text{. } J(s) \text{ is the current distribution on the perfectly conducting vibrator, and } F_S(s, J(s)) \text{ is the functional (17) defining the boundary influence.}

Proof. Let us consider the equation similar to Eq. (18)

\[
\left( \frac{d^2}{ds^2} + k_1^2 \right) \int_{-L}^{L} J(s') \frac{e^{-ik_1 R(s,s')}}{R(s, s')} ds' = -i\omega \varepsilon_1 [E_0(s) + F_S(s, J(s)) - z_i(s)J(s)],
\]

where $z_i(s)$ is complex surface impedance which varies along the vibrator axis. If the equality $F_S(s, J(s)) = z_i(s)J(s)$ on the right hand side of Equation (19) holds, the vibrator current is determined only by the fields of extraneous sources. Hence, Equation (19) can be formally represented as a system of two equations

\[
\begin{cases}
\left( \frac{d^2}{ds^2} + k_1^2 \right) \int_{-L}^{L} J(s') \frac{e^{-ik_1 R(s,s')}}{R(s, s')} ds' = -i\omega \varepsilon_1 E_0(s), \\
F_S(s, J(s)) - z_i(s)J(s) = 0.
\end{cases}
\]

The first equation is Equation (4) for a perfectly conducting vibrator, $z_i = 0$, located in an infinite homogeneous medium. The second equation is the functional equation, which can be solved using the current found from the first equation. These equations can be used to obtain the distribution of variable complex impedance along the vibrator axis

\[
z_i(s) = \frac{F_S(s, J(s))}{J(s)},
\]

where $z_i(s)$ can be a generalized function. Thus, the influence of the boundaries can be fully compensated by “applying” the impedance in Eq. (21) to the vibrator surface. The current distribution of such
impedance vibrator corresponds now to that of perfectly conducting vibrator, located in an infinite homogeneous medium with material parameters \((\varepsilon_1, \mu_1)\). Thus, Theorem 2 is proved.

4.1. Corollaries from Theorem 2

**Corollary 2.1.** Since the surface of the scattering irregularities, located in electrodynamic volume, can be interpreted as components of the common surface \(S\), and the Theorem 2 is valid for this case.

**Commentary.** The main difficulty of the Theorem 2 application concerns determination of the functional \(F_S(s, J(s))\). However, if Green’s function of the electromagnetic volume is known, the problem is greatly simplified, since the fields at the boundary surface \(S\) in Eq. (17) can be found using the Green’s function.

**Corollary 2.2.** Input resistance of the impedance vibrator with the distribution in Eq. (21), positioned in the electrodynamic volume, is equivalent to that of perfectly conductive vibrator of the same geometry located in free space.

4.2. Example of Theorem 2 Application

The application of the Theorem 2 can be demonstrated by the problem of the electro-magnetic radiation by a horizontal vibrator in a semi-infinite material medium [3]. Geometry of the structure and notation are shown in Fig. 1. Here \(\{x, y, z\}\) is a Cartesian coordinate system associated with a perfectly conducting plane, a cylindrical vibrator allocated in a medium at a distance \(h\) from the plane with material parameters \((\varepsilon_1, \mu_1)\). The vibrator length is \(2L\) and its radius is \(r\).

![Figure 1. The geometry of the vibrator structure.](image)

Let us substitute Green’s function for the half-space

\[
G_s(s, s') = \frac{e^{-ik_1\sqrt{(s-s')^2+r^2}}}{\sqrt{(s-s')^2+r^2}} - \frac{e^{-ik_1\sqrt{(s-s')^2+(2h+r)^2}}}{\sqrt{(s-s')^2+(2h+r)^2}}.
\]

in Equation (4). Then the equation for the vibrator current can be written as

\[
\left(\frac{d^2}{ds^2} + k_1^2\right)\int_{-L}^{L} J(s') e^{-ik_1\sqrt{(s-s')^2+r^2}} \frac{ds'}{\sqrt{(s-s')^2+r^2}} = -i\omega \varepsilon_1 E_{0a}(s) + \left(\frac{d^2}{ds^2} + k_1^2\right)\int_{-L}^{L} J(s') \frac{e^{-ik_1\sqrt{(s-s')^2+(2h+r)^2}}}{\sqrt{(s-s')^2+(2h+r)^2}} ds' + i\omega \varepsilon_1 z_i(s) J(s).
\]
Without loss of generality, we assume that the vibrator is excited at the center of a lumped voltage generator with an amplitude $V_0$. Let us apply the Theorem 2. First, we find the vibrator current in the infinite material medium by solving the first equation of the system (20) using the averaging method. The solution [3, 4, 7] is

$$J(s) = -\alpha V_0 \left( \frac{i \omega \varepsilon_1}{2k_1} \right) \frac{\sin k_1 (L - |s|) + \alpha P_s^r(k_1 r, k_1 s)}{\cos k_1 L + \alpha P_s^L(k_1 r, k_1 L)}. \quad (24)$$

Here $P_s^r(k_1 r, k_1 s) = P_s[k_1 r, k_1 (L + s)] - (\sin k_1 s + \sin k_1 |s|)P_s^L(k_1 r, k_1 L)$, the constants $P_s[k_1 r, k_1 (L + s)]$ and $P_s^L(k_1 r, k_1 L)$ are defined by the analytical formulas [3, 4, 7].

Then, substituting the current $J(s)$ from (24) into the formula (21), we obtain

$$z_i(s) = \left( \frac{d^2}{ds^2} + k_1^2 \right) \int_{-L}^{L} \frac{\sin k_1 (L - |s'|) + \alpha P_s^r(k_1 r, k_1 s')}{\sin k_1 (L - |s|) + \alpha P_s^L(k_1 r, k_1 s)} \frac{e^{-ik_1 \sqrt{(s-s')^2 + (2h+r)^2}}}{\sqrt{(s-s')^2 + (2h+r)^2}} ds'. \quad (25)$$

Thus, the functional $F_2(\bar{r}, \bar{J}(\bar{r}))$ in (17), wherein the integration is done over an infinite boundary surface, is replaced by the functional, wherein integration is performed only over the surface of the mirror vibrator image. Since the kernel of the integral operator is a smooth function, the differentiation operator can be moved under the integral sign. The final expression for the impedance distribution function can be written as

$$z_i(s) = \frac{1}{F_i(s)} \int_{-L}^{L} F_i(s') F_e(s, s') ds', \quad (26)$$

were

$$F_i(s) = \sin k_1 (L - |s|) + \alpha P_s^r(k_1 r, k_1 s), \quad R_1(s, s') = \sqrt{(s-s')^2 + (2h+r)^2},$$

$$F_e(s, s') = \frac{e^{-ik_1 R_1(s, s')}}{(R_1(s, s'))^4} \left[ \frac{(s-s')^2}{3ik_1 - k_1^2} \left( 3ik_1 - \frac{k_1^2}{R_1(s, s')} \right) - R_1(s, s') - ik_1 (R_1(s, s'))^2 + k_1^2 (R_1(s, s'))^3 \right].$$

As might be expected, if $h \to \infty$ in expression (26), $|z_i(s)| \to 0$ for the any point on the vibrator surface.

5. CONCLUSION

The paper presents a generalization of the theory of thin impedance vibrators which can be found in a number of publications devoted to vibrator radiators with a lumped excitation in free space or in electrodynamic volumes with coordinate boundaries. Taking into account methodological aspects of the problem, the authors decided to make such a generalization in the form of two theorems. On the one hand, the approach allows systematically assessing already known results, and, on the other hand, extend the methodology to the solutions of new boundary value problems with compound surface boundaries. Requirements for the boundary surface are as follows: it must be a Lyapunov surface, which does not cross current sources, or in a more general sense, it has to be passive and does not possess the properties of mutual transformation of spatial harmonics of the electric and magnetic fields.

The first theorem proves the limited influence of the external electrodynamic medium upon the current distribution on the radiating vibrator. Quantification of this effect was related to the natural small parameter $\alpha = \frac{1}{2 \min \{r, (2L)\}}$; therefore, the result would not exceed the small parameters $\alpha$. The theorem proof is based on the lemma, which has independent significance. Four corollaries disclose more fully the theorems’ scope. The theorem can be used for selection of the current distribution in the vibrator for the solution of complicated boundary value problems and assessing the accuracy of this approximation.

The second theorem states that the influence of the spatial boundaries, including borders of scattering irregularities on the current distribution in a perfectly conducting radiator can be
compensated by “applying” an extended complex impedance having the required distribution to the vibrator surface. The theorems’ application was demonstrated by solving the problem for the horizontal vibrator located above a perfectly conducting plane.

The results presented in the paper can be useful both for electromagnetic theory of thin impedance vibrators and for solution of boundary value problems with vibrator excitations including the questions related physical interpretation of mathematical modeling results.

REFERENCES