

Curved Space-Time for Light by an Anisotropic Medium: Media with the Variable Optical Axes

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Abstract—An optical impedance-matched medium with a gradient refractive index can resemble a geometrical analogy with an arbitrary curved space-time. In this paper, we show that a non-impedance-matched medium with a varying optical axis can also resemble the features of a space of non-trivial metric for the light. The medium with a varying optical axis is an engineered stratified slab of material, in which the orientation of the optical axis in each layer slightly differs from the other layers, while the magnitude of refractive index remains constant. Instead of the change in refractive index, the inhomogeneity of such a medium is induced by the local anisotropy. Therefore, the propagation of light depends on the local optical axis. We study the conditions that make the analogy between curved space-time and a medium with a varying optical axis. Extension of the transformation optics to the media with optical axis profile might ease some fabrication difficulties of materials with gradient refractive index.

1. INTRODUCTION

Transformation optics [1–4] works based on the diffeomorphic map between a virtual space and physical space. For the sake of simplicity, in most of the applications, physical space is constructed from an isotropic medium with a refractive index profile varying in position, while the optical axis remains fixed. This simplification restricts the underlying diffeomorphic map to the family of quasi-conformal maps [5]. However, inspired by optical axes grating in liquid crystals [6, 7], one can show [8] that the diffeomorphic map between the virtual space and physical space might extend to the family of the area-preserving maps by sequentially manipulating the direction of optical axes instead of gradually changing the amplitude of refractive index.

In theory, an array of homogeneous anisotropic thin layers, where in each of the layers the direction of the principal axes is controllable, can form an inhomogeneous medium with optical axes profile. The inhomogeneity is induced by the local anisotropy and gradual change in the direction of the optical axes over space. The global inhomogeneity is responsible for curving the light trajectories. This kind of medium can be realized, for example, by applying the external field or internally charged particles on the designed arrays of liquid crystals [9] or by other techniques that combine the arrays of homogeneous anisotropic layers. Such materials show the capacity of being used in transformation optics designs when impedance-matched materials are costly. Unlike most of the artificial metamaterials that are constructed from two kinds of meta-atom, the medium with variable optical axis [8] can be fabricated from one kind of homogeneous anisotropic material.

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From the practical perspective, unlike the impedance-matched media, anisotropic materials [10] are accessible in nature and even much cheaper to design artificially. Birefringent crystals [11] are the best-known materials that perform electrical anisotropy based on their particular crystal structure and the symmetry of their space-group. Many plastics are also anisotropic [12], because their molecules are ‘frozen’ in a stretched conformation when the plastic is molded or extruded.

In this research, considering a medium with variable optical axis [8], we construct an optical metric in the plane of propagation, for extraordinary rays. Therefore, a medium with the variable optical axis can be considered as a curved space-time for extraordinary light rays. In the final part of the paper, we study the anisotropic, impedance-matched medium. We show in details, instead of demanding the impedance-matched condition, for some applications, that it is enough to restrict the concern only to the electrical response of the material and utilize the optical properties of an anisotropic birefringent medium.

The paper is structured as follows. In Section 2, we briefly summarize the method applied. In Section 3, we explain in detail a technique, called “eigenvalue wave equation method” to solve the Maxwell equations in the anisotropic medium. In Section 4, we apply the method to study the behavior of light in a normal incident on the single anisotropic slab. In Section 5, we derive the light trajectory for two examples where the array of the non-magnetic slabs forms a variable optical axes media. Further, we construct the optical metric for such media and discuss the corresponding curvature in the optical plane for ray trajectories. Additionally, we have studied the functionality of the birefringent medium in transformation optics.

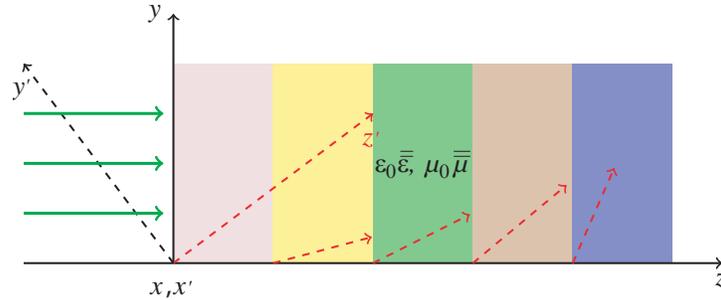


Figure 1. Schematic of anisotropic slabs with $\bar{\epsilon}$ and $\bar{\mu}$ as a tensor, light incidents from left and incident plane is y - z plane; two principal axes of each slab, z' and y' , do not lay on the z and y coordinate. Red dashed lines indicate third principal axis of the medium; The orientation of the principal axis changes in each slab.

2. METHOD

The aim of this paper is to derive the effective optical metric for a stratified medium in which the anisotropy and homogeneity are only local. The medium as a whole is inhomogeneous. We consider the medium consisting of many thin slabs. Each slab assumes to be anisotropic and homogeneous. We will derive the solutions of the wave equation in any single anisotropic layer with an arbitrary direction of the principle axes. Having the wave solutions for each slab, we can calculate the ray path and consequently, achieve the ray trajectory in the system. According to the Fermat principle, light in a medium follows the geodesics. By studying the properties of the light geodesics in the medium, we can associate a geometry to the medium. Particularly, it is possible to study the curvature of the effective metric.

3. LIGHT IN ANISOTROPIC MEDIA

In mathematical term, the electric permittivity and magnetic permeability of anisotropic materials are described by a tensorial quantity. For most of the anisotropic media, there are two different refractive indices associated with two normal modes. One of these refractive indices, called extraordinary, depends

on the orientation of the principle axes [13,14]. Therefore, it is convenient to start studying the propagation of light in an anisotropic slab.

Maxwell equations in an inhomogeneous anisotropic source-free materials, $\rho = 0$, $\bar{J} = 0$, are given by [15]:

$$\nabla \cdot \bar{D} = 0, \quad \nabla \cdot \bar{B} = 0, \quad \nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}, \quad \nabla \times \bar{H} = \frac{\partial \bar{D}}{\partial t}. \quad (1)$$

The constitutive relations are hold as:

$$\bar{D} = \varepsilon_0 \bar{\varepsilon} \bar{E}, \quad \bar{B} = \mu_0 \bar{\mu} \bar{H}, \quad (2)$$

where $\bar{\varepsilon}$ and $\bar{\mu}$ are respectively the electric permittivity and magnetic permeability tensors for an arbitrary optical axis.

Explicitly assuming electric and magnetic anisotropy, wave equation can be written as

$$\bar{\nabla} \times \bar{\mu}^{-1} (\bar{\nabla} \times \bar{E}) = -\mu_0 \frac{\partial^2 \bar{D}}{\partial t^2}. \quad (3)$$

In a homogeneous anisotropic medium, the plane waves, $\bar{E} \propto \exp[i\bar{k} \cdot \bar{r} - i\omega t]$, is usually assumed as the general solution of the Maxwell wave equation [16]. Using this assumption, Equation (3) becomes:

$$\bar{k} \times \bar{\xi} (\bar{k} \times \bar{E}) = -k_0^2 \bar{\varepsilon} \bar{E}, \quad (4)$$

where $\bar{\xi} = \bar{\mu}^{-1}$. We rewrite Equation (4) in a matrix equation form,

$$M \bar{E} = -k_0^2 \bar{\varepsilon} \bar{E},$$

in which the matrix elements of M are obtained as

$$\begin{aligned} M_{ii} &= 2\xi_{jk} k_j k_k - (\xi_{kk} k_j^2 + \xi_{jj} k_k^2), \\ M_{ij} &= \xi_{ij} k_k^2 - (\xi_{ki} k_j + \xi_{kj} k_i) k_k + \xi_{kk} k_j k_i, \end{aligned} \quad (5)$$

where $\{i \neq j \neq k\} = \{1, 2, 3\}$.

In an anisotropic material, the electric and magnetic components of the field are not necessarily perpendicular to the wave vector. The direction of \bar{E} and \bar{H} -fields can vary as light propagates through the medium. However, electric and magnetic inductions are always perpendicular to the wave vector according to the relations, $\nabla \cdot \bar{D} = 0$ and $\nabla \cdot \bar{B} = 0$. So, it is useful to write the eigenvalue equation in term of \bar{D} [13]. By defining the phase refractive index as a ratio between wave number in medium and in vacuum, $n = k/k_0$ and substituting $\bar{k} = nk_0 \hat{U}$ in Equation (4), we get eigenvalue equation,

$$\hat{U} \times \bar{\xi} (\hat{U} \times \bar{\eta} \bar{D}) = -\frac{1}{n^2} \bar{D}, \quad (6)$$

where $\hat{U} = \bar{k}/k$ is a direction of the wave vector and $\bar{\eta} = \bar{\varepsilon}^{-1}$. Accordingly, we find two directions for \bar{D} corresponding to the wave vector. Equation (6) can be written in the operator form,

$$L \bar{D} = -\frac{1}{n^2} \bar{D}. \quad (7)$$

In a general coordinate, matrix L might have a complicated form. Nevertheless, knowing that the light propagates in a plane, we can establish a fixed coordinate system such that the propagation plane coincides with one of the principal planes of the coordinate system. As shown in Fig. 1, we choose y - z as the propagation plane, so that the x component of the wave vector vanishes. Thus, matrix L is simplified to the following non-zero components:

$$L_{1i} = (\xi_{31}\eta_{3i} - \xi_{33}\eta_{1i})u_2^2 + (-\xi_{22}\eta_{1i} + \xi_{21}\eta_{2i})u_3^2 + (2\xi_{23}\eta_{1i} - \xi_{31}\eta_{2i} - \xi_{21}\eta_{3i})u_2u_3, \quad (8)$$

$$L_{2i} = (\xi_{12}\eta_{1i} - \xi_{11}\eta_{2i})u_3^2 + (-\xi_{13}\eta_{1i} + \xi_{11}\eta_{3i})u_2u_3, \quad (9)$$

$$L_{3i} = (\xi_{13}\eta_{1i} - \xi_{11}\eta_{3i})u_2^2 + (-\xi_{12}\eta_{1i} + \xi_{11}\eta_{2i})u_2u_3, \quad (10)$$

where $i = 1, 2, 3$.

One can obtain electrical displacement \overline{D} by solving Equation (7) through the matrix algebra:

$$\det(\overline{L} + \frac{1}{n^2}\overline{I}) = 0. \quad (11)$$

Having the components of \overline{D} , other fields and Poynting vector can be calculated easily [15]

$$\overline{E} = \frac{\overline{\eta}}{\varepsilon_0}\overline{D}, \quad \overline{B} = n\sqrt{\mu_0\varepsilon_0}\hat{U} \times \overline{E}, \quad \overline{H} = \frac{\overline{\xi}}{\mu_0}\overline{B}, \quad \overline{S} = \overline{E} \times \overline{H}, \quad (12)$$

where, we have assumed that fields $\overline{B}, \overline{H}$ are harmonic in time; $\propto \exp[i\overline{k} \cdot \overline{r} - i\omega t]$. Finally, the direction of the field propagation for each mode is determined from the Poynting vector as:

$$\frac{d\overline{r}}{dl} = \frac{\overline{S}}{S}. \quad (13)$$

4. NORMAL INCIDENT OF LIGHT ON A PIECE OF ELECTROMAGNETIC SLAB

In this section, we study the behavior of light which strikes perpendicularly on a single anisotropic slab of material, shown in Fig. 1. Our analysis is extendable to an arbitrary angle of incidence. We consider the y - z plane as the propagation plane, where the light travels parallel to the z axis. In the normal incidence, the wave vector has no any projection on the boundary; $k_y = 0$. The phase matching condition on the boundaries requires that k_y vanishes inside the medium. In general, in the normal incidence, the direction of the wave vector does not change.

$$|\overline{k}| = k_z. \quad (14)$$

Now we want to achieve the normal modes for this direction of the wave vector. We assume that two principal axes y' and z' of the slab lay in the y - z plane, shown in Fig. 1. In this case, the dielectric tensor can be obtained from following relation [17],

$$\overline{\varepsilon} = A\overline{\varepsilon}'A^T, \quad (15)$$

where $\overline{\varepsilon}' = \text{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ is the principle permittivity tensor, and A is the rotation matrix, which describes the rotation of the coordinate axes with respect to the crystal principal axes.

Suppose that two principal axes y' and z' of the slab lay in this plane, shown in Fig. 1, then the rotation matrix is given by

$$A = R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}, \quad (16)$$

where θ is an angle between z and z' . From Equation (15), we can obtain the rotated permittivity tensor,

$$\overline{\varepsilon} = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 \cos^2\theta + \varepsilon_3 \sin^2\theta & -(\varepsilon_2 - \varepsilon_3) \sin\theta \cos\theta \\ 0 & -(\varepsilon_2 - \varepsilon_3) \sin\theta \cos\theta & \varepsilon_2 \sin^2\theta + \varepsilon_3 \cos^2\theta \end{pmatrix}. \quad (17)$$

and also the inverse permittivity tensor,

$$\overline{\eta} = \frac{1}{\varepsilon_2\varepsilon_3} \begin{pmatrix} \frac{\varepsilon_2\varepsilon_3}{\varepsilon_1} & 0 & 0 \\ 0 & \varepsilon_2 \sin^2\theta + \varepsilon_3 \cos^2\theta & (\varepsilon_2 - \varepsilon_3) \sin\theta \cos\theta \\ 0 & (\varepsilon_2 - \varepsilon_3) \sin\theta \cos\theta & \varepsilon_2 \cos^2\theta + \varepsilon_3 \sin^2\theta \end{pmatrix}. \quad (18)$$

From Maxwell equations we have $\nabla \cdot \overline{D} = 0$, hence, for the normal incidence it reads as $D_z = 0$. For other components of \overline{D} by replacing relation (18) in Equation (6), we will have:

$$\begin{pmatrix} -\eta_{11}\xi_{22} & \xi_{21}\eta_{22} \\ \eta_{11}\xi_{12} & -\xi_{11}\eta_{22} \end{pmatrix} \begin{pmatrix} D_x \\ D_y \end{pmatrix} = -\frac{1}{n^2} \begin{pmatrix} D_x \\ D_y \end{pmatrix}. \quad (19)$$

By applying the condition in Equation (11), the eigenvalues of Equation (19) are obtained from the following equation:

$$\left(\frac{1}{n^2} - \eta_{11}\xi_{22}\right) \left(\frac{1}{n^2} - \xi_{11}\eta_{22}\right) - \xi_{21}^2\eta_{22}\eta_{11} = 0. \quad (20)$$

This is a quadratic equation in terms of $1/n^2$ which has two eigenvalues;

$$\frac{1}{n^2} = \begin{cases} \frac{1}{2} \left(\eta_{11}\xi_{22} + \xi_{11}\eta_{22} + \sqrt{(\eta_{11}\xi_{22} - \xi_{11}\eta_{22})^2 + 4\xi_{21}^2\eta_{11}\eta_{22}} \right) \\ \frac{1}{2} \left(\eta_{11}\xi_{22} + \xi_{11}\eta_{22} - \sqrt{(\eta_{11}\xi_{22} - \xi_{11}\eta_{22})^2 + 4\xi_{21}^2\eta_{11}\eta_{22}} \right) \end{cases} \quad (21)$$

Corresponding eigenvectors will determine the physical components of the field.

In the next part, we investigate two special examples: a layer of a non-magnetic medium, with $\bar{\mu} = 1$, and a slab of impedance-matched material, with $\bar{\mu} = \bar{\epsilon}$.

4.1. Non-Magnetic Anisotropic Medium

For the purely electric medium, the refractive indices in Equation (21) are given by

$$n_o^2 = \epsilon_1, \quad (22)$$

$$n_e^2(\theta) = \frac{\epsilon_2\epsilon_3}{\epsilon_2 \sin^2 \theta + \epsilon_3 \cos^2 \theta}. \quad (23)$$

Relation (23) shows that the refractive index depends on principal values of the permittivity tensor, i.e., ϵ_2 and ϵ_3 , and, θ , the angle between the direction of the wave vector and third principle axis. Whereas the refractive index in Equation (22) depends only on ϵ_1 .

For a specific angle of incidence, in this example perpendicular incidence, there are two modes associated with the above refractive indices profiles. For refractive index in Equation (22), we can obtain \bar{D} by solving the eigenvalue equation (19):

$$\bar{D} = D_x(1 \ 0 \ 0)^T, \quad (24)$$

and for the refractive index in Equation (23) we have:

$$\bar{D} = D_y(0 \ 1 \ 0)^T. \quad (25)$$

By applying the normalization condition, $\bar{E} \cdot \bar{E} = 1$, we can determine components D_x and D_y . For the first normal mode in Equation (24), electric and magnetic fields are written in the from of Equation (12),

$$\bar{E} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \bar{H} = \left(\frac{\epsilon_0}{\mu_0}\right)^{\frac{1}{2}} (\epsilon_1)^{\frac{1}{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \quad (26)$$

As the electrical field of this particular mode is perpendicular to the propagation plane, we can conclude that this normal mode is a TE polarized component of the field. For second normal mode in Equation (25), \bar{E} and \bar{H} are given by,

$$\bar{E} = (\epsilon_2^2 \sin^2 \theta + \epsilon_3^2 \cos^2 \theta)^{-\frac{1}{2}} \begin{pmatrix} 0 \\ \epsilon_2 \sin^2 \theta + \epsilon_3 \cos^2 \theta \\ (\epsilon_2 - \epsilon_3) \sin \theta \cos \theta \end{pmatrix}, \quad (27)$$

$$\bar{H} = \left(\frac{\epsilon_0}{\mu_0}\right)^{\frac{1}{2}} \left(\frac{\epsilon_2\epsilon_3 (\epsilon_2 \sin^2 \theta + \epsilon_3 \cos^2 \theta)}{\epsilon_2^2 \sin^2 \theta + \epsilon_3^2 \cos^2 \theta}\right)^{\frac{1}{2}} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}. \quad (28)$$

with a similar argument, when the electrical field of the mode lies in the propagating plane y - z , the mode is the TM-polarized component. On the other hand, the TE-polarized fields propagate along the

electric flux density, but modes with TM polarization do not. The Poynting vectors of TE and TM polarizations can derive as

$$\bar{S}_{TE} = \left(\frac{\varepsilon_0}{\mu_0} \right)^{\frac{1}{2}} (\varepsilon_1)^{\frac{1}{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad (29)$$

$$\bar{S}_{TM} = \left(\frac{\varepsilon_0}{\mu_0} \right)^{\frac{1}{2}} \frac{(\varepsilon_2 \varepsilon_3)^{\frac{1}{2}} (\varepsilon_2 \sin^2 \theta + \varepsilon_3 \cos^2 \theta)^{\frac{1}{2}}}{\varepsilon_2^2 \sin^2 \theta + \varepsilon_3^2 \cos^2 \theta} \begin{pmatrix} 0 \\ -(\varepsilon_2 - \varepsilon_3) \sin \theta \cos \theta \\ \varepsilon_2 \sin^2 \theta + \varepsilon_3 \cos^2 \theta \end{pmatrix}. \quad (30)$$

The ray direction is given by the angle of the ray with respect to the z -axis, determined by the relations (29), (30) and (13). For the TE mode, we can write:

$$\tan \phi = \frac{S_y}{S_z} = 0, \quad \frac{d\bar{r}}{dl} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (31)$$

For the TM mode, we achieve:

$$\tan \phi = \frac{-(\varepsilon_2 - \varepsilon_3) \sin \theta \cos \theta}{\varepsilon_2 \sin^2 \theta + \varepsilon_3 \cos^2 \theta}, \quad (32)$$

$$\frac{d\bar{r}}{dl} = \frac{1}{\sqrt{\varepsilon_2^2 \sin^2 \theta + \varepsilon_3^2 \cos^2 \theta}} \begin{pmatrix} 0 \\ -(\varepsilon_2 - \varepsilon_3) \sin \theta \cos \theta \\ \varepsilon_2 \sin^2 \theta + \varepsilon_3 \cos^2 \theta \end{pmatrix}. \quad (33)$$

Since the wave vector is along the z -axis, ϕ is equivalent to the deviation angle of the light from the wave vector direction. In relation (32), the propagation direction of TM polarization is not identical to the wave vector. Therefore, it represents extraordinary ray, whereas the ray direction of TE polarization is along the wave vector, and therefore, it is the ordinary ray.

5. MEDIUM WITH VARIABLE OPTICAL AXES

In this section, we investigate the space-time metric in a designed medium [8], in which the orientation of the principal axis changes in position. For simplicity, we consider the case where the variation of the direction depends only on the z parameter: $\theta = \theta(z)$. In such a medium inhomogeneity is induced by the variation of $\theta(z)$. Equivalently, we can assume the medium as thin stratified layers of homogeneous, anisotropic slabs in the z direction, shown in Fig. 1. The direction of the principle axis in each slab is constant, but the overall orientation of the principle axis in the whole medium is a function of z . Each layer in the y - z plane is a rectangle, shown in Fig. 1. Therefore, in the normal incident on the boundary, $\bar{k} = k_z \hat{z}$, we have $k_y = 0$. As seen in the previous sections, the phase matching condition on each boundary guarantees that the direction of the wave vector through each layer does not change, and the wave vector is along the z direction.

The ray direction in each layer follows relation (32). Consequently, we can easily trace the light trajectory in this layered medium.

5.1. Ray Tracing

We apply the ray-tracing method to trace the light geodesics in two examples of layered media with variable optical axes: First, a layered medium in which the orientation of optical axis varies with z as $\theta = z$, and the second one where the dependency of the optical axis to z direction follows the relation: $\theta = \sqrt{z}$. Also, we assume that the principal values of permittivity are equal to $\varepsilon_2 = 2.75$ and $\varepsilon_3 = 2.21$, which are associated to the calcite crystal principal permittivities [14].

According to relation (32), we can trace the extraordinary ray in layered anisotropic media by solving the following equation:

$$dy = \frac{-(\varepsilon_2 - \varepsilon_3) \sin \theta(z) \cos \theta(z)}{\varepsilon_2 \sin^2 \theta(z) + \varepsilon_3 \cos^2 \theta(z)} dz \quad (34)$$

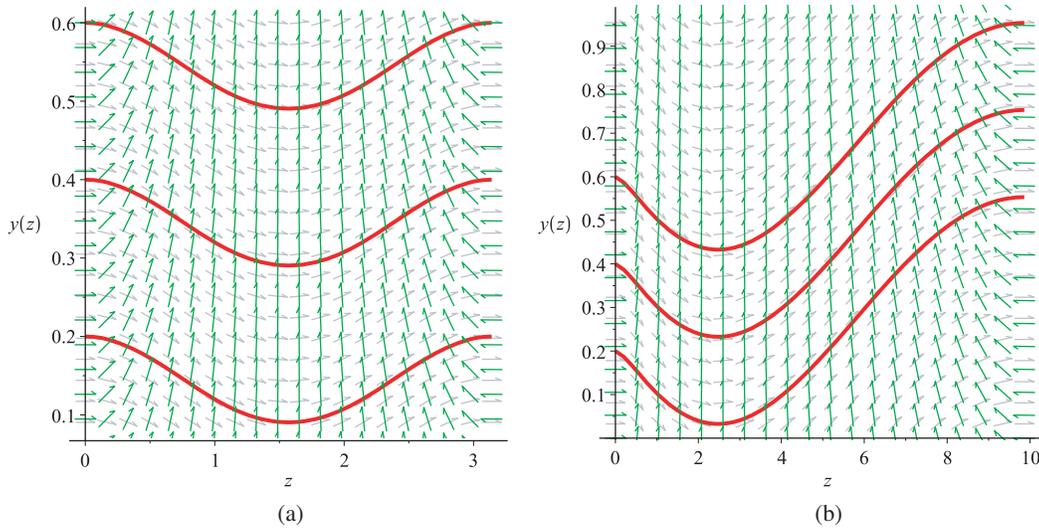


Figure 2. Ray tracing in anisotropic media with variable optical axes, (a) $\theta = z$ and (b) $\theta = \sqrt{z}$, the green arrows indicate orientations of the third principle axis and the red lines indicate ray trajectories in the media.

In Fig. 2, the plotted trajectories of the family of extraordinary rays are shown. Left and right diagrams are corresponding to the $\theta = z$ and $\theta = \sqrt{z}$, respectively, which indicate the normal incidence of light.

As we can see in Fig. 2, the light geodesics through these media are not straight lines, and moreover, it seems that these surfaces are under tension. Therefore, we expect a non-zero Riemannian tensor for the corresponding metric.

5.2. Metric

We can construct the space-time metric of the two dimensional distorted surfaces, as shown in Fig. 2. The first step to construct the metric is to choose three bases:

$$e_0 = g_{00}^{1/2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad (35)$$

where coefficient g_{00} is a scalar function which provides sufficient condition for null geodesics $ds^2 = 0$. Using the bases in Equation (35), we can derive the metric components [4],

$$g_{\mu\nu} = e_\mu \cdot e_\nu. \quad (36)$$

The space-time metric in the matrix form becomes,

$$\bar{g} = \begin{pmatrix} g_{00} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (37)$$

or equivalently in the form of the Riemann line element,

$$ds^2 = g_{00}c^2dt^2 + dy^2 + dz^2 \quad (38)$$

The null geodesics condition, $ds^2 = 0$, requires that,

$$dy^2 + dz^2 + g_{00}c^2dt^2 = 0. \quad (39)$$

Therefore, the time component of the metric becomes:

$$g_{00} = -\frac{dl^2}{c^2dt^2} \quad (40)$$

where $dl^2 = dy^2 + dz^2$. On the other hand, for the light fields, the surfaces of the equal phases are defined as the solutions of following equation:

$$d\varphi(r, t) = 0. \quad (41)$$

The condition in Equation (41) results in:

$$\bar{k} \cdot d\bar{r} - \omega dt = 0. \quad (42)$$

or equally:

$$cdt = n\hat{U} \cdot d\bar{r}. \quad (43)$$

From Equation (32) we have

$$cdt = n \frac{\varepsilon_2 \sin^2 \theta + \varepsilon_3 \cos^2 \theta}{\sqrt{\varepsilon_2^2 \sin^2 \theta + \varepsilon_3^2 \cos^2 \theta}} dl. \quad (44)$$

Substituting Equation (44) in relation (40), coefficient g_{00} is easily determined,

$$g_{00} = - \frac{\varepsilon_2^2 \sin^2 \theta + \varepsilon_3^2 \cos^2 \theta}{n^2 (\varepsilon_2 \sin^2 \theta + \varepsilon_3 \cos^2 \theta)^2}, \quad (45)$$

and by using these results in Equation (38) the line element of the propagation plane can be written as

$$ds^2 = - \left(\frac{\varepsilon_2^2 \sin^2 \theta + \varepsilon_3^2 \cos^2 \theta}{n^2 (\varepsilon_2 \sin^2 \theta + \varepsilon_3 \cos^2 \theta)^2} \right) c^2 dt^2 + dx^2 + dy^2. \quad (46)$$

Using the refractive index in Equation (23), achieved in the previous section, we can construct the following optical metric for the extraordinary ray:

$$ds^2 = - \left(\frac{\varepsilon_2^2 \sin^2 \theta + \varepsilon_3^2 \cos^2 \theta}{\varepsilon_2 \varepsilon_3 (\varepsilon_2 \sin^2 \theta + \varepsilon_3 \cos^2 \theta)} \right) c^2 dt^2 + dy^2 + dz^2. \quad (47)$$

Relation (47) explicitly shows that the time component does not vanishes, $g_{00} \neq 0$. Indeed, we can rewrite g_{00} as

$$g_{00} = - \frac{1}{n_e^2 \cos^2 \phi}, \quad (48)$$

where ϕ is the deviation angle given by Equation (32). In anisotropic media, usually the deviation angle is small. So, it seems that to create the analogue event horizon with variable optical axes media such as Fig. 1 is hard. However, these media can be useful to bending light in the partially invisibility devices.

5.3. Deviation Angle

By using relation (32), we can calculate the maximum possible deviation angle for the extraordinary ray:

$$\phi_{\max} = \arctan \left(\frac{\varepsilon_2 - \varepsilon_3}{2\sqrt{\varepsilon_2 \varepsilon_3}} \right), \quad (49)$$

The maximum deviation occurs when the relative angle θ became:

$$\theta = \arctan \left(\sqrt{\frac{\varepsilon_3}{\varepsilon_2}} \right). \quad (50)$$

We can rewrite relation (49) as:

$$\phi_{\max} = \arctan \left[\frac{1}{2} \left(\frac{n_2}{n_3} - \frac{n_3}{n_2} \right) \right]. \quad (51)$$

For natural anisotropic media, the maximum deviation angle is small. For example, in Calcite ϕ_{\max} is almost 6° . However, this deviation angle could be greater in anisotropic metamaterials. In

an artificial anisotropic slab, made of isotropic thin layers with ε' and ε , the principle values of the permittivity tensor are given by [10]:

$$\varepsilon_{\parallel} = f'\varepsilon' + f\varepsilon, \tag{52}$$

$$\varepsilon_{\perp} = \frac{\varepsilon'\varepsilon}{f\varepsilon' + f'\varepsilon}, \tag{53}$$

where $f' + f = 1$. For the cases that $f' = f = \frac{1}{2}$, we can obtain maximum deviation angle for the extraordinary rays as:

$$\frac{\varepsilon' - \varepsilon}{4\sqrt{\varepsilon'\varepsilon(\varepsilon' + \varepsilon)}}. \tag{54}$$

In artificial material, ϕ_{\max} is larger than natural anisotropic media, but, still, it is a small amount. In the hyperbolic metamaterial in which one of the ε' or ε is negative, ϕ_{\max} reaches large values.

5.4. Ray Refractive Index

A careful look on relation (47) reveals the conformally flat nature of the two-dimensional space, which associates with the spatial part of the metric

$$ds^2 = -c^2 dt^2 + n \frac{(\varepsilon_2 \sin^2 \theta + \varepsilon_3 \cos^2 \theta)^2}{\varepsilon_2^2 \sin^2 \theta + \varepsilon_3^2 \cos^2 \theta} dl^2. \tag{55}$$

with the following refractive index:

$$n_{\text{ray}}(\theta) = \frac{\varepsilon_2 \varepsilon_3 (\varepsilon_2 \sin^2 \theta + \varepsilon_3 \cos^2 \theta)}{(\varepsilon_2^2 \sin^2 \theta + \varepsilon_3^2 \cos^2 \theta)}, \tag{56}$$

$$= n_e^2(\theta) \cos^2 \phi. \tag{57}$$

Index n_{ray} is the effective refractive indices perceived by the light rays, called ray refractive index while the extraordinary refractive index n_e is phase refractive index.

In Fig. 3, we plot the ray refractive index n_{ray} and phase refractive index n_e , respectively, versus the position for two cases: first when $\theta = z$ and second when $\theta = \sqrt{z}$. As shown in Fig. 3, the proximity of the two curves confirms the results of the previous subsection: the deviation angle ϕ is small. The curves show that these refractive indices can take the values between $\sqrt{\varepsilon_2}$ and $\sqrt{\varepsilon_3}$. Although this range is limited in the natural anisotropic crystals, it can get larger in the anisotropic metamaterials. Moreover, the graphs show how the medium appears for light as gradient index media.

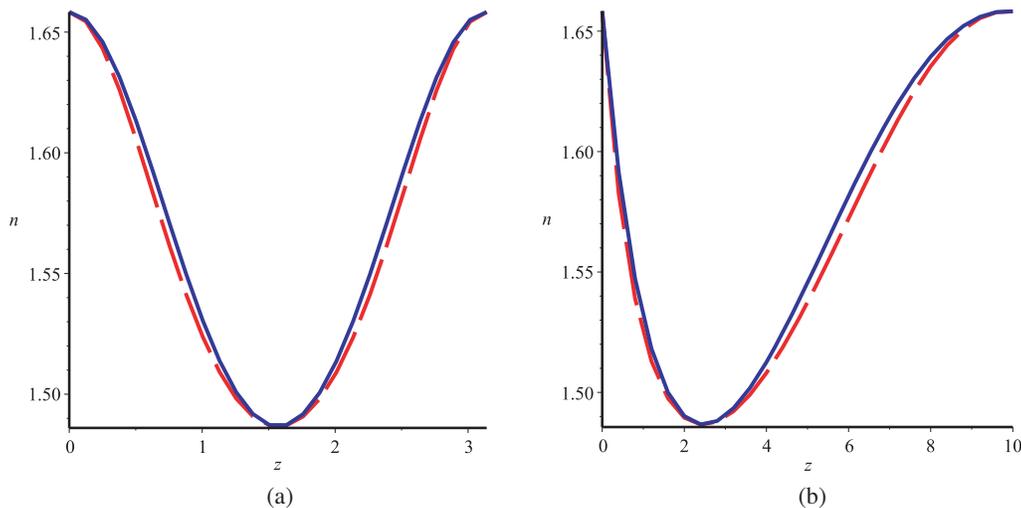


Figure 3. Variation of the ray refractive index and phase refractive index as a function of position, (a) $\theta = z$ and (b) $\theta = \sqrt{z}$, the blue line indicate ray refractive index, n_{ray} and the red dashed lines indicate to phase refractive index, n_e .

5.5. Curvature

The line element in Equation (47) indicates that the propagation plane is a conformally flat space with (possibly) non-zero curvature. We can investigate the curvature by using the differential geometry relations.

Riemann curvature tensor is given by [4]

$$R_{jkl}^i \equiv \Gamma_{jl,k}^i - \Gamma_{jk,l}^i + \Gamma_{mk}^i \Gamma_{jl}^m - \Gamma_{ml}^i \Gamma_{jk}^m, \quad (58)$$

where Γ_{jk}^i is the Christoffel symbol, and the comma notation “,” refers to partial differentiation. The Christoffel symbol can be expressed in terms of metric components as,

$$\Gamma_{jk}^i = \frac{1}{2} g^{il} (g_{lj,k} + g_{lk,j} - g_{jk,l}). \quad (59)$$

For the metric in Equation (47), which is a two-dimensional conformally flat space, $g_{ij} = n^2 \delta_{ij}$, we can obtain Christoffel symbol from relation,

$$\Gamma_{ij}^i = \frac{1}{2n^2} n_{,j}^2. \quad (60)$$

Using relation (60), Riemann curvature tensor, Equation (58) is achieved as:

$$R_{11} = R_{22} = \frac{1}{2n^4} \left\{ \left(\frac{\partial}{\partial z} n^2 \right)^2 + \left(\frac{\partial}{\partial y} n^2 \right)^2 \right\} - \frac{1}{2n^2} \left\{ \left(\frac{\partial}{\partial z} \frac{\partial}{\partial z} n^2 \right) + \left(\frac{\partial}{\partial y} \frac{\partial}{\partial y} n^2 \right) \right\}. \quad (61)$$

We calculate this tensor for the above mentioned example of the medium with variable axes, $\theta = z$. Consequently, we find that the curvature is non-zero,

$$R_{ii} \neq 0. \quad (62)$$

The media with $\theta = f(z)$ have a non-zero Riemann curvature tensor and appear as a curved space for the extraordinary light. Therefore, it is possible to use the variable optical axes media for realizing the curved space-time in the laboratory.

6. IMPEDANCE-MATCHED ANISOTROPIC MEDIUM

In this section, by using the relations achieved in Section 4, we investigate the behavior of the normal incident light on the impedance-matched anisotropic slab, $\mu_{ij} = \varepsilon_{ij}$ and then compare its results with the case of the non-magnetic medium. For the impedance-matched, anisotropic medium, $\mu_{ij} = \varepsilon_{ij}$, two refractive indices in Equation (21) are reduced to one relation,

$$n_{\text{imp}}^2(\theta) = \frac{\varepsilon_1 \varepsilon_2 \varepsilon_3}{\varepsilon_2 \sin^2 \theta + \varepsilon_3 \cos^2 \theta}. \quad (63)$$

Therefore, like the birefringent media, for the impedance-matched medium we have one eigenvalue, i.e., the impedance-matched media do not birefringence. For more investigation we compare this medium with the results achieved in the case of non-magnetic medium in previous subsection. By comparing the refractive indices in Equation (63) with Equation (23), we find that,

$$n_{\text{imp}}(\theta) = \sqrt{\varepsilon_1} n_e(\theta), \quad (64)$$

where index e (imp) stands for extra ordinary (impedance-matched). ε_1 govern the electrical responses of the ordinary mode in birefringent medium,

$$n_{\text{imp}}^2(\theta) = n_o^2 n_e^2(\theta). \quad (65)$$

Also, we investigate the behavior of the birefringent medium normal mode, TE and TM polarizations, in the impedance-matched medium. Following Equation (12), for the TM mode, the electric and magnetic fields are given by,

$$\overline{E} = (\varepsilon_2^2 \sin^2 \theta + \varepsilon_3^2 \cos^2 \theta)^{-\frac{1}{2}} \begin{pmatrix} 0 \\ \varepsilon_2 \sin^2 \theta + \varepsilon_3 \cos^2 \theta \\ (\varepsilon_2 - \varepsilon_3) \sin \theta \cos \theta \end{pmatrix}, \quad (66)$$

$$\overline{H} = \left(\frac{\varepsilon_0}{\mu_0} \right)^{\frac{1}{2}} \frac{(\varepsilon_1 \varepsilon_2 \varepsilon_3)^{\frac{1}{2}} (\varepsilon_2 \sin^2 \theta + \varepsilon_3 \cos^2 \theta)^{\frac{1}{2}}}{(\varepsilon_2^2 \sin^2 \theta + \varepsilon_3^2 \cos^2 \theta)^{\frac{1}{2}}} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}. \quad (67)$$

On the other hand, for the TE polarization, the electric and magnetic fields become,

$$\vec{E} = (1 \ 0 \ 0)^T, \quad (68)$$

$$\vec{H} = \left(\frac{\varepsilon_0}{\mu_0} \right)^{\frac{1}{2}} \frac{(\varepsilon_1 \varepsilon_2 \varepsilon_3)^{\frac{1}{2}}}{(\varepsilon_2 \sin^2 \theta + \varepsilon_3 \cos^2 \theta)^{\frac{1}{2}} \varepsilon_2 \varepsilon_3} \begin{pmatrix} 0 \\ \varepsilon_2 \sin^2 \theta + \varepsilon_3 \cos^2 \theta \\ (\varepsilon_2 - \varepsilon_3) \sin \theta \cos \theta \end{pmatrix}. \quad (69)$$

Expressions (68) and (69) show that for the TE mode, the electric field \vec{E} and displacement field \vec{D} are in the same direction. On the other hand, the magnetic field \vec{H} is not along the induction \vec{B} , and vice versa in the case of TM mode. From relation (13), we obtain the Poynting vector for the TM and TE modes in impedance-matched medium

$$\vec{S}_{\text{TM}} = \left(\frac{\varepsilon_0}{\mu_0} \right)^{\frac{1}{2}} \frac{(\varepsilon_1 \varepsilon_2 \varepsilon_3)^{\frac{1}{2}} (\varepsilon_2 \sin^2 \theta + \varepsilon_3 \cos^2 \theta)^{\frac{1}{2}}}{\varepsilon_2^2 \sin^2 \theta + \varepsilon_3^2 \cos^2 \theta} \begin{pmatrix} 0 \\ -(\varepsilon_2 - \varepsilon_3) \sin \theta \cos \theta \\ \varepsilon_2 \sin^2 \theta + \varepsilon_3 \cos^2 \theta \end{pmatrix}, \quad (70)$$

$$\vec{S}_{\text{TE}} = \left(\frac{\varepsilon_0}{\mu_0} \right)^{\frac{1}{2}} \frac{(\varepsilon_1 \varepsilon_2 \varepsilon_3)^{\frac{1}{2}}}{(\varepsilon_2 \sin^2 \theta + \varepsilon_3 \cos^2 \theta)^{\frac{1}{2}} \varepsilon_2 \varepsilon_3} \begin{pmatrix} 0 \\ -(\varepsilon_2 - \varepsilon_3) \sin \theta \cos \theta \\ \varepsilon_2 \sin^2 \theta + \varepsilon_3 \cos^2 \theta \end{pmatrix}. \quad (71)$$

It is clear that TE and TM modes have the same Poynting vector. Consequently, light in impedance-matched media is not divided into ordinary and extraordinary rays as expected. The angle between ray and z -axis can be written as,

$$\tan \phi_{(\text{TE})} = \tan \phi_{(\text{TM})} = \frac{-(\varepsilon_2 - \varepsilon_3) \sin \theta \cos \theta}{\varepsilon_2 \sin^2 \theta + \varepsilon_3 \cos^2 \theta}. \quad (72)$$

Also, we can write the ray direction in the impedance-matched slab as

$$\frac{d\vec{r}}{dl} = \frac{1}{\sqrt{\varepsilon_2^2 \sin^2 \theta + \varepsilon_3^2 \cos^2 \theta}} \begin{pmatrix} 0 \\ -(\varepsilon_2 - \varepsilon_3) \sin \theta \cos \theta \\ \varepsilon_2 \sin^2 \theta + \varepsilon_3 \cos^2 \theta \end{pmatrix}. \quad (73)$$

Relation (73) shows important fact about the impedance-matched media: *In these media although the birefringent effect does not occur, the ray direction in this medium is of extraordinary ray. The ray direction in this medium does not necessarily go along the wave vector.*

Comparison between Equations (32) and (73) shows

$$\left(\frac{d\vec{r}}{dl} \right)_{el}^{\text{TM}} = \left(\frac{d\vec{r}}{dl} \right)_{\text{imp}}, \quad (74)$$

i.e., the direction of the rays in impedance-matched media is the same as the direction of extraordinary rays in non-magnetic anisotropic media.

6.1. Ray Tracing and Metric

Figure 2 (drawn according to relation (74)) shows the ray trajectory in the impedance-matched, inhomogeneous medium with $\theta = z$ and $\theta = \sqrt{z}$, while the corresponding effective metric of the propagation plane can be achieved from relation (46),

$$ds^2 = -c^2 dt^2 + n \frac{(\varepsilon_2 \sin^2 \theta + \varepsilon_3 \cos^2 \theta)^2}{\varepsilon_2^2 \sin^2 \theta + \varepsilon_3^2 \cos^2 \theta} dl^2. \quad (75)$$

The impedance matched-media appear as following metric for the light ray:

$$ds^2 = -c^2 dt^2 + \frac{\varepsilon_1 \varepsilon_2 \varepsilon_3 (\varepsilon_2 \sin^2 \theta + \varepsilon_3 \cos^2 \theta)}{(\varepsilon_2^2 \sin^2 \theta + \varepsilon_3^2 \cos^2 \theta)} (dy^2 + dz^2). \quad (76)$$

On the other hand, the spatial metric felt by light in an impedance-matched medium given by [4]

$$\bar{g} = (\det \bar{\varepsilon}) \bar{\varepsilon}^{-1}. \quad (77)$$

This spatial metric in our example, Fig. 1, takes the following form:

$$\bar{g}_{imp}(y-z) = \begin{pmatrix} \varepsilon_1(\varepsilon_2 \sin^2 \theta + \varepsilon_3 \cos^2 \theta) & \varepsilon_1(\varepsilon_2 - \varepsilon_3) \sin \theta \cos \theta \\ \varepsilon_1(\varepsilon_2 - \varepsilon_3) \sin \theta \cos \theta & \varepsilon_1(\varepsilon_2 \cos^2 \theta + \varepsilon_3 \sin^2 \theta) \end{pmatrix}. \quad (78)$$

At the first glance, the metric in Equation (78) looks different from our constructed metric in Equation (76), but by using the following relation:

$$\varepsilon_2 \varepsilon_3 = (\varepsilon_2 \cos^2 \theta + \varepsilon_3 \sin^2 \theta)(\varepsilon_2 \sin^2 \theta + \varepsilon_3 \cos^2 \theta) - [(\varepsilon_2 - \varepsilon_3) \sin \theta \cos \theta]^2, \quad (79)$$

we show that the two metrics in Equations (76) and (78) are equivalent. This equivalence can be considered as a verification of our method.

6.2. Comparing between the Metrics

In this part, we discuss how transformation optics will be applied to the non-impedance-matched media. Remember from Section 4, the direction of the extraordinary ray in non-magnetic medium in Equation (32) coincides with direction of the ray in corresponding impedance-matched version in Equation (73). Relation (64) shows that the two refractive indices are conformally equal. Therefore, we expect a similarity in their optical path length if $\varepsilon_1 = \text{constant}$.

Mathematically, we consider the metrics in Equations (76) and (47). Using null geodesics condition, we can write the following metric, Equation (80) as a conformal equivalent of the metric in Equation (47),

$$ds_e^2 = -\varepsilon_1 c^2 dt^2 + \frac{\varepsilon_1 \varepsilon_2 \varepsilon_3 (\varepsilon_2 \sin^2 \theta + \varepsilon_3 \cos^2 \theta)}{(\varepsilon_2^2 \sin^2 \theta + \varepsilon_3^2 \cos^2 \theta)} dl^2. \quad (80)$$

Now, the spatial part of this metric in Equation (80), is the same as the spatial part of the metric in Equation (76),

$$ds_{imp}^2 = -c^2 dt^2 + \frac{\varepsilon_1 \varepsilon_2 \varepsilon_3 (\varepsilon_2 \sin^2 \theta + \varepsilon_3 \cos^2 \theta)}{(\varepsilon_2^2 \sin^2 \theta + \varepsilon_3^2 \cos^2 \theta)} dl^2. \quad (81)$$

Therefore, if $\varepsilon_1 = \text{constant}$, two metrics are conformally equivalent. A non-magnetic anisotropic medium, which is not impedance-matched, appears for the TM polarized light (extraordinary light), as the impedance-matched medium for the non-polarized light. It is shown that we can write this equivalence between these metrics as the following general relation:

$$\bar{g}_e(2+1) = \begin{pmatrix} -\varepsilon_1 & 0 \\ 0 & \bar{g}_{imp}(y-z) \end{pmatrix}. \quad (82)$$

As a result, for suitably polarized light, we can use non-magnetic medium instead of impedance-matched one to make a desired metric. More elaborated theory using non-impedance-matched media as an alternative to impedance-matched media will be discussed in the further publications.

7. CONCLUSION

We have studied the propagation of normal incident light in two kinds of medium with a similar anisotropy in their refractive index profiles: One fulfils the impedance-matched condition, and the other has only electrical anisotropy. Although in the impedance-matched media the birefringent effect does not occur, the ray of light coincides in the trajectory with the extraordinary ray in a corresponding birefringent medium, the impedance-matching condition eliminates the ordinary ray. Second, we conclude that this extraordinary direction in the impedance-matched medium coincides with the direction of the extraordinary rays in the non-magnetic medium when both have the same anisotropy in their refractive index. A non-magnetic anisotropic medium, which is not impedance-matched, appears for the TM polarized light (extraordinary light), as the impedance-matched medium for the non-polarized light. As a result, for suitably polarized light, we can use non-magnetic medium instead of impedance-matched one to make a desired metric.

By using the optical fact that the direction of wave propagation depends on the orientation of the optical axis, we have studied the behaviour of light in an optical medium, in which the optical

axis is controllable at each point. Medium with varying optical axis is an engineered inhomogeneous material, e.g., a stratified medium, formed from arrays of thin homogeneous layers. By controlling the direction of optical axis in each layer, the propagation of electromagnetic waves is controlled through the medium. We have shown that this uncommon form of inhomogeneity would also result in the emergence of effective geometry in the medium. The array of homogeneous, anisotropic slabs can appear for the light as a curved space-time. We indicate the advantage of the variable optical axis medium as an alternative to gradient refractive index medium.

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REFERENCES

1. Leonhardt, U. and T. G. Philbin, "General relativity in electrical engineering," *New J. Phys.*, Vol. 8, No. 10, 247, 2006.
2. Pendry, J. B., D. Schurig, and D. R. Smith, "Controlling electromagnetic fields," *Science*, Vol. 312, No. 5781, 1780–1782, 2006.
3. Thompson, R. T., S. A. Cummer, and J. Frauendiener, "A completely covariant approach to transformation optics," *J. Opt.*, Vol. 13, No. 2, 024008, 2011.
4. Leonhardt, U. and T. Philbin, *Geometry and Light: The Science of Invisibility*, Dover, New York, 2012.
5. McCall, M., "Transformation optics and cloaking," *Contem. Phys.*, Vol. 54, No. 6, 273–286, 2013.
6. Sarkissian, H., S. V. Serak, N. V. Tabiryan, L. B. Glebov, V. Rotar, and B. Y. Zeldovich, "Polarization-controlled switching between diffraction orders in transverse-periodically aligned nematic liquid crystals," *Opt. Lett.*, Vol. 31, 2248–2250, 2006.
7. Nersisyan, S. R., N. V. Tabiryan, D. M. Steeves, and B. R. Kimball, "Optical axis gratings in liquid crystals and their use for polarization insensitive optical switching," *Journal of Nonlinear Optical Physics and Materials*, Vol. 18, No. 1, 1–47, 2009.
8. Liang, Z., X. Jiang, F. Miao, S. Guenneau, and J. Li, "Transformation media with variable optical axes," *New J. of Phys.*, Vol. 14, No. 10, 103042, 2012.
9. Sluijter, M., *Ray-optics Analysis of Inhomogeneous Optically Anisotropic Media*, Delft University of Technology, TU Delft, 2010.
10. Cai, W. and V. M. Shalaev, *Optical Metamaterials*, Springer, Berlin, Germany, 2010.
11. Born, M. and E. Wolf, *Principles of Optics: Electromagnetic Theory of Propagation, Interference, and Diffraction of Light*, Cambridge University Press, London, 1999.
12. Neves, N. M., A. S. Pouzada, J. H. D. Voerman, and P. C. Powell, "The use of birefringence for predicting the stiffness of injection molded polycarbonate discs," *Polym. Eng. Sci.*, Vol. 38, No. 10, 1770–1777, 1998.
13. Saleh, B. E. A., M. C. Teich, and B. E. Saleh, *Fundamentals of Photonics*, Vol. 22, Wiley, New York, 1991.
14. Yariv, A. and P. Yeh, *Optical Waves in Crystals*, Vol. 10, Wiley, New York, 1984.
15. Jackson, J. D., *Classical Electrodynamics*, Wiley, New York, 1962.
16. Hao, J. and L. Zhou, "Electromagnetic wave scatterings by anisotropic metamaterials: Generalized 4×4 transfer-matrix method," *Phy. Rev. B*, Vol. 77, No. 094201, 2008.
17. Yeh, P., "Optics of anisotropic layered media: A new 4×4 matrix algebra," *Surf. Science*, Vol. 96, No. 1, 41–53, 1980.