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Large Linear Random Symmetric Arrays

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Abstract—In this work linear random arrays are studied. It is shown that random symmetric linear arrays can be more easily characterised (with respect to the asymmetric ones) in terms of the first and second order statistics of the array factor magnitude. In particular, the non-stationarity of the array factor can be taken into account while studying the array response. Accordingly, this leads to more accurate predictions as far as the side-lobe level is concerned.

1. INTRODUCTION

It has been recognised for a long time that non-uniform arrays give advantages in terms of resilience to grating-lobes and bandwidth enlargement [1]. Moreover, especially for large sized arrays, the number of radiators can be reduced without degrading resolution and with no significant increasing in the side-lobe level. This can be roughly justified by noting that aperiodic arrays have more degrees of freedom to be exploited while shaping the array factor than the uniform ones. In this framework, random arrays are those where the positions of radiators are established according to some probabilistic law. This entails that the array factor is a random process as the radiators' position are random variables.

Results about the probabilistic theory of arrays can be found in the work of Maffett [2]. However, it is Lo who first developed this study in a systematic way [3–7].

While studying random arrays, the statistics of the array factor magnitude is of paramount importance as it is related to the side-lobe level. This is often achieved by considering array factor as being wide sense stationary in the side-lobe region [3]. This indeed is a reasonable assumption. However, removing such an assumption can lead to better side-lobe estimations, but this is not in general an easy task. Actually, this problem is still an open issue [8, 9].

Although the probabilistic theory allows providing optimal synthesis of arrays only in a probabilistic sense [10], it seems still the most complete analytical theory nowadays for obtaining *a priori* information on the achievable performance before proceeding to the synthesis. This is particularly true for large arrays (in terms of aperture and number of antennas) for which central limit theorem can be invoked in order to significantly help the analytical treatment. Therefore, herein, we are concerned with such a kind of array, which, however, are interesting for important applications as radio astronomy [11] or for microwave power transmission [12]. Also, the addressed topic is of interest for other different fields as it is strongly linked to the theory of non-uniform sampling, as shown by Bilinskis and Lagunas [13].

In this paper, we show the advantage of studying linear symmetric random arrays, where the radiators are deployed symmetrically with respect to the centre of the array aperture. In particular, we succeed in finding the cumulative distribution function of the array factor magnitude in closed form. Moreover, the non-stationarity of the array factor process is accounted for.

In order to characterise side-lobes, two literature approaches, the sampling [3] and up-crossing [15] methods are considered. Eventually, it is shown that by accounting for the array factor non-stationarity

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leads to a better estimation of the side-lobe level distribution. Also the computation of the average number of up-crossings is numerically simpler than that for the asymmetric case. Furthermore, taking advantage of the array symmetry, a simple formula is introduced to estimate the side-lobe level which avoids computation related to the two previous methods.

Finally, it is worth asking about the loss in performance (that is how side-lobes grow up) that symmetric arrays experience compared to the asymmetric ones. Numerical simulations show that side-lobe increasing is around one dB.

2. BASICS ON RANDOM ARRAYS

Consider a linear array of N equally-excited isotropic antennas placed along the x axis within the segment $[-L/2, L/2]$ (see Fig. 1). The corresponding normalised array factor is [6]

$$F(u) = \frac{1}{N}\sum_{n=1}^{N} e^{j2\pi X_n u} \tag{1}$$

where

- θ and θ_o are the observation angle and steering angle measured from the normal to the array axis;
- L is the aperture of the array measured in meter;
- $u = \sin\theta - \sin\theta_o$;
- $x_n \in [-L/2; L/2]$ is the position of the n-th radiator from the origin of the axis;
- $X_n = x_n/\lambda$.

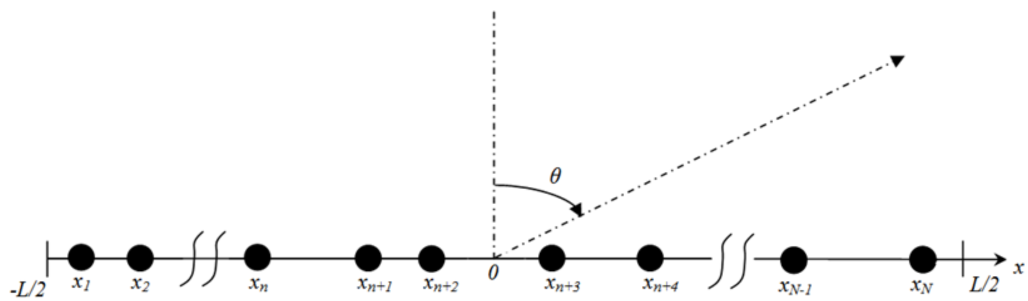

Figure 1. Geometry of a generic random asymmetric arrays.

The positions of the radiators are assumed to be random variables. Consequently, the array factor is a stochastic process whose characterisation requires the joint probability density function (PDF) of any n of its samples (with respect to u). However, useful array properties can be obtained by means of low order and somehow partial statistics [3]. In particular, by assuming the X_n as independent and identically distributed (*i.i.d.*) random variables with PDF, f_X, defined over the array aperture [3–5][†], the expected array factor and the variance are found to be

$$\begin{aligned}
\phi(u) &= E\{F(u)\} = \frac{1}{N}\sum_{n=1}^{N} E\{e^{j2\pi X_n u}\} \\
&= E\{e^{j2\pi X u}\} = \int_{-L/(2\lambda)}^{L/(2\lambda)} f(\xi)\cdot e^{j2\pi\xi u}\, d\xi \\
\sigma^2(u) &= E[|F(u)-\phi(u)|^2] = E[|F(u)|^2] - |\phi(u)|^2 \\
&= \frac{1}{N^2}\sum_{n=1}^{N}\sum_{m=1}^{N} E\left\{e^{j2\pi X_n u}\cdot e^{-j2\pi X_m u}\right\} - |\phi(u)|^2
\end{aligned} \tag{2}$$

[†] More generally, each radiator's position can have its own different distribution.

$$= \frac{1}{N} + \sum_{n=1}^{N} \sum_{\substack{m=1 \\ m \neq n}}^{N} E\left\{e^{j2\pi X_n u}\right\} \cdot E\left\{e^{-j2\pi X_m u}\right\} - |\phi(u)|^2 = \frac{1 - |\phi(u)|^2}{N} \quad (3)$$

Note that f_X and $\phi(u)$ are linked in the same way as the far-field pattern and a continuous line-source [14] are. Therefore, the mean array factor does not suffer from grating lobes as the array factor of uniform arrays does. Now, assume f_X to be an even function. When N is sufficiently high, the Lyapunov's central limit theorem can be invoked so that the real $F_R(u)$ and imaginary $F_I(u)$ parts of $F(u)$ can be considered as independently and normally distributed, that is $F_R(u) \sim \mathcal{N}[\phi(u), \sigma_R^2(u)]$ and $F_I(u) \sim \mathcal{N}[0, \sigma_I^2(u)]$. Accordingly, the cumulative distribution function (CDF) of the array factor magnitude is given by [3]

$$P(|F(u)| < \alpha) = \iint\limits_{|F(u)|<\alpha} \frac{e^{-\left\{\frac{[F_R - \phi(u)]^2}{2\sigma_R^2(u)} + \frac{F_I^2}{2\sigma_I^2(u)}\right\}}}{2\pi\sigma_R(u)\sigma_I(u)} dF_R dF_I \quad (4)$$

This is a generalised non-central chi-square distribution with two degrees of freedom, which does not admit a closed form expression [3]. In the side-lobe region, it is generally assumed that $\phi(u)=0$ and $\sigma_R^2(u) = \sigma_I^2(u) = 1/2N$. This greatly simplifies the matter as the magnitude array factor becomes Rayleigh distributed for each u. However, this introduces an error while estimating the side-lobe level. To overcome this problem, an asymptotic expansion of Eq. (4) was introduced in [4] by resorting to the Gaussian error function. This is valid when $\alpha \gg \sigma_R(u), \sigma_I(u)$. Unfortunately, this generally holds in the main-beam region, hence it is not useful for side-lobe level study. In the same paper, this constraint was relaxed, and Eq. (4) was approximated by resorting to the tabulated incomplete gamma function.

In this work, we try to overcome such a drawback. To this end, we assume that the antennas are symmetrically arranged about the array centre. This greatly simplifies the analytical treatment also for the side-lobe level estimation, the latter being of paramount importance for array performance evaluation.

3. SYMMETRIC ARRAYS

As reminded above, to get simplified analytical characterisation of random arrays, previous researchers have resorted to some approximations, such as the Gaussianity and weak stationarity of the array factor ‡. Moreover, partial stationary [15] (i.e., only the variance is shift-invariant) still leads to a 2D numerical integration. In this section, we want to show that for symmetric arrays, by assuming only the Gaussianity of the array factor (stationary hypothesis is relaxed) allows obtaining a closed-form expression of the array factor magnitude CDF and hence a more accurate characterisation of the side-lobe level.

3.1. Symmetric Arrays Distribution

Starting from the working hypotheses of the previous section, it is furthermore assumed that $\forall\ X_n > 0\ \exists\ X_m < 0 : X_m = -X_n$. Also, for odd N, an antenna is located (deterministically) right at the symmetry centre ($X = 0$) of the array. Accordingly, the array factor is rewritten as

$$F(u) = \begin{cases} \dfrac{2}{N} \displaystyle\sum_{n=1}^{N/2} \cos(2\pi X_n u) & \text{if } N \text{ even} \\ \dfrac{1}{N} + \dfrac{2}{N} \displaystyle\sum_{n=1}^{(N-1)/2} \cos(2\pi X_n u) & \text{if } N \text{ odd} \end{cases} \quad (5)$$

‡ Stationarity means that the statistical properties of the array factor do not change over the variable u. More in detail, strong stationarity concerns the shift-invariance (in u) of any its finite-dimensional distributions, whereas weak stationarity only concerns the shift-invariance of first and second moments of a process. Note that for Gaussian processes strong and weak stationarities coincide.

If the antennas' positions are $i.i.d.$ random variables (except for the antenna placed at $X = 0$ when N is odd), and N is sufficiently high, then $F(u) \sim \mathcal{N}[\phi(u), \sigma^2(u)]$ with

$$\phi(u) = \begin{cases} \displaystyle\int_0^{L/(2\lambda)} f_X(X)\cos(2\pi X u)\, dX & \text{if } N \text{ even} \\ \displaystyle\frac{1}{N} + \frac{N-1}{N}\int_0^{L/(2\lambda)} f_X(X)\cos(2\pi X u)\, dX & \text{if } N \text{ odd} \end{cases} \tag{6}$$

$$\sigma^2(u) = \begin{cases} \displaystyle\frac{1}{N}[1+\phi(2u)] - \frac{2}{N}\phi^2(u) & \text{if } N \text{ even} \\ \displaystyle\frac{N^2-3N}{N^2(N-1)} + \frac{\phi(2u)}{N} - \frac{2}{N-1}\phi^2(u) + \frac{4}{N(N-1)}\phi(u) & \text{if } N \text{ odd} \end{cases} \tag{7}$$

In order to study the side-lobe level, what matters is the probability distribution of the array factor magnitude. The latter, as opposed to the asymmetric case, can be obtained without any approximation through a simple transformation. Hence, the PDF of the array factor magnitude is

$$\begin{aligned} f_{|F|}(|F|, u) &= f_F(F, u) + f_F(-F, u) \\ &= \frac{1}{\sqrt{2\pi}\sigma(u)} e^{-\frac{[F-\phi(u)]^2}{2\sigma^2(u)}} + \frac{1}{\sqrt{2\pi}\sigma(u)} e^{-\frac{[F+\phi(u)]^2}{2\sigma^2(u)}}, \quad F \geq 0 \end{aligned} \tag{8}$$

from which the CDF easily follows

$$P\{|F(u)| \leq y\} = Q\left(\frac{-y-\phi(u)}{\sigma(u)}\right) - Q\left(\frac{y-\phi(u)}{\sigma(u)}\right) \tag{9}$$

where the definition of $Q(\cdot)$ is related to the so-called error function and is reported in Appendix A along with a polynomial approximation used in the sequel.

Equation (9) can be rewritten as

$$P\{|F(u)| \leq y\} = 1 - Q\left(\frac{y+|\phi(u)|}{\sigma(u)}\right) - Q\left(\frac{y-|\phi(u)|}{\sigma(u)}\right) \tag{10}$$

when $y \geq |\phi(u)|$ and as

$$P\{|F(u)| \leq y\} = Q\left(\frac{|\phi(u)|-y}{\sigma(u)}\right) - Q\left(\frac{|\phi(u)|+y}{\sigma(u)}\right) \tag{11}$$

when $y < |\phi(u)|$. Accordingly, Eq. (9) can be computed by resorting to some analytical approximation for Q (see Appendix A) which holds for positive arguments and have a maximum absolute error of 0.27%. Therefore, no tabulated function is required. Also, the p-per-cent level curve $r_p(u)$, that is

$$P\{|F(u)| \leq r_p(u)\} = p\% \tag{12}$$

can be conveniently computed due to symmetry of the array. In fact, considering Eqs. (10) and (11), for the generic p-per-cent level curve, when $r_p(u) \geq |\phi(u)|$

$$P\{|F(u)| \leq r_p(u)\} = 1 - Q\left(\frac{r_p(u)+|\phi(u)|}{\sigma(u)}\right) - Q\left(\frac{r_p(u)-|\phi(u)|}{\sigma(u)}\right) = p\% \tag{13}$$

while when $r_p(u) < |\phi(u)|$

$$P\{|F(u)| \leq r_p(u)\} = Q\left(\frac{|\phi(u)|-r_p(u)}{\sigma(u)}\right) - Q\left(\frac{|\phi(u)|+r_p(u)}{\sigma(u)}\right) = p\% \tag{14}$$

In this case, determining r_p requires to solve (for each u) a nonlinear equation which, however, is polynomial if the approximate representation of Q is used.

A further advantage with respect to the asymmetric arrays is that the mean and variance of the magnitude array factor can be evaluated in closed form as

$$\phi_{|F|}(u) = \sqrt{\frac{2}{\pi}}\sigma(u) e^{-\frac{|\phi(u)|^2}{2\sigma^2(u)}} + |\phi(u)|\left[1 - 2Q\left(\frac{|\phi(u)|}{\sigma(u)}\right)\right] \tag{15}$$

$$\sigma^2_{|F|}(u) = \sigma^2(u) + \phi^2(u) - \phi^2_{|F|}(u) \tag{16}$$

3.2. Side-Lobe Level Distribution

So far we have shown how the array factor magnitude distribution can be more accurately determined under the assumption of symmetric radiators' positions. However, this only regards first-order (i.e., somehow *punctual*) characterisation. In order to characterise the side-lobe level (SLL), the supremum of the array factor magnitude $\max\{|F(u)|\}$ in the SL region (SLR) must be studied. This is a difficult task still unsolved under general conditions. For this reason, approximate approaches have been proposed in literature. The sampling [3] and up-crossing [15] approaches are among the most used in the literature, and therefore, they will be exploited herein.

The sampling approach arises from the observation that the array factor (as a function of u) is an entire function of exponential type with probability one. Accordingly, the array factor can be characterised someway by its samples. In particular, the statistics of the random variables arising from sampling the array factor is used to estimate SLL as [3]

$$\begin{aligned} P(SLL \leq y) &= P(|F(u)| \leq y, \forall u \in SLR) \\ &\approx P(|F(u_1)| \leq y, |F(u_2)| \leq y, \dots, |F(u_M)| \leq y) \end{aligned} \tag{17}$$

where, from the operational point of view, $u_1, u_2, \dots, u_M$ are the points in which $\phi(u)$[§] reaches its extreme values in half of the side-lobe region (i.e., outside the main beam and within the visible domain only form one side with respect to $u = 0$). [‖] Alternatively, in [6] the sampling points have been selected according to the Nyquist rate. The rationale subtended in Eq. (17) is the interpretation of the interpolating sampling series as a first order Lagrange interpolation. This way, the array factor between the sampling points is bounded between these sample values. Hence, information at the sampling points should be enough to characterise the maximum of the array factor magnitude. Finally, assuming the samples independent yields

$$P\{|F(u)| \leq y,\ \forall\ u \in SLR\} \approx \prod_{m=1}^{M} P\{|F(u_m)| \leq y\} \tag{18}$$

where $P\{|F(u_m)| \leq y\}$ is given by Eq. (8). Note that compared to the analogues for asymmetric arrays [3], there was no need to further assume the samples identically distributed in order to have a mathematically tractable expression. Hence, it is expected that Eq. (18) returns better SLL estimations than the ones previously reported in [3] and [6].

It is clear that the sampling approach as sketched above is formally questionable. Indeed, the side-lobe maxima of $|\phi(u)|$ does not in general correspond to the side-lobe maxima of the array factor magnitude. Moreover, the linear Lagrange interpolation hardly matches the side-lobe behaviour between two sampling points. Nonetheless, the sampling approach is very simple and shows that it returns results in good agreement with some numerical experiments [3]. Also, SLL estimation can be a little bit refined by removing the independence hypothesis and considering only adjacent samples to be correlated [6].

A different way to characterise the SSL is to estimate how many times the magnitude array factor up-crosses (i.e., crosses with positive slope) a given level in the SLR. This approach does not suffer from the formal objections aforementioned while discussing the sampling method. However, in general it is difficult to obtain analytical results [15].

Let y be the level with respect to which the number of up-crossings has to be estimated and say $\mathcal{N}_y$ the random variable that counts the times $|F(u)|$ up-crosses it in the side-lobe region. In order to find the SLL, one is interested in finding the threshold for which the probability of no crossing is one. This is equivalent to estimating $P(\mathcal{N}_y \geq 1)$, i.e., that at least there is one up-cross. This task is greatly simplified if we are content to find an upper bound for the previous probability. To this end, Markov's inequality can be used to show that $P(\mathcal{N}_y \geq 1) \leq E[\mathcal{N}_y]$. Indeed, the relationship $P\{\mathcal{N}_y \geq 1\} \leq E\{\mathcal{N}_y\}$ is useful when the mean of up-crossings is sufficiently low as this corresponds to a low probability that the SLL exceeds the y level. Accordingly, we focus on $E(\mathcal{N}_y)$ which has a known expression since the

§ For asymmetric arrays in [3] and [5], $\phi(u)$ is real because f_X is assumed even. Instead, for symmetric arrays, $\phi(u)$ is always real.

‖ As $F(-u) = F^*(u)$, it is enough considering only $u > 0$. This point seems neglected in Lo's works where twice as many as sampling points have been indeed considered.

seminal work of Rice [17]. In detail, it results in that

$$E\{\mathcal{N}_y\} = \int_{\delta}^{2} du \int_{0}^{+\infty} |F|' f_{|F||F|'}(y, |F|', u) d|F|' \tag{19}$$

where $f_{|F||F|'}$ is the joint PDF between the array magnitude $|F|$ and the derivative of the array magnitude $|F|'$, and δ is the nominal starting point of the (positive, i.e., $u > 0$) SLR. Generally, δ is assumed in correspondence of the first null of $\phi(u)$. Accordingly, the (positive) SLR is $[\lambda/L, 2]$. In the case of asymmetric random arrays Eq. (19) admits an easy computation only when $F(u)$ is assumed weakly stationary. In fact, in this case $F_R(u)$, $F_R'(u)$, $F_I(u)$ and $F_I'(u)$ are uncorrelated, and Gaussian and $f_{|F||F|'}$ can be readily found from $f_{F_R F_R' F_I F_I'}$. In [15], the case $E[F_R(u)]$ and $E[F_R'(u)]$ being not constant with u has been considered as well, but the variances are still assumed constant, and the $f_{F_R F_R' F_I F_I'}$ is still built up by simply multiplying the marginal $PDFs$. This, however, is not rigorously justified as now $F_R(u)$, $F_R'(u)$, $F_I(u)$ and $F_I'(u)$ are no more uncorrelated.

Here, for symmetric arrays, the expected number of up-crossings is found without resorting to none of the assumptions mentioned above. In fact, since the array factor is a real process, determining the up-crossings of the array factor magnitude is equivalent to simultaneously studying how $F(u)$ and $\tilde{F}(u) = -F(u)$ up-cross the level $y > 0$. Since $F(u)$ and $F'(u)$ are jointly Gaussian, and of course the same holds true for $\tilde{F}(u)$ and $\tilde{F}'(u)$. The mean number of up-crossings can be written as

$$\begin{aligned} E\{\mathcal{N}_y\} &= \int_{\delta}^{2} du \int_{0}^{+\infty} F' f_{FF'}(y, F', u) dF' + \int_{\delta}^{2} du \int_{0}^{+\infty} \tilde{F}' f_{\tilde{F}\tilde{F}'}(y, \tilde{F}', u) d\tilde{F}' \\ &= \int_{\delta}^{2} du \int_{0}^{\infty} F' \left\{ \frac{e^{-\frac{1}{2[1-\rho_{FF'}^2(u)]}\left[\frac{[y-\phi(u)]^2}{\sigma^2(u)} - \frac{2\rho_{FF'}(u)[y-\phi(u)][F'-\phi_{F'}(u)]}{\sigma(u)\sigma_{F'}(u)} + \frac{[F'-\phi_{F'}(u)]^2}{\sigma_{F'}^2(u)}\right]}}{2\pi\sigma(u)\sigma_{F'}(u)\sqrt{1-\rho_{FF'}^2(u)}} \right. \\ &\quad \left. + \frac{e^{-\frac{1}{2[1-\rho_{FF'}^2(u)]}\left[\frac{[y+\phi(u)]^2}{\sigma^2(u)} - \frac{2\rho_{FF'}(u)[y+\phi(u)][F'+\phi_{F'}(u)]}{\sigma(u)\sigma_{F'}(u)} + \frac{[F'+\phi_{F'}(u)]^2}{\sigma_{F'}^2(u)}\right]}}{2\pi\sigma(u)\sigma_{F'}(u)\sqrt{1-\rho_{FF'}^2(u)}} \right\} dF' \end{aligned} \tag{20}$$

with $\phi_{\tilde{F}(u)} = -\phi(u)$, $\phi_{\tilde{F}'(u)} = -\phi_{F'}(u)$, $\sigma_{\tilde{F}}^2(u) = \sigma_F^2(u)$, $\sigma_{\tilde{F}'}^2(u) = \sigma_{F'}^2(u)$. Also, $\phi_{F'}(u)$ and $\sigma_{F'}^2(u)$ are the mean and variance of $F'(u)$; $\rho_{FF'}(u)$ is the correlation coefficient between $F(u)$ and $F'(u)$ (see Appendices B and C for their derivation), while $\phi_{\tilde{F}(u)}$ and $\sigma_{\tilde{F}}^2(u)$ are the mean and variance of $\tilde{F}(u)$; $\phi_{\tilde{F}(u)}$ and $\sigma_{\tilde{F}'}^2(u)$ are the mean and variance of $\tilde{F}'(u)$; $\rho_{\tilde{F}\tilde{F}'}(u)$ is the correlation coefficient between $\tilde{F}(u)$ and $\tilde{F}'(u)$.

Equation (20) entails solving numerically a 2D integration. However, such an integration takes into account the *exact* $F(u)$ and $F'(u)$ joint PDF. Also, should the approximation adopted in [15] be exploited (i.e., vanishing correlation coefficients) in our case the numerical integration becomes 1D.

Before concluding this section, we go back to the problem of directly estimating $\max\{|F(u)|\}$ by inquiring whether the symmetry assumption about the radiators' locations can be further exploited. We have already shown that for symmetric arrays the array factor is a real and Gaussian process. Therefore, the following standard result holds true

$$P\{\phi(u) - 4\ \sigma(u) \leq F(u) \leq \phi(u) + 4\ \sigma(u)\} \times 100 \approx 99.99\% \tag{21}$$

This of course is a punctual (i.e., for a fixed u) relationship. Nonetheless, it is reasonable to infer that with probability *almost* one

$$SLL \approx \max\left\{\max_{u\in[\delta,2]}\{|\phi(u) - 4\ \sigma(u)|\}, \max_{u\in[\delta,2]}\{|\phi(u) + 4\ \sigma(u)|\}\right\} \tag{22}$$

Since $\phi(u)$ and $\sigma(u)$ are known in closed form, this allows easy computation of a SLL estimation for given aperture and number of antennas.

4. NUMERICAL ASSESSMENTS

In this section, some numerical examples are presented in order to check Eqs. (18), (20) and (22).

A large array with $L = 300\lambda$ is considered whereas the number of antennas varies from 200 to 600 with a step of 50. This corresponds to an average spacing between antennas, approximately ranging from 1.5λ to $\lambda/2$. The PDF of the positions is chosen uniform. The experimental array factor in SLR is computed via Montecarlo simulations (20000 trials are used) by employing a sampling step of $\lambda/(20L)$, which is ten times below the one required from the bandwidth of a generic array factor squared magnitude sample and used in the sampling approach.

Figure 2 compares the experimental $CDFs$ of the side-lobe level with the estimation returned by Eq. (18) with $M = 4L/\lambda$ (i,e., according to the Nyquist criterion)¶ by sampling the array factor within $u = [\lambda/L, 2]$. The case of the array factor is assumed stationary, which is with $\phi(u) = 0$ and $\sigma^2(u) = 1/N$, as commonly done in literature, and is reported as well for comparison purposes. As can be seen, estimation in Eq. (18) works better. This definitely proves that accounting for the exact PDF is of great importance. The SLL level can then be estimated by such curves as the threshold y where first $P(SLL \leq y) \approx 1$. Finally, it is obviously noted that when the right number of radiators is allowed (i.e., $\lambda/2$ spaced a part), random elements collocation is not better than the uniform one.

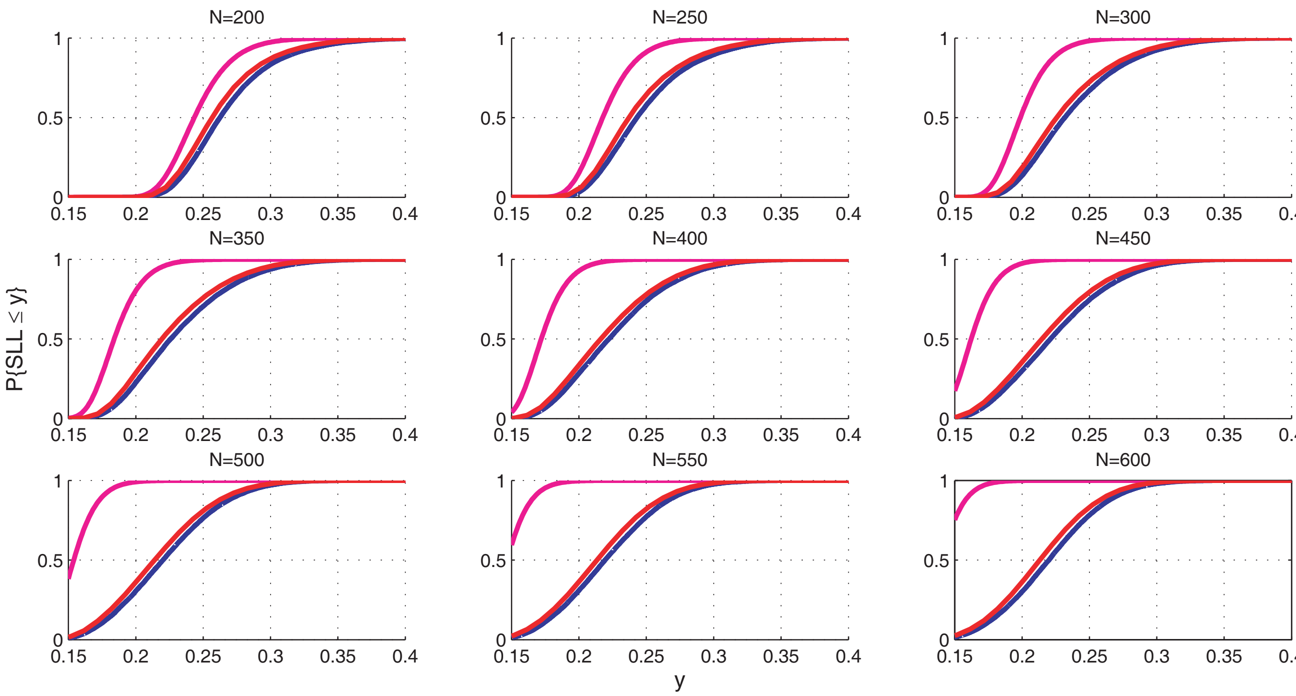

Figure 2. Estimating the SLL by the sampling approach. Comparison between the experimental side-lobe level CDF (blue line), the theoretical estimation returned by Eq. (18) (red) and the same in the stationary case, i.e., with $\phi(u) = 0$ and $\sigma^2(u) = 1/N$ with $M = 4(L/\lambda)$ (magenta line), for a 300λ long array and various values of N.

In Fig. 3, Eq. (20) is checked. In this case, we set $E[\mathcal{N}_y] = 0.1$ and find the corresponding level y. This is equivalent to finding the level for which the probability of up-crossing (or equivalently that the SLL stay above) is ≤ 0.1. Once again, it can be seen that accounting for the non-stationarity of the array factor (as done herein) returns very accurate results.

Finally, in Table 1 results concerning the validation of Eq. (22) are shown. The comparison is done between experimental values and those estimated via Eq. (22). As can be seen, despite the simplicity

¶ Note that we have chosen the sampling approach because, as shown, it returns better estimation than by fixing the sampling points over the local extreme of $\phi(u)$ [7].

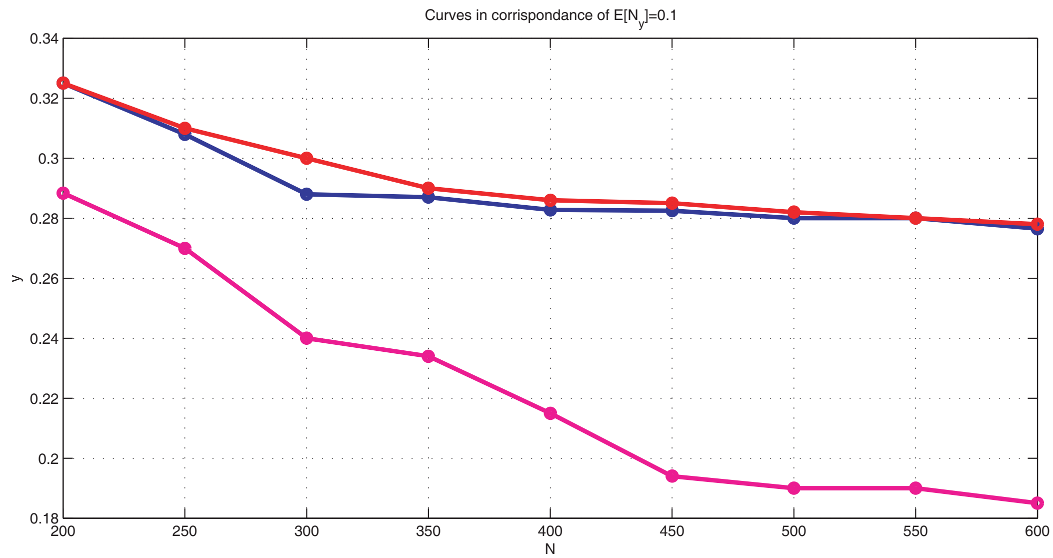

Figure 3. Estimating the threshold y for $E[\mathcal{N}_y] = 0.1$. Experimental results are in blue line, the ones retuned by Eq. (20) are in red line whereas the magenta line refers to the stationary case.

Table 1. Checking Eq. (22) for the array of $L = 300\lambda$.

N	Experimental (dB)	Theoretical (dB)
200	−6.3133	−6.1026
250	−7.0334	−6.6360
300	−6.5153	−7.0504
350	−7.3131	−7.3874
400	−7.3197	−7.6705
450	−7.8676	−7.9090
500	−8.1325	−8.1188
550	−8.4213	−8.3021
600	−8.5289	−8.4663

Table 2. Statistics (in dB) of the SLL for the symmetric array with the aperture of 300λ.

N	min$\{SLL\}$	mean$\{SLL\}$	max$\{SLL\}$
200	−14.4010	−11.4063	−6.3133
250	−15.3787	−12.0375	−7.0334
300	−15.9392	−12.4439	−6.5153
350	−16.5153	−12.6832	−7.3131
400	−16.9208	−12.8873	−7.3197
450	−17.5019	−12.9694	−7.8676
500	−17.9850	−13.0501	−8.1325
550	−17.9170	−13.0936	−8.4213
600	−18.1205	−13.1131	−8.4663

Table 3. Statistics (in dB) of the SLL for the equivalent asymmetric array with the aperture of 300λ.

N	min$\{SLL\}$	mean$\{SLL\}$	max$\{SLL\}$
200	−15.7747	−12.5477	−7.9282
250	−16.9474	−12.8244	−8.1979
300	−17.0505	−12.9532	−8.3501
350	−17.4256	−13.0251	−8.5583
400	−17.7323	−13.0907	−9.7150
450	−17.5910	−13.1158	−9.3516
500	−17.9212	−13.1269	−9.7054
550	−17.9610	−13.1540	−9.7635
600	−17.8968	−13.1579	−9.9312

of Eq. (22), the estimated values are very close to the experimental ones, more or less with an accuracy similar to the previous methods.

As a final point, we address the question whether the symmetry of the antennas' positions can lead to a relevant degradation of the array response. This point arises naturally as symmetric arrays allow for half of the degrees of freedom with respect to the asymmetric case. As can be observed from Tables 2 and 3, which compare the side-lobe level statistics for the symmetric and asymmetric arrays (of course keeping the array aperture and number of antennas the same), when the number of elements is high, the deviation between the performance of the two types of arrays is not very significant. This corroborates previous conclusions [18] which state that the number of radiators plays a major role compared to the way they are deployed.

5. CONCLUSIONS

In this paper, we address the problem of analysing random arrays. In particular, we focus on large (in terms of number of elements and size) linear symmetric arrays.

It is known that random arrays allow mitigating the problem of grating lobes and enlarging the working frequency band. However, the crucial question is to foreseen how the SLL grows up. To this end, different methods have been presented in the literature, which are often based on some assumptions, as stationarity, which greatly simplify the analytical treatment. In order to remove such an assumption, we are confined to consider random symmetric arrays. This allows getting the exact CDF of the SLL. Moreover, by using two literature methods, we show that very accurate results are obtained, especially as compared to the ones that the stationary assumption leads to. Also, the symmetry allows the derivation of a very simple formula for the SLL estimation (see Eq. (22)) which proves to be very close to the experimental results.

Of course, the clear advantages obtained while estimating the SLL must be a traded-off with the loss of performance which can be expected due to renouncing to half of the degrees of freedom allowed from a more general asymmetric layout. However, numerical simulations show that actually the symmetric element arrangement is worth considering as only (more or less) one dB is lost in the SLR.

APPENDIX A. ANALYTICAL APPROXIMATION OF THE Q-FUNCTION

The function $Q(x)$ is defined in the following way

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{\lambda^2}{2}} d\lambda \tag{A1}$$

which can be approximated in the following way

$$Q(x) \approx \left[\frac{1}{(1-0.339)x + 0.339\sqrt{x^2+5.510}} \right] \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \tag{A2}$$

with an absolute maximum error of 0.27% for $x \geq 0$ [16].

APPENDIX B. MEAN AND VARIANCE OF THE DERIVATIVE OF THE ARRAY FACTOR

The mean of the derivative of array factor is simply the derivative of the mean array factor

$$\begin{aligned} \phi_{F'}(u) &= E\left\{ \frac{dF(u)}{du} \right\} = \frac{dE\{F(u)\}}{du} \\ &= \begin{cases} -2\pi \; E\{X \; \sin(2\pi X u)\} & \text{if } N \text{ even} \\ -2\pi \; \dfrac{N-1}{N} \; E\{X \; \sin(2\pi X u)\} & \text{if } N \text{ odd} \end{cases} \end{aligned} \tag{B1}$$

The variance of $F'(u)$, for N even, is

$$\begin{aligned}\sigma^2_{F'}(u) &= E\left\{F'^2(u)\right\} - \phi^2_{F'}(u)\\ &= \frac{16\pi^2}{N^2}\sum_{n=1}^{N/2}\sum_{m=1}^{N/2} E\{X_n\ X_m \sin(2\pi X_n u)\sin(2\pi X_m u)\} - 4\pi^2\ E^2\{X\sin(2\pi X u)\}\\ &= \frac{4\pi^2}{N}\ E\{X^2\} - \frac{4\pi^2}{N}\ E\{X^2\cos(4\pi X u)\} - \frac{8\pi^2}{N}\ E^2\{X\sin(2\pi X u)\}\end{aligned} \tag{B2}$$

while for N odd, in analogous manner, it is obtained

$$\sigma^2_{F'}(u) = \frac{4\pi^2}{N^2}(N-1)\ E\{X^2\} - \frac{4\pi^2}{N^2}(N-1)\ E\{X^2\cos(4\pi X u)\} - \frac{8\pi^2}{N^2}(N-1)\ E^2\{X\sin(2\pi X u)\} \tag{B3}$$

APPENDIX C. PUNCTUAL CORRELATION COEFFICIENT OF THE ARRAY FACTOR AND ITS DERIVATIVE

The covariance of $F(u)$ and $F'(u)$ is

$$\begin{aligned}Cov[F(u),F'(u)] &= E\{F(u)F'(u)\} - \phi(u)\phi_{F'}(u)\\ &= \frac{1}{2}E\left\{\frac{dF^2(u)}{du}\right\} - \phi(u)\phi_{F'}(u)\\ &= \frac{1}{2}\frac{dE\{F^2(u)\}}{du} - \phi(u)\phi_{F'}(u)\\ &= \frac{1}{2}\frac{d[\phi^2(u)+\sigma^2(u)]}{du} - \phi(u)\phi_{F'}(u)\\ &= \frac{1}{2}\frac{d\sigma^2(u)}{du}\\ &= \frac{1}{2N}\frac{d\phi(2u)}{du} - \frac{2}{N}\phi(u)\phi_{F'}(u)\\ &= \frac{1}{2N}\phi_{F'}(2u) - \frac{2}{N}\phi(u)\phi_{F'}(u)\end{aligned} \tag{C1}$$

and the correlation coefficient is

$$\rho_{FF'}(u) = \frac{Cov[F(u),F'(u)]}{\sigma(u)\sigma_{F'}(u)} \tag{C2}$$

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