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Simple Method to Calculate the Force between Thin Walled Solenoids

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Abstract—We developed a simple method to calculate the axial force between concentric thin walled solenoids. To achieve this, the force between them was mapped as a function of their geometrical relations based on separation-to-diameter ratios. This resulted in an equation and a set of data. We used them together to calculate axial forces between two coaxial thin walled solenoids. With this method, direct evaluation of elliptical integrals was circumvented, and the forces were obtained with a simple expression. The results were validated against solutions obtained with an existing semi-analytical method and force measurements between high coercivity permanent magnets.

1. INTRODUCTION

The calculation of magnetic forces has a wide range of applications. It can be used, with certain limitations, to calculate the force between coils, magnets, and superconductors. That is why their calculation and simplification is the subject of many interesting scientific works. Beleggia et al. derive analytical expressions for the force between a permanent magnet and a soft magnetic plate [1]. Robertson et al. publish a simplified force equation for coaxial cylindrical magnets and thin coils [2]. Ravaud et al. present a synthesis of analytical calculation of magnetic parameters for cylindrical magnets and coils [3], expressions to calculate the force between thick rectangular coils [4], as well as analytical expressions for the magnetic field of radially magnetized permanent magnets [5]. Akoun and Yonnet derive analytical expressions for the calculation of forces between cuboidal magnets [6]. Yang and Hull present expressions to obtain force and torque between square loops [7]. Babic et al. present formulas for the calculation of magnetic forces between: coaxial cylindrical magnets and thin coils [8], thick rectangular coils and thin walled solenoids [9], inclined circular coils [10], and misaligned thick coils with parallel axes [11]. Shiri and Shoulaie present a method that is useful to calculate the force between planar spiral coils [12], and also calculates the electromagnetic force distribution on a coil [13]. Iwasa has a chapter devoted to magnets, fields, and forces [14].

In this contribution, we will concentrate our efforts on simplifying the calculation of axial forces between concentric thin walled solenoids with circular cross section. We will start from an expression for the axial force between a pair of concentric thin coaxial coils. In order to find an expression for the force between concentric thin walled solenoids, it will be integrated two times. We will derive a simple equation that to be used will need a function that takes into consideration the geometrical relations between the solenoids. Starting from existing semi-analytical expressions, we will find the data needed to map that function. With the function and simple equation, we will be able to calculate the force between concentric thin walled solenoids with good accuracy. Finally, we will compare the results with those obtained using existing methods and with force measurements between high coercivity permanent magnets.

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2. METHOD

2.1. Axial Force between a Pair of Thin Coaxial Coils

Consider two thin coaxial coils where their diameter is defined as d, the separation between them as s, and their separation-to-diameter ratio as $\gamma = s/d$. A schematic drawing of the coils is shown in Fig. 1(a). We express the axial force between them as:

$$F_{ab} = \frac{\mu_0}{2} i_a i_b f''(\gamma), \tag{1}$$

where μ_0 is the permeability of free space; i_a and i_b are electrical currents that respectively circulate in each of the coils; $f''(\gamma)$ is a function of their separation-to-diameter ratio.

2.2. First Integration, Axial Force between a Thin Coil and a Thin Walled Solenoid

We approximated the axial force between a thin coil and a thin walled solenoid, as those in Fig. 1(b), as the summation of forces between n number of thin coil pairs with the following expression:

$$F_{as} = \frac{\mu_0}{2} i_a \sum_1^n i_{sn} f''(\gamma_n). \tag{2}$$

The surface current in the thin walled solenoid was defined as:

$$I_s = i_{sn} n. \tag{3}$$

And the interval $\Delta\gamma$ partitioned according to:

$$\frac{1}{n} = \frac{\Delta\gamma}{\gamma_\beta - \gamma_\alpha}, \tag{4}$$

where γ_α and γ_β are, respectively, the separation-to-diameter ratios between the first and the last pair of coils of the summation. They are also the separation-to-diameter ratios between the thin coil and the top and bottom parts of the thin walled solenoid. Combining Equations (2), (3) and (4), we rewrote the expression in the form of a Riemman integral:

$$F_{as} = \lim_{n\to\infty} \frac{\mu_0}{2} \frac{i_a I_s}{\gamma_\beta - \gamma_\alpha} \sum_1^n f''(\gamma_n)\Delta\gamma = \frac{\mu_0}{2} \frac{i_a I_s}{\gamma_\beta - \gamma_\alpha} \int_{\gamma_\alpha}^{\gamma_\beta} f''(\gamma) d\gamma. \tag{5}$$

Since $f'(\gamma)$ is the primitive function of $f''(\gamma)$, the axial force between a thin coil and a thin walled solenoid was rewritten as:

$$F_{as} = \frac{\mu_0}{2} \frac{i_a I_s}{\gamma_\beta - \gamma_\alpha} \left[f'(\gamma_\beta) - f'(\gamma_\alpha) \right]. \tag{6}$$

2.3. Second Integration, Axial Force between a Pair of Thin Walled Solenoids

In a similar way as for the axial force between a coil and a thin walled solenoid, the axial force between two thin walled solenoids, as those in Fig. 1(c), was expressed as a summation of forces, this time between n number of coils and thin walled solenoids. We utilized Equation (6) for this purpose, the summation becomes:

$$F_{sz} = \frac{\mu_0}{2} I_s \sum_1^n i_{zn} \frac{[f'(\gamma_{\beta n})]}{(\gamma_{\beta n} - \gamma_{\alpha n})} - \frac{\mu_0}{2} I_s \sum_1^n i_{zn} \frac{[f'(\gamma_{\alpha n})]}{(\gamma_{\beta n} - \gamma_{\alpha n})}. \tag{7}$$

The surface current in the second solenoid was defined as:

$$I_z = i_{zn} n. \tag{8}$$

The interval $\Delta\gamma$ was partitioned for the first part of the summation according to:

$$\frac{1}{n} = \frac{\Delta\gamma}{\gamma_\chi - \gamma_\beta}. \tag{9}$$

And for the second part:

$$\frac{1}{n} = \frac{\Delta\gamma}{\gamma_\beta - \gamma_\alpha}, \tag{10}$$

where γ_χ is the separation-to-diameter ratio between the furthermost edges of the thin walled solenoids. In other words, γ_χ is the separation-to-diameter ratio of the magnetic gap. Combining Equations (7), (8), (9) and (10), we rewrote the expression in the form of two Riemman integrals:

$$\begin{aligned} F_{sz} &= \lim_{n\to\infty} \frac{\mu_0}{2} \frac{I_s I_z \sum_1^n f'(\gamma_{\beta n})\Delta\gamma}{(\gamma_\beta - \gamma_\alpha)(\gamma_\chi - \gamma_\beta)} - \lim_{n\to\infty} \frac{\mu_0}{2} \frac{I_s I_z \sum_1^n f'(\gamma_{\alpha n})\Delta\gamma}{(\gamma_\beta - \gamma_\alpha)(\gamma_\beta - \gamma_\alpha)} \\ &= \frac{\mu_0}{2} \frac{I_s I_z \int_{\gamma_\beta}^{\gamma_\chi} f'(\gamma_\beta) d\gamma}{(\gamma_\beta - \gamma_\alpha)(\gamma_\chi - \gamma_\beta)} - \frac{\mu_0}{2} \frac{I_s I_z \int_{\gamma_\alpha}^{\gamma_\beta} f'(\gamma_\alpha) d\gamma}{(\gamma_\beta - \gamma_\alpha)(\gamma_\beta - \gamma_\alpha)}. \end{aligned} \tag{11}$$

Since $f(\gamma)$ is the primitive function of $f'(\gamma)$, the force between a pair of thin walled solenoids was rewritten in the following form:

$$F_{sz} = \frac{\mu_0}{2} \frac{I_s I_z}{(\gamma_\beta - \gamma_\alpha)} \left[\frac{f(\gamma_\chi) - f(\gamma_\beta)}{(\gamma_\chi - \gamma_\beta)} - \frac{f(\gamma_\beta) - f(\gamma_\alpha)}{(\gamma_\beta - \gamma_\alpha)} \right]. \tag{12}$$

We found that with Equation (12), it is possible to calculate the axial force between two thin walled solenoids provided that the surface currents, separation-to-diameter ratios γ_α, γ_β, γ_χ, and $f(\gamma)$ are known. In the special case when the heights h of both solenoids are identical, the separation-to-diameter ratios become:

$$\gamma_\chi = 2\gamma_\beta - \gamma_\alpha. \tag{13}$$

By combining Equations (12) and (13), we found that the force between concentric thin walled solenoids with the same height is:

$$F_{sz} = \frac{\mu_0}{2} \frac{I_s I_z}{(\gamma_\beta - \gamma_\alpha)^2} \left[f(\gamma_\chi) - 2f(\gamma_\beta) + f(\gamma_\alpha) \right]. \tag{14}$$

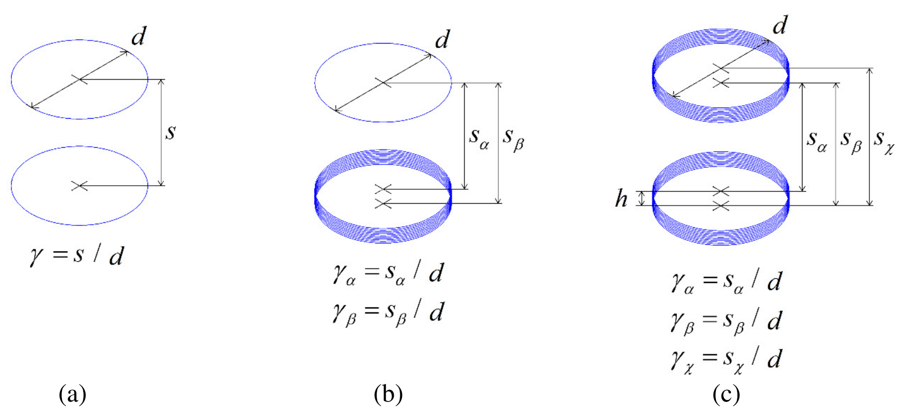

Figure 1. (a) A pair of thin coaxial coils. (b) A thin coil and a thin walled solenoid. (c) A pair of thin walled solenoids.

2.4. Finding $f(\gamma)$

The function $f''(\gamma)$ defined in Equation (1) can be obtained from a carefully designed experiment, from finite element simulations, or from an existing solution. For this contribution, we obtained it from existing semi-analytical solutions [14–16]. We rewrote the expressions in the following form:

$$f''(\gamma) = \sqrt{1/\gamma^2 + 1}\left\{k^2 K(k) + (k^2 - 2)\left[K(k) - E(k)\right]\right\}, \tag{15}$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind. The modulus k is given by:

$$k^2 = 1/(1 + \gamma^2). \tag{16}$$

From Equations (15) and (16), we calculated the values of $f''(\gamma)$ for the interval $0.01 \leq \gamma \leq 5.5$. We utilized 5000 evenly distributed points. Afterwards, we numerically integrated the results twice to find $f(\gamma)$ for the same interval with the adaptive quadrature technique available in Matlab R2014a. With the data at hand, we proceeded to curve fit $f(\gamma)$.

2.5. Curve Fitting $f(\gamma)$

To further simplify the method, we have curve fitted $f(\gamma)$. After evaluating thousands of possibilities with the aid of pyeq2 and Matlab R2014a, we found that the following rational function, numerated by a quartic polynomial and denominated by a linear binomial, offered a good balance between simplicity and accuracy.

$$f(\gamma) = (A\gamma^4 + B\gamma^3 + C\gamma^2 + D\gamma + E)/(\gamma + F). \tag{17}$$

With Equations (17) and (12), we calculated the axial force between thin walled solenoids using only basic arithmetic operations. The relevant characteristics of the curve fit that we selected are summarized in Table 1.

Table 1. Curve fit information.

Coefficient Values	
$A = 0.001032$	$D = 0.209100$
$B = -0.014560$	$E = -0.005901$
$C = 4.069000$	$F = 0.294100$
Fit statistics	
The sum of squares due to error: 0.0001786	R-square: 1
Adjusted R-square: 1	Root mean squared error: 0.0001891

2.6. Force Calculations between High Coercivity Permanent Magnets Using a Semi-Analytical Method

Consider two identical thin walled solenoids, as those in Fig. 1(c). Their height is defined as h and the separation between them as s_α. They have a height-to-diameter ratio $\alpha = h/d$ and separation-to-diameter ratio $\gamma_\alpha = s_\alpha/d$. Since high coercivity permanent magnets can be modelled as thin walled solenoids, the force between two identical coaxial thin walled solenoids, or between two high coercivity permanent magnets, can be expressed as [17]:

$$\begin{aligned} F = (B_r^2/2\mu_0)\mathrm{A} \times \Big\{ &\sqrt{1 + (\alpha + \gamma_\alpha)^2}(8/\pi)(\alpha + \gamma_\alpha)\left[K(k_1) - E(k_1)\right] \\ &- \sqrt{1 + (2\alpha + \gamma_\alpha)^2}(4/\pi)(2\alpha + \gamma_\alpha)\left[K(k_2) - E(k_2)\right] - \sqrt{1 + \gamma_\alpha^2}(4/\pi)(\gamma_\alpha)\left[K(k_3) - E(k_3)\right]\Big\}, \end{aligned} \tag{18}$$

where B_r is the remanent field and A the facing cross sectional area of the thin walled solenoids or permanent magnets. The moduli k_1, k_2, and k_3 for the complete elliptic integrals of the first and second kind are:

$$k_1^2 = 1/(1 + (\alpha + \gamma_\alpha)^2). \tag{19}$$

$$k_2^2 = 1/(1 + (2\alpha + \gamma_\alpha)^2). \tag{20}$$

$$k_3^2 = 1/(1 + \gamma_\alpha^2). \tag{21}$$

2.7. Force Measurements between High Coercivity Permanent Magnets

For this part of the method, we measured the axial forces between one pair of Nd-Fe-B permanent magnets with a remanence of 1.35 T, diameter of 20 mm, and height of 10 mm. To keep the magnets concentric at all times, they were firmly mounted on aluminum holders that were further mounted on a drill bench as shown in Fig. 2. The force was measured with a Z-type load cell arranged in line with the direction of the force. The displacement was measured with a digital gauge built in the drill bench.

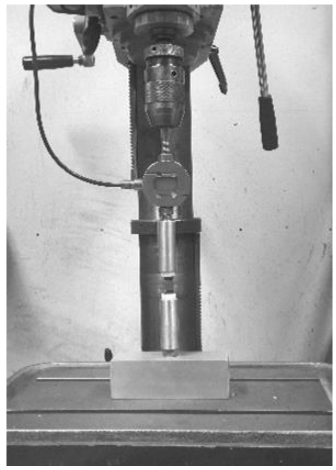

Figure 2. Setup utilized to measure the force between permanent magnets.

3. RESULTS

3.1. $f(\gamma)$ vs γ

In Fig. 3, the data of $f(\gamma)$ vs γ obtained from the double integration of $f''(\gamma)$ as described in Section 2.4 are plotted with solid red lines. The curve fit achieved, according to Section 2.5, is shown with blue dots. With $f(\gamma)$ and Equation (12), we were able to calculate the force between thin walled solenoids, on one limit, when the gap s_α was 100 times smaller than their diameter and on the other limit, when the magnetic gap s_χ was 5.5 times larger than their diameter.

3.2. Force Calculations and Force Measurements

The results of the measurements and calculations are shown in Fig. 4. The calculations were performed utilizing the methods described in Section 2, and the equations were implemented in Matlab R2014a. For the results plotted with a red solid line, we utilized the semi-analytical method that involves elliptical integrals according to Equations (18)–(21). The results plotted in a blue dot-dashed line and the results in a black dashed line correspond to calculations performed with the method proposed in this article. For both cases, we utilized Equation (14). The difference between them is that for the first case, we evaluated $f(\gamma)$ from the curve fit presented in Section 2.5, and for the other case, we obtained $f(\gamma)$ directly from the data presented in Fig. 3. In all three cases, we performed 1001 evenly distributed calculations for the interval $0.15 \leq s_\alpha \leq 20.5$ mm. The computational time needed to calculate the forces utilizing the curve fit was around 10 times faster than the time needed with the semi-analytical method. On the other hand, the time needed to calculate the forces directly from the data took around 7 times longer than the semi-analytical method. As for the measurements, we performed 154 individual measurements for the same range (interval $0.15 \leq s_\alpha \leq 20.5$ mm). They are plotted in Fig. 4 with green dots. As can be seen from the figure, there is good agreement with the measurements and calculations.

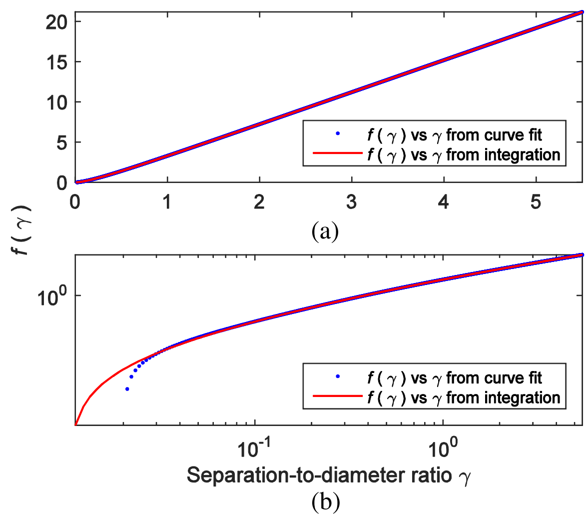

Figure 3. (a) $f(\gamma)$ vs γ from the integrations and its corresponding curve fit. (b) Data plotted in logarithmic scale. In both insets, the solid line is the data obtained from the integrations and the dots are the curve fit.

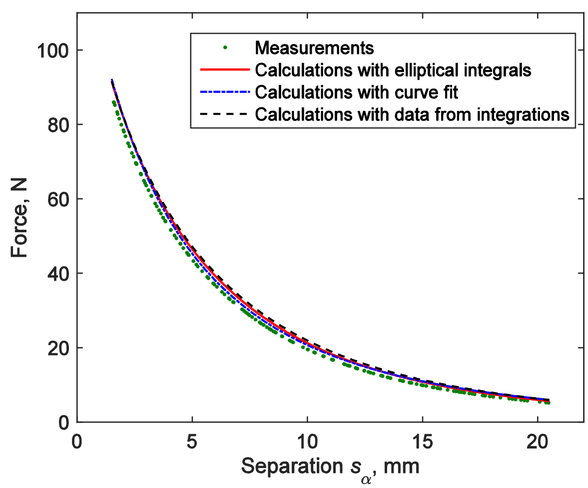

Figure 4. Measurements and calculations using different methods for the force between high coercivity permanent magnets.

4. VALIDATION

4.1. Comparison between the Force Calculation Methods

We have compared the calculation methods presented in this article. For this purpose, we chose identical solenoids with a surface current density equivalent to a 1 T homogeneously magnetized permanent magnet (795.8 kA/m) and calculated the force between them for a given range utilizing the formulas described. In order to have realistic values, the range of height-to-diameter ratios utilized for the solenoids selected was $0.2 < \alpha < 2$. Thinner magnets are easily demagnetized, and thicker magnets are too far from the optimal yielding force ratio $\alpha \approx 0.4$ [17, 19–21]. The separation-to-diameter ratio γ_α was varied from the point in which the separation between them was 100 times smaller than their diameter ($\gamma_\alpha = 0.01$) until the force was reduced to 5%. As can be seen in Fig. 5, the results showed that there was virtually no difference between the forces calculated with the semi-analytical method and the proposed method when we evaluated $f(\gamma)$ directly from the data obtained from the integrations. When we utilized the curve fit, there was a maximum overestimation of 6.8%.

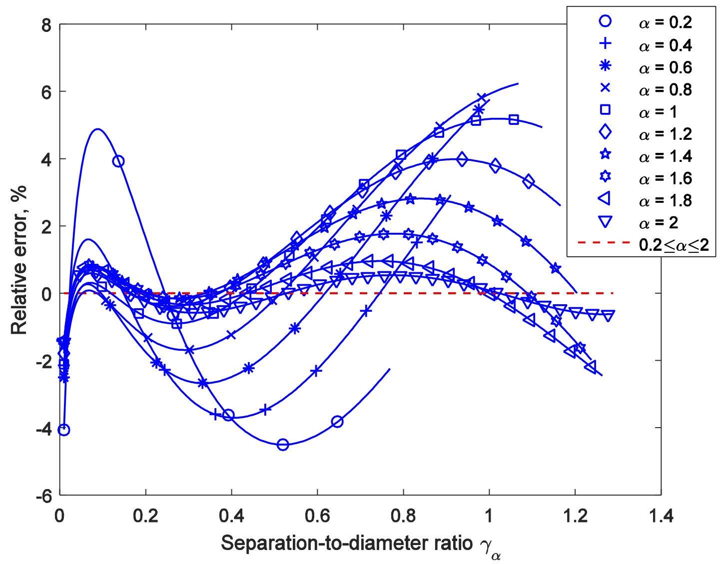

Figure 5. Relative error between the force calculated with the equations for the semi-analytical (18) and the proposed method (12). The lines in red (dashed) are the relative error when the forces were calculated using $f(\gamma)$ from the data presented in Fig. 3. The results in blue (lines with markers), are the relative force errors obtained when the forces were calculated using $f(\gamma)$ from the curve fit presented in (17). In both cases, the separation-to-diameter ratio was varied from $\gamma_\alpha = 0.01$ until the force was reduced to 5%. Each curve corresponds to solenoids with different height-to-diameter ratios (α). In the case of the dashed lines, all of them overlap.

5. CONCLUSION

We presented a method to calculate the force between thin-walled solenoids. The resulting equations were fairly compact, and there was no need for complicated expressions. It showed very high accuracy when the force was calculated directly from the data obtained after numerically integrating the behavior of a pair of thin coils two times. It also offered simplicity and fast computational times when the force was calculated using a curve fit. This is an advantage in situations where computational power is limited as in industrial robots. These machines have a limited memory and processors that favor reliability over computational power. If control actions are needed in real time, it can be a challenge to use a computationally heavy model. Since the use of elliptical integrals was circumvented, the presented equations are relatively easy to implement especially in low level programming languages. An example of a potential application is the automatic assembly of ferrite magnets for wave energy converters with industrial robots [18]. If needed, the method can be easily extended to solenoids that do not have a circular cross section.

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REFERENCES

1. Beleggia, M., D. Vokoun, and M. De Graef, "Forces between a permanent magnet and a soft magnetic plate," *IEEE Magn. Lett.*, Vol. 3, 0500204, 2012.
2. Robertson, W., B. Cazzolato, and A. Zander, "A simplified force equation for coaxial cylindrical magnets and thin coils," *IEEE Trans. Magn.*, Vol. 47, No. 8, 2045–2049, 2011.

3. Ravaud, R., G. Lemarquand, S. Babic, V. Lemarquand, and C. Akyel, "Cylindrical magnets and coils: Fields, forces, and inductances," *IEEE Trans. Magn.*, Vol. 46, No. 9, 3585–3590, 2010.
4. Ravaud, R., G. Lemarquand, V. Lemarquand, S. Babic, and C. Akyel, "Mutual inductance and force exerted between thick coils," *Progress In Electromagnetics Research*, Vol. 102, 367–380, 2010.
5. Ravaud, R. and G. Lemarquand, "Comparison of the Coulombian and Amperian current models for calculating the magnetic field produced by radially magnetized arc-shaped permanent magnets," *Progress in Electromagnetics Research*, Vol. 95, 309–327, 2009.
6. Akoun, G. and J.-P. Yonnet, "3D analytical calculation of the forces exerted between two cuboidal magnets," *IEEE Trans. Magn.*, Vol. 20, No. 5, 1962–1964, 1984.
7. Yang, Z. and R. Hull, "Forces and torques between two square current loops," *Int. J. of Applied Electromagnetics and Mechanics*, Vol. 10, No. 1, 63–76, 1999.
8. Babic, S., C. Akyel, and N. Boudjada, "The simplest formulas for self inductance, mutual inductance and magnetic force of coaxial cylindrical magnets and thin coils," *Recent Researches in Circuits and Systems*, 44–4, WSEAS Press8, Kos Island, Greece, 2012.
9. Babic, S., C. Akyel, Y. Ren, and W. Chen, "Calculation of the mutual inductance and the magnetic force between a thick circular coil of the rectangular cross section and a thin wall solenoid (integro-differential approach)," *Progress In Electromagnetics Research B*, Vol. 33, 221–237, 2011.
10. Babic, S. and C. Akyel, "Magnetic force between inclined circular loops (Lorentz Approach)," *Progress In Electromagnetics Research B*, Vol. 38, 333–349, 2012.
11. Babic, S., C. Akyel, Y. Ren, and W. Chen, "Magnetic force calculation between circular coils of rectangular cross section with parallel axes for superconducting magnet," *Progress In Electromagnetics Research B*, Vol. 37, 275–288, 2012.
12. Shiri, A. and A. Shoulaie, "A new methodology for magnetic force calculations between planar spiral coils," *Progress In Electromagnetics Research*, Vol. 95, 39–57, 2009.
13. Shiri, A., M. R. Alizadeh Pahlavani, H. A. Mohammadpour, and A. Shoulaie, "Electromagnetic force distribution on cylindrical coils' body," *PIERS Proceedings*, 1102–1105, Xi'an, China, March 22–26, 2010.
14. Iwasa, Y., *Case Studies in Superconducting Magnets*, 2nd Edition, 88–89, Springer, Boston, MA, USA, 2009.
15. Garrett, M. W., "Calculations of fields, forces, and mutual inductances of current systems by elliptic integrals," *J. Appl. Phys.*, Vol. 34, No. 9, 2567–2573, 1963.
16. Maxwell, J., *A Treatise on Electricity and Magnetism*, Vol. 2, 3rd Edition, Arts. 615 and 833, Oxford University Press, London, U.K., 1891 (Reprinted 1954, Dover, New York).
17. Pérez-Loya, J. J. and U. Lundin,, "Optimization of force between cylindrical permanent magnets," *IEEE Magn. Lett.*, Vol. 5, 1–4, 2014.
18. Hultman, E., B. Ekergård, T. Kamf, D. Salar, and M. Leijon, "Preparing the Uppsala University wave energy converter generator for large-scale production," *5th International Conference on Ocean Energy*, Halifax, Canada, 2014.
19. Robertson, W., B. Cazzolato, and A. Zander, "Maximizing the force between two cuboid magnets," *IEEE Magn. Lett.*, Vol. 1, 9000102, 2010.
20. Cooper, R., V. Neil, and W. Woodruff, "Optimum permanent-magnet dimensions for repulsion applications," *IEEE Trans. Magn.*, Vol. 9, No. 2, 125–127, 1973.
21. Kinouchi, Y., T. Ushita, H. Tsutsui, and Y. Yoshida, "Optimization and design of permanent magnet devices for retaining dental prosthetic appliances," *IEEE Trans. Biomed. Eng.*, Vol. 30, No. 4, 201–207, 1983.