Fast and Stable Integration Method for the Aperture Admittance of an Open-Ended Coaxial Probe Terminated into Low-Loss Dielectrics

Licheng Zhou¹, Yang Ju², Peiyu Wang³, and Yongmao Pei³. *

Abstract—The utilization of an open-ended coaxial probe for characterization of dielectric properties or quantitative nondestructive detection of defects in materials firstly requires evaluating the aperture admittance. For the case that the probe is terminated into low-loss dielectrics backed by a conducting sheet, however, the admittance expression encounters poles in the vicinity of the path of integration, resulting in low convergence rate or even overflow in numerical quadrature. In this study, locations and properties of the singularities of the integral formulation for generally lossy, low-loss, and lossless dielectric slabs backed by a perfectly conducting sheet are investigated above all. Subsequently, making use of the contour integral technique, a fast and stable integration method is put forward to calculate the admittance integral formulation. Finally, numerical experiments are conducted to justify the validity and efficiency of the proposed integration method for low-loss dielectric cases by comparison with the traditional integration method as well as commercial FEM software.

1. INTRODUCTION

Open-ended coaxial probes have been extensively used for characterization of dielectric properties and nondestructive detection of defects in materials, owing to their capability of broadband measurement at microwave frequencies along with relatively high spatial resolution [1–5]. Information about the materials under test (MUT), such as complex dielectric properties, thickness of thin dielectric slabs, and disbond in layered media, can be extracted from the reflected microwaves inside the coax [3–8]. These features have made coaxial probes as useful tools for quantitative testing and nondestructive evaluation in microwave engineering, biomedicine, agriculture, geotechnology, etc. [9–15].

In the procedure of quantitative evaluation, one should at first determine the aperture admittance of an open-ended coaxial probe terminated into different MUTs. Up to now, several techniques have been developed for this purpose, such as analytical analysis, semi-analytical full-wave method, and numerical simulation [16–29]. For the analytical method, it is conventionally assumed a probe flange of an infinite area and only fundamental TEM mode propagation inside the coax [16–18]. Moreover, most of the models studied in these methods pertain to an aperture terminated into an infinite dielectric half-space [8, 30–33]. A more general analytical formulation for the cases of layered media backed by a conducting sheet or an infinite half-space was derived by Bakhtiari et al. [18, 34]. However, the aperture expression confronts poles on the real axis along the path of integration when coping with lossless MUTs [18, 34]. Even for low-loss dielectrics, some of the poles in complex domain are quite close to the positive real axis, leading to the possibility of low convergence rate or overflow error in the integration procedure by traditional integration methods (TIMs) (e.g., Gauss-Legendre method, the midpoint rule, etc.) [18, 38]. One promising approach to resolve this issue is to apply the contour integral technique
(CIT) to calculate the aperture admittance. To the authors’ knowledge, however, most of the studies reported in the literature were focused on the cases of generally lossy materials [17–19, 34–37]. In such situations, one can arrive at accurate results for the aperture admittance very efficiently by any traditional numerical integration routine.

In this paper, locations and properties of the singularities of the formulation for the aperture admittance terminated into a conducting sheet backed single-layered dielectric of general loss, low loss, and non-loss are investigated. Based on CIT, a fast and stable numerical method which carries out integration in complex domain is proposed for efficient calculation of the aperture admittance. Numerical simulations are conducted to demonstrate CIT is more stable and efficient when coping with low-loss dielectrics in comparison with TIM as well as commercial FEM software.

2. CIT FOR THE APERTURE ADMITTANCE

Figure 1 depicts the geometry of an open-end coaxial probe with a flange of an infinite diameter terminated into a dielectric slab backed by a perfect conductor. The filling dielectric inside the coax is assumed to have a relative complex permittivity of \( \varepsilon_{rc} \). The outer radius of the inner conductor and inner radius of the outer conductor for the coax are denoted by \( a \) and \( b \), respectively. A nonmagnetic dielectric slab with relative permittivity \( \varepsilon_{r1} = \varepsilon'_r - j\varepsilon''_r \) and thickness \( d_1 \) backed by a conducting sheet is treated as the MUT with infinitely transverse dimensions. Taking into account only the fundamental TEM mode, Bakhtiari et al. [18] derived the terminating admittance normalized with respect to the characteristic admittance of the coax as

\[
y_s = \frac{1 - R}{1 + R} = \int_{0}^{+\infty} g(\zeta) \cdot f(\zeta) \, d\zeta
\]

in which

\[
g(\zeta) = \frac{\varepsilon_{r1}}{\sqrt{\varepsilon_{rc}} \ln (b/a)} \cdot \frac{[J_0(k_0\zeta b) - J_0(k_0\zeta a)]^2}{\zeta}
\]

where \( R \) refers to the complex reflection coefficient, \( J_0 \) the first-kind Bessel function of order zero, and \( k_0 \) the wave number in free space. In view of a conducting sheet backed single-layered MUT, the function \( f(\zeta) \) in Eq. (1) is in terms of \( d_1, k_0, \) and \( \varepsilon_{r1}, \) and takes the form

\[
f(\zeta) = \frac{1}{\sqrt{\varepsilon_{r1} - \zeta^2}} \cdot \frac{1}{j \cdot \tan \left( k_0 d_1 \sqrt{\varepsilon_{r1} - \zeta^2} \right)}
\]

Conventionally, the dielectric slab in Fig. 1 is replaced with a reference liquid of general loss with well-known dielectric properties, such that the MUT serves as a known load for calibration of the coaxial probe. Under such circumstances, it is quite straightforward and easy to calculate the aperture admittance through Eq. (1) by TIMs, because there are no poles in the vicinity of the positive real axis.

![Open-ended Coaxial Line](image)

**Figure 1.** Open-ended coaxial probe terminated into a dielectric slab backed by a perfect conductor.
along the path of integration. As for lossless materials, the path of integration will encounter poles as indicated by Eq. (3). However, most studies were emphasized on MUTs of general loss, and low-loss or lossless dielectrics were not well considered [18, 37]. In this work, we will investigate the locations and properties of the isolated singularities of Eq. (1) for the cases of both \( \varepsilon''_{r1} > 0 \) and \( \varepsilon''_{r1} = 0 \), and then put forward a fast and stable integration method based on CIT.

In order to find out the poles, the integrant \( g(\zeta) \cdot f(\zeta) \) in Eq. (1) should be investigated. For \( g(\zeta) \) as expressed by Eq. (2), there exists a singularity \( (\zeta = 0) \) on the real axis along the path of integration. However, the limit of \( g(\zeta) \) as \( \zeta \) approaches zero is found to be zero, proving that \( \zeta = 0 \) is merely a removable singularity [39]. During the numerical integration procedure, the removable singularity \( \zeta = 0 \) can be ignored. As for \( f(\zeta) \) in Eq. (3), one can achieve the locations of the isolated singularities in complex domain by solving the equations \( \sqrt{\varepsilon_{r1} - \zeta^2} = 0 \) and \( \tan(k_0d_1\sqrt{\varepsilon_{r1} - \zeta^2}) = 0 \), resulting in

\[
\zeta_n^s = \sqrt{\varepsilon_{r1}' - [(n\pi)/(k_0d_1)]^2} - j \cdot \varepsilon_{r1}'' \tag{4}
\]

where \( n = 0, 1, 2, 3, \ldots \). In consideration of \( \varepsilon_{r1}'' > 0 \) (loss tangent \( \tan \delta = \varepsilon_{r1}''/\varepsilon_{r1}' > 0 \)), all of the singularities \( \zeta_n^s \) are in the second and fourth quadrants, and none of them falls on the path of integration for Eq. (1).

For the case of low-loss MUTs presented in Fig. 1, the imaginary part of relative permittivity \( \varepsilon''_{r1} \) and the loss tangent \( \tan \delta \) will approach zero. In such a situation, the singularities \( \zeta_n^s \) tend to be a real number \( \sqrt{\varepsilon_{r1}' - [(n\pi)/(k_0d_1)]^2} \). Considering the dielectric constant should be greater than 1.0, namely \( \varepsilon_{r1}' \geq 1.0 \), there must exist an integer \( N \) satisfying

\[
(N \cdot \pi)/(k_0d_1) \leq \sqrt{\varepsilon_{r1}' < [(N + 1) \cdot \pi]/(k_0d_1)} \tag{5}
\]

One can observe that there are \( (N + 1) \) singularities near the real axis along the path of integration, and they are close to these points right on the real axis:

\[
A_n = \sqrt{\varepsilon_{r1}' - [(n\pi)/(k_0d_1)]^2} \tag{6}
\]

where \( n = 0, 1, 2, \ldots, N \). Further inspection of the above equation shows that all of these points \( A_n \) on the real axis fall within the interval \([0, A_0]\). In other words, all of the poles are located nearby a segment of the integration path in Eq. (1). If the MUT in Fig. 1 is lossless \( (\varepsilon_{r1}'' = 0) \), the singularities \( \zeta_n^s \) subsequently can be directly derived from Eq. (4) as \( \zeta_n^s = \sqrt{\varepsilon_{r1}' - [(n\pi)/(k_0d_1)]^2} \), and they are located exactly on the real axis.

The straightforward approach to calculate the terminating admittance expressed by Eq. (1) is to carry out quadrature directly along the positive real axis by TIMs. Since the integral interval ranges from zero to infinite, the integral should be divided into two portions at an interior breakpoint at the first step. For TIM, the integral interval of Eq. (1) can be divided into two portions \( C_1 : [0, 2A_0] \) and \( C_2 : [2A_0, +\infty] \) as presented in Fig. 2(a). Subsequently, application of transformation of variables [38]

**Figure 2.** (a) Paths of integration by TIM. (b) Paths of integration by CIT.
on the aperture admittance $y_s$ renders an integral with an integral interval of $[0, 1]$, as expressed by

$$y_s = \int_0^{2A_0} Y_s(\zeta) d\zeta + \int_{2A_0}^{+\infty} Y_s(\zeta) d\zeta = \int_0^1 \left[ 2A_0 \cdot Y_s (2A_0 \cdot t) + 2A_0/t^2 \cdot Y_s (2A_0/t) \right] dt \tag{7}$$

in which $Y_s(\zeta) = g(\zeta) \cdot f(\zeta)$. To obtain the numerical results of Eq. (7), one can employ any TIM such as the midpoint rule to readily carry out the integration [38].

As mentioned above, however, there exist $(N + 1)$ isolated singularities that are close to the path $C_1 : [0, 2A_0]$ if the MUT is of low loss. The integral may be difficult to converge to a value with relatively high accuracy when utilizing a TIM to solve it by iteration. Moreover, TIMs are not available to do the integration for the cases of lossless dielectrics due to the existence of singularities on the integrating path. In this study, a fast and stable numerical method based on CIT is proposed to execute the integration for Eq. (1). As described above, the singularities of Eq. (1) fall either within the second and fourth quadrants for the cases of MUTs with loss, or on the real axis for those without loss. This means none of the singularities is located in the first quadrant, indicating that the integral of CIT is smooth and analytical in this region. In accordance with Cauchy’s theorem [39], the integral along the path $C_1 : [0, 2A_0]$ is equivalent to the one along any path lying within the first quadrant with a starting point at the origin and an ending point at $2A_0$ on the real axis. For the sake of flexibility and simplicity, the integration path in the first quadrant, as presented in Fig. 2(b), can be assumed a semi-ellipse with a center at $A_0$ and a semi-axis of $\alpha A_0$ parallel to the imaginary axis. The integration path in the first quadrant is denoted by $C_{12}$ as depicted in Fig. 2(b), and is expressed as $C_{12} : z = A_0 + A_0 \cdot (\cos \theta + j\alpha \sin \theta)$ with $\theta \in [\pi, 0]$. Since Bessel functions exhibit a divergence behavior as the imaginary part of the argument increases, it is better to choose $\alpha < 1$ to avoid numerical errors [40]. By using variable substitution, the integral along the path $C_1$ can be derived as

$$\int_{C_1} Y_s(\zeta) d\zeta = \int_{C_{12}} Y_s(z) dz = -\int_0^\pi Y_s[z(\theta)] d\theta = \int_0^1 \pi A_0 [\sin (\pi \cdot t) - j\alpha \cos (\pi \cdot t)] \cdot Y_s[z(\pi \cdot t)] dt \tag{8}$$

Because no singularity lies in the vicinity of the integration path of $C_2$, the integral along this path can be operated the same as presented in Eq. (7). Consequently, with the aid of CIT the integral of Eq. (1) finally takes the form

$$y_s = \int_{C_{12}} Y_s(z) dz + \int_{C_2} Y_s(\zeta) d\zeta \tag{9}$$

It should be noted that Eq. (9) is applicable to any cases of $\varepsilon''_{r1} \geq 0$, no matter the MUT is of general loss, low loss or non-loss.

3. RESULTS AND DISCUSSION

In this section, a series of numerical simulations will be presented to examine the efficiency and stability of the proposed CIT compared to TIM as well as commercial FEM software. In all the computation cases, the filling material inside the coax depicted in Fig. 1 is assumed to be Teflon with complex dielectric constant $\varepsilon_{rc} = 2.08 \cdot (1 - 0.0006j)$. Dimensions for the coax are chosen as $a = 0.52$ mm and $b = 1.2$ mm. Note that the feasible operating frequencies of the coax ranges from DC to 40 GHz.

To begin with, the stability and efficiency of CIT is compared to TIM as the loss tangent of the MUT is varied. In all the simulations, the midpoint rule is adopted as the TIM to calculate the aperture admittance expression of Eq. (7). In this case, the dielectric of the MUT in Fig. 1 is assumed to have a dielectric constant of $\varepsilon'_{r1} = 2.08$ with a thickness of 2.0 mm. The operating frequency is arbitrarily chosen as 10 GHz in the numerical experiments. For both methods, the number of sample points for calculating the integrals of Eq. (7) and Eq. (9) at the first iteration step is chosen as 100, and that of the next $i$th iteration step would be $100 \cdot 2^{i-1}$. A computational accuracy of 0.0001 is selected as well. Table 1 presents the effect of MUT’s loss tangent on the efficiency and stability of the two integration methods as the loss tangent is varied between $10^{-7}$ and $10^1$. One may observe from the calculation
results that, for generally lossy cases (\(\tan \delta > 10^{-2}\)), the iteration steps of CIT and TIM are nearly the same, indicating the two methods yield similar computational efficiency. It can also be seen that the iteration steps are less than 8 for both methods. Clearly, the calculations for MUTs of general loss are easy to converge by either CIT or TIM. As a decrease in the MUT’s loss tangent from \(10^{-2}\) to \(10^{-5}\) occurs, however, the iteration steps for TIM substantially increase. As for CIT, those values maintain a constant of 7 as the loss tangent changes from \(10^{-2}\) to \(10^{-7}\), proving that CIT is less sensitive to low-loss dielectrics and has higher numerical efficiency compared to TIM. Further inspection of the results in the last two rows of Table 1 displays that TIM renders incorrect results as \(\tan \delta\) descends to less than \(10^{-6}\). The problem arises from the existence of the poles in the vicinity of the real axis and resulting overflow error encountered in the computational procedure. As a result, one can infer that the proposed CIT in this study is more stable and efficient than TIM for calculating the coax aperture admittance involving low-loss MUTs.

### Table 1. Comparison of the two integration methods for generally lossy and lossless materials.

<table>
<thead>
<tr>
<th>(\tan \delta)</th>
<th>Integration by Eq. (7) (TIM)</th>
<th>Integration by Eq. (9) (CIT)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Iteration Steps</td>
<td>Aperture Admittance</td>
</tr>
<tr>
<td>(10^4)</td>
<td>7</td>
<td>(1.1279 + 0.0450j)</td>
</tr>
<tr>
<td>(10^9)</td>
<td>7</td>
<td>(0.1245 - 0.1161j)</td>
</tr>
<tr>
<td>(10^{-1})</td>
<td>7</td>
<td>(0.0145 - 0.1216j)</td>
</tr>
<tr>
<td>(10^{-2})</td>
<td>7</td>
<td>(0.0034 - 0.1220j)</td>
</tr>
<tr>
<td>(10^{-3})</td>
<td>8</td>
<td>(0.0023 - 0.1221j)</td>
</tr>
<tr>
<td>(10^{-4})</td>
<td>11</td>
<td>(0.0022 - 0.1221j)</td>
</tr>
<tr>
<td>(10^{-5})</td>
<td>14</td>
<td>(0.0021 - 0.1221j)</td>
</tr>
<tr>
<td>(10^{-6})</td>
<td>7</td>
<td>(0.0000 - 0.1221j)</td>
</tr>
<tr>
<td>(10^{-7})</td>
<td>7</td>
<td>(0.0000 - 0.1221j)</td>
</tr>
</tbody>
</table>

In order to further examine the properties of the proposed CIT in this study, the convergence rate of the two methods at different accuracies is investigated as well. In this case, the calculation parameters depicted in Fig. 1 are chosen as \(\varepsilon'_1 = 2.08\), \(\tan \delta = 10^{-4}\), \(d_1 = 2.0\) mm and an operating frequency of 10 GHz. It should be noted that the loss tangent of low-loss materials is typically in the order of \(10^{-4}\), such as those of quartz and Teflon [41]. The simulation results are shown in Table 2, in which the influence of truncation error from \(10^{-4}\) to \(10^{-7}\) on the convergence rate of the two methods is displayed. One can see that a decrease in truncation error from \(10^{-4}\) to \(10^{-7}\) leads to an increase in the iteration steps for both TIM and CIT. However, CIT requires less iteration steps and computational cost as compared to TIM for the same accuracy. It can be concluded that CIT inherently possesses higher computational efficiency in contrast to traditional integration routine.

### Table 2. Efficiency of the two integration methods with diverse computational accuracies.

<table>
<thead>
<tr>
<th>Truncation Error</th>
<th>Iteration Steps by Eq. (7) (TIM)</th>
<th>Iteration Steps by Eq. (9) (CIT)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10^{-4})</td>
<td>11</td>
<td>7</td>
</tr>
<tr>
<td>(10^{-5})</td>
<td>11</td>
<td>8</td>
</tr>
<tr>
<td>(10^{-6})</td>
<td>12</td>
<td>11</td>
</tr>
<tr>
<td>(10^{-7})</td>
<td>15</td>
<td>15</td>
</tr>
</tbody>
</table>

In the meanwhile, reflection coefficient of three distinct low-loss materials versus frequency in the range of 1 GHz to 40 GHz is calculated by TIM, CIT, as well as the commercial software HFSS. In the numerical simulations, air, Teflon, and quartz are chosen as the low-loss dielectrics with complex dielectric constant \(1.0 \cdot (1 - 0.0001j)\), \(2.08 \cdot (1 - 0.0006j)\), and \(3.82 \cdot (1 - 0.0004j)\), respectively. Each dielectric is assumed to have a similar thickness of 1.0 mm. The simulation results for the three low-loss
MUTs are shown in Fig. 3. Firstly, comparison between TIM and CIT indicates that they render the same calculated values in the whole frequency range for different dielectrics. This is attributed to the same theoretical formulation of Eq. (1) in conjunction with Eqs. (2) and (3) is employed in both TIM and CIT, although the former one executes integration along the real axis whereas the later one does so in complex domain. Calculations by HFSS are also performed to verify the numerical results of the theoretical formulation by TIM and CIT. One may observe from Fig. 3 that the reflection coefficient obtained by TIM and CIT is consistent with that by the FEM software. However, higher frequencies and greater dielectric constant values result in relatively greater difference between the theoretical and FEM results. Nevertheless, the simulations verify the validity and applicability of the proposed CIT for the purpose of broadband measurement. In order to investigate the efficiency, the amount of computational time for the three approaches is counted. In the computations, the reflection coefficient of each dielectric case is calculated in the 1 GHz to 40 GHz range with an interval of 0.5 GHz, namely 79 frequency samples are simulated for each low-loss MUT. The statistic results are displayed in Table 3. It is evident that for all dielectric cases, CIT is the least time-consuming method while FEM software requires maximum computational time. One may see that time cost by TIM is approximately three times larger than that by CIT. For FEM software, a dramatically ascending trend in time consumption is also observed as the dielectric constant raises. In contrast, TIM and CIT exhibit stable time cost for different dielectrics, owing to the fact that they both pertain to analytical methods. As a summary, all the simulation results demonstrate that the proposed CIT in this study has higher efficiency than both TIM and FEM software.

![Figure 3](image-url)

Figure 3. Reflection coefficient amplitude and phase of different dielectrics calculated by TIM, CIT, and HFSS at frequencies ranging from 1 GHz to 40 GHz: (a) air, (b) Teflon, (c) quartz.
Table 3. Computational time cost by TIM, CIT, and HFSS.

<table>
<thead>
<tr>
<th>Dielectric</th>
<th>Time Cost by TIM</th>
<th>Time Cost by CIT</th>
<th>Time Cost by HFSS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air</td>
<td>40 s</td>
<td>10 s</td>
<td>1554 s</td>
</tr>
<tr>
<td>Teflon</td>
<td>18 s</td>
<td>8 s</td>
<td>7057 s</td>
</tr>
<tr>
<td>Quartz</td>
<td>24 s</td>
<td>7 s</td>
<td>9568 s</td>
</tr>
</tbody>
</table>

4. CONCLUSION

The formulation for aperture admittance of an open-ended coaxial probe terminated into low-loss dielectrics backed by a perfectly conducting sheet encounters poles close to the path of integration, leading to low convergence rate or even overflow error in numerical calculations. In this paper, locations and properties of the singularities of the integral for the aperture admittance involving generally lossy, low-loss, and lossless dielectrics are studied. The contour integral method by which the integral is done in complex domain is proposed to calculate the aperture admittance. Numerical calculations demonstrate that the proposed integral method, as compared to traditional integration routines and commercial FEM software, is more stable and efficient for low-loss dielectric cases. The numerical method proposed here can be further used for utilizing an open-ended coaxial probe to fast quantitatively evaluate low-loss dielectrics.

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