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Symmetric Extension of Steering Vectors and Beamforming

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Abstract—Aiming at problems that interpolated array has large amount of computation and high sensitivity to transformation angle and interpolated step, a new array extension algorithm which is symmetric extension steering vector is proposed. In this paper, two properties of the conjugate of received data and the source covariance matrix being a real diagonal matrix are exploited to extend the dimensions of the covariance matrix. However, the essence of this extension method is the symmetric extension of the steering vector. The high complexity and degradation of the performance of interpolated array beamforming caused by the sensitivity of angle and interpolated step are improved. Numerical simulations confirm the validity of the proposed algorithm. Compared with existing algorithms, the proposed algorithm is not affected by the angle range of transformation and interpolated step. Besides, the complexity of array extension using this proposed algorithm is much lower than the interpolated transformation method.

1. INTRODUCTION

In array signal processing, the more elements, the better resolution when the element spacing is constant. But in real life, the larger array, the higher production cost and the more difficult to maintain. Therefore, many scholars have considered the method of extending the array aperture by virtual elements [1]. In 1990s, the interpolation technology was proposed by Friedlander firstly [2]. The freedom of the array is increased, and the irregular array is transformed into a uniform linear array (ULA). Since then, interpolated array has been used in beamforming [3–5]. However, it is found that there are two main problems of the interpolated array beamforming: firstly, the interference cannot be suppressed when it falls outside the transform area; besides, when the transform area is too large, there are zero drift and mainlobe offset. In other words, the 'angle-sensitive' exists [6]. Secondly, when the interpolated step is greater than 1°, the interpolated array cannot achieve beamforming accurately even at a suitable angle range, that is to say, the 'interpolated step-sensitive' exists [6]. Much improved algorithms have been proposed to solve 'angle-sensitivity' [7–10], but most of them are done by sacrificing computational complexity. However, no good idea to solve the second problem continues. Affected by the above problems, the computational complexity of interpolated array beamforming is pretty high, which is not suitable for receiving and processing the real-time signal.

In recent years, with the application of virtual arrays in practical systems, some new extended methods have been proposed to solve the complexity problem. In 2004, the conjugate ESPRIT algorithm based on signals with non-circular symmetry was proposed by Tayem. Although array extended is implemented, it can only be applied to specific incident signals [11]. In 2006, an extension method of constructing a Toeplitz matrix using diagonal elements of cross-correlation matrix of L-shaped array was proposed by Kikuchi [12]. In 2007, an extension algorithm segmenting the cross-correlation matrix of an L-shaped array and constructing a new matrix by using the rotation invariance of the array manifold was proposed by Gu [13]. In 2015, an array aperture extended algorithm for two-dimensional

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DOA estimation with an L-shaped array was proposed by Nie [14]. The rotation in-variance and signal covariance matrix is a real diagonal matrix used to extend the dimension of the covariance matrix. In 2016, a method of constructing a new matrix using the conjugate symmetry of the array manifold was proposed by Dong et al., and the correct ratio of the two-dimensional estimated angle pairing was improved [15]. In 2018, in the structure of a co-prime array, extended two sub-arrays by using properties of unitary transformation and rotation invariance were proposed by Li [16].

Although the above algorithm has low complexity, it is mainly applied to a specific two-dimensional array structure. The advantage of a two-dimensional array is that the effect of noise can be eliminated, but one-dimensional ULA cannot. Therefore, for a one-dimensional ULA, the method using the received data conjugate information and source covariance matrix being a real diagonal matrix is proposed. This method not only realizes large-angle extended array beamforming, but also overcomes the 'interpolated step-sensitive', and the complexity is much lower than the interpolated array beamforming.

2. ARRAY EXTENSION BASED ON STEERING VECTOR CONJUGATE SYMMETRY

2.1. Signal Model

A general real ULA with M isotropic elements and d spacing is shown in the upper part of Fig. 1. Mutual coupling is ignored. In order to realize better extension, this paper makes a minor adjustment to the real ULA. The new array geometry is shown in the lower part of Fig. 1.

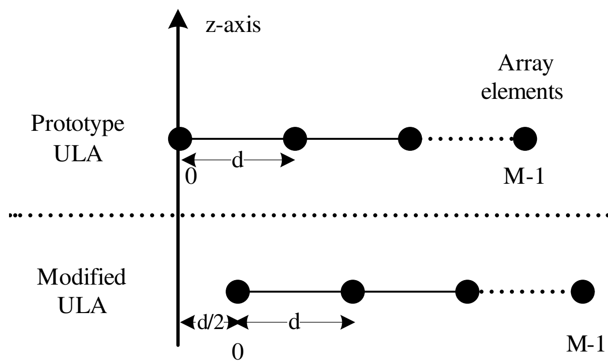

Figure 1. The geometries of prototype ULA and modified ULA.

If N radiating narrowband sources are observed by this array and the sources are uncorrelated with each other, the received signal vector can then be written as:

$$x(t) = As(t) + n(t) \tag{1}$$

where $s(t)$ denotes a signals vector; $n(t)$ denotes a white Gaussian noise with zero-mean and variance σ_n^2 and is statistically independent of sources; $A = [a(\theta_1), a(\theta_2), \ldots, a(\theta_N)]$ is defined as $M \times N$ array manifold; $a(\theta)$ represents the modified steering vector, and it is $a(\theta) = \left[e^{-jk(d/2)\sin\theta}, e^{-jk(3d/2)\sin\theta}, \ldots, e^{-jk((2M-1)d/2)d\sin\theta}\right]^T$, $k = \frac{2\pi}{\lambda}$.

2.2. The Method of Array Extension

Common array extension, such as array interpolation [2] and cumulant [17], requires lot of calculation and has high complexity. In this paper, a low-complexity method is proposed by using the steering vector conjugate symmetric extension and incoherent sources.

According to the modified ULA geometry, the covariance matrix is given by

$$\begin{aligned} R &= E\left[x(t)\,x(t)^H\right] \\ &= AE\left\{s(t)\,s(t)^H\right\}A^H + E\left\{n(t)\,n(t)^H\right\} \\ &= AR_sA^H + \sigma_n^2 I \end{aligned} \tag{2}$$

where $E[\cdot]$ denotes expectations, and $(\cdot)^H$ denotes conjugation transpose. Source covariance matrix is $R_s = \text{diag}\{p_1, p_2, \ldots, p_N\}$, and p_i is the i-th signal power, $i = 1, 2, \ldots N$. I is $M \times M$ identity matrix.

Then its conjugate matrix can be expressed as:

$$R^* = \left(AR_sA^H\right)^* + \left(\sigma_n^2 I\right)^* = A^* R_s^* A^T + \sigma_n^2 I \tag{3}$$

where $(\cdot)^*$ denotes conjugation, and $(\cdot)^T$ denotes transpose. When the incident signal is uncorrelated, R_s is a real diagonal matrix [7], i.e., $R_s^* = R_s$, and the Equation (3) can be reduced to:

$$R^* = A^* R_s A^T + \sigma_n^2 I \tag{4}$$

Combining Equations (2) and (4), the new data matrix can be written as:

$$Y^H = \begin{bmatrix} R^* J_M & R \end{bmatrix} \tag{5}$$

where $J_M = \begin{bmatrix} 0 & \ldots & 0 & 1 \\ 0 & \ldots & 1 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ 1 & \ldots & 0 & 0 \end{bmatrix}$ is an $M \times M$ reverse identity matrix.

Bringing Equations (2) and (4) into Equation (5), Equation (5) can be reduced to:

$$\begin{aligned} Y^H &= \begin{bmatrix} A^* R_s A^T J_M & AR_sA^H \end{bmatrix} + \begin{bmatrix} \sigma_n^2 I J_M & \sigma_n^2 I \end{bmatrix} \\ &= \begin{bmatrix} A^* R_s A^T J_M & AR_sA^H \end{bmatrix} + \sigma_n^2 \begin{bmatrix} I J_M & I \end{bmatrix} \end{aligned} \tag{6}$$

Then new covariance matrix can be expressed as:

$$\begin{aligned} \bar{R} = \mathrm{E}\left[YY^H\right] &= \begin{bmatrix} J_M A^* R_s A^T \\ AR_sA^H \end{bmatrix} \begin{bmatrix} A^* R_s A^T J_M & AR_sA^H \end{bmatrix} + \sigma_n^4 \begin{bmatrix} J_M I \\ I \end{bmatrix} \begin{bmatrix} I J_M & I \end{bmatrix} \\ &= \begin{bmatrix} J_M A^* R_s A^T A^* R_s A^T J_M & J_M A^* R_s A^T A R_s A^H \\ AR_sA^H A^* R_s A^T J_M & AR_sA^H AR_sA^H \end{bmatrix} + \sigma_n^4 \begin{bmatrix} J_M I \\ I \end{bmatrix} \begin{bmatrix} I J_M & I \end{bmatrix} \end{aligned} \tag{7}$$

Let $\bar{A} = \begin{bmatrix} J_M A^* \\ A \end{bmatrix}$ be the extended steering vector, then Eq. (7) is the data covariance matrix after extension, which can be expressed as:

$$\bar{R} = \bar{A} \begin{bmatrix} R_s A^T A^* R_s & R_s A^T A R_s \\ R_s A^H A^* R_s & R_s A^H A R_s \end{bmatrix} \bar{A}^H + \sigma_n^4 \bar{N} \bar{N}^H \tag{8}$$

where $\bar{N} = \begin{bmatrix} J_M I \\ I \end{bmatrix}$ denotes the extended noise covariance.

Then the extended array received signal is expressed as:

$$\bar{x}(t) = \begin{bmatrix} J_M A^* \\ A \end{bmatrix} s(t) + \begin{bmatrix} J_M n(t) \\ n(t) \end{bmatrix} \tag{9}$$

From the above analysis, the real array covariance matrix is extended to $\bar{R}$, and the array manifold A is extended to $\bar{A}$.

2.3. The Complexity Analysis of Array Extension

It can be seen from the above analysis that the extended method proposed in this paper does not need to calculate the higher order matrix or solve transformation relationship by setting the interpolated points in a specific angle region, so it has lower complexity.

Take the complex multiplication times of the matrix as an example. The detailed analysis process of proposed algorithm is summarized as Table 1.

Where, M denotes the real elements, $\bar{M}$ denotes the extended elements. L are snapshots.

However, in order to compare the complexity, complex multiplication times of conventional interpolated method are also given [2], where b is the interpolated point. If transformation angle is $[-30^\circ, 0^\circ]$, and interpolated step is 0.1°, then $b = 301$. The detailed analysis process is summarized in Table 2.

Table 1. The complex multiplication times of proposed algorithm.

Proposed Algorithm main steps	Complex multiplication times	Total complex multiplication times
Real array covariance matrix R	$o\left(L^2M^2\right)$	$o\left(L^2M^2+M^2+M^3+2M\bar{M}+LM\bar{M}^2\right)$
new data matrix Y^H	$o\left(M^2\right)+o\left(M^3\right)$	
$Y=\left[Y^H\right]^H$	$o\left(M\bar{M}\right)+o\left(M\bar{M}\right)$	
Extension covariance matrix $\bar{R}$	$o\left(LM\bar{M}^2\right)$	

Table 2. The complex multiplication times of conventional interpolated [2].

Proposed Algorithm main steps	Complex multiplication times	Total complex multiplication times
Real array covariance matrix R	$o\left(L^2M^2\right)$	$o\begin{pmatrix} L^2M^2+4Mb+\bar{M}b+Mb^2+b^3+ \\ \bar{M}b^2+\bar{M}M^2+2\bar{M}M+M\bar{M}^2 \end{pmatrix}$
Transformation matrix B	$o\begin{pmatrix} 2Mb+Mb^2+b^3 \\ +\bar{M}b^2+\bar{M}b+2Mb \end{pmatrix}$	
Extension covariance matrix $\bar{R}$	$o\left(\bar{M}M^2+2\bar{M}M+M\bar{M}^2\right)$	

3. BEAMFORMING BASED ON STEERING VECTOR CONJUGATE SYMMETRY ARRAY

3.1. Beamforming Based on Steering Vector Conjugate Symmetry Array

The main beamforming algorithms include Least Mean Square (LMS) [18], Sample Matrix Inversion (SMI) [19], and Minimum Variance Distortionless Response (MVDR) [18, 20]. In this paper, the proposed array extension technique is combined with MVDR.

It can be seen from Equation (8) that $\sigma_n^4\bar{N}\bar{N}^H\neq\sigma_n^2 I$. That is to say, the Gaussian white noise is contaminated after extension. It is necessary to whiten the color noise.

Let $\bar{R}_n=\sigma_n^4\bar{N}\bar{N}^H$ and decompose its eigenvalues into:

$$\bar{R}_n=U_n\Lambda_n U_n^H \tag{10}$$

where $\Lambda_n=\operatorname{diag}\left(\lambda_1,\lambda_2,\ldots,\lambda_{\bar{M}}\right)$ is a diagonal matrix composed of $\bar{M}$ eigenvalues, and $U_n=[u_1,u_2,\ldots,u_{\bar{M}}]$ is the corresponding eigenvectors.

Assuming that the whitening matrix is Z, whitening $\bar{R}_n$ can be expressed as:

$$\tilde{\bar{R}}_n=Z\bar{R}_n Z^H=\sigma_n^2 I \tag{11}$$

Bring Eq. (10) to Eq. (11), Eq. (12) can be simplified as:

$$Z\left(U_n\Lambda_n U_n^H\right)Z^H=\sigma_n^2 I \tag{12}$$

Solving the above formula and getting the whitening matrix is:

$$Z=\left(\sigma_n^2\right)^{1/2}\Lambda_n^{-1/2}U_n^H \tag{13}$$

So the whitened covariance matrix can be described as:

$$\tilde{\bar{R}}=\bar{A}\begin{bmatrix} R_sA^TA^*R_s & R_sA^TAR_s \\ R_sA^HA^*R_s & R_sA^HAR_s \end{bmatrix}\bar{A}^H+Z\bar{R}_nZ^H \tag{14}$$

Combined with the MVDR algorithm, the optimal weight can be obtained by:

$$\tilde{\bar{w}}_{opt} = \frac{\tilde{\bar{R}}^{-1} \bar{a}(\theta_0)}{\bar{a}^H(\theta_0)\tilde{\bar{R}}^{-1}\bar{a}(\theta_0)} \tag{15}$$

where $\bar{a}(\theta_0)$ is an extended steering vector of the desired direction.

3.2. Summary Algorithm

According to the proposed array extension method and referring to the beamforming algorithm, the algorithm is summarized as follows:

1) Design the appropriate array geometry as shown in the lower part of Fig. 1.
2) The covariance matrix R of real array is obtained according to Equation (2).
3) Construct a new data matrix Y^H according to Equations (3), (4), (5), (6).
4) The extended covariance matrix $\bar{R}$ is obtained according to Equations (7) and (8).
5) Calculate $\tilde{\bar{R}}$ in the white noise background using Equations (10), (11), (13), (14).
6) Calculate the optimal weight $\tilde{\bar{w}}_{opt}$ according to Equation (15).

4. SIMULATION AND RESULT ANALYSIS

In this section, the beamforming performance and complexity of the proposed algorithm are compared with the conventional interpolation virtual array (Conventional IVA) [2] and multi-region interpolation virtual array (Multi Region IVA) [7] by setting five simulation experiments. In this experiment, the real array uses 8-element spacing $\lambda/2$ ULA, while the other algorithms in this paper are extended form the 4-element to 8-element ULA. Signal to noise ratio, interference to noise ratio and signal-to-interference-and-noise ratio are abbreviated as SNR, INR and SINR, respectively.

4.1. The Situation of No Burst Interference

4.1.1. Small Transformation Angle

Simulation 1: analysis beamforming performance of extended arrays. The desired signal direction is set to $\theta_0 = 0°$, and add one independent interference from $-20°$. SNR = 0 dB, INR = 40 dB. $[-90°, 90°]$ is divided evenly into 6 sections as the transformation area of Multi-Region IVA [7]. The conventional

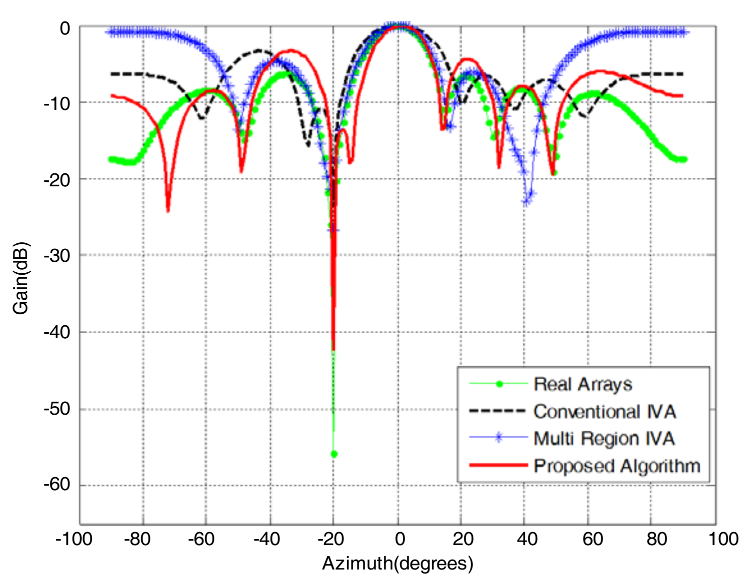

Figure 2. Analysis beamforming under small transformation angle.

IVA [2] transformation areas are defined as [−30°, 0°], interpolated step selected as 0.1°, and snapshots are 200.

Figure 2 shows that beamforming is accurately implemented by conventional IVA [2] and the proposed method. But the proposed algorithm has a deeper depression in −20°, which can suppress interference better.

Simulation 2: analysis output SINR for different input SNR. Set input SNR range from 0 dB to 20 dB. Other conditions are such as simulation 1.

Figure 3 shows that the conventional IVA [2] and the proposed algorithm have almost the same performance. But the combined error introduced by the region division has led to the worst performance of multi-region IVA [7].

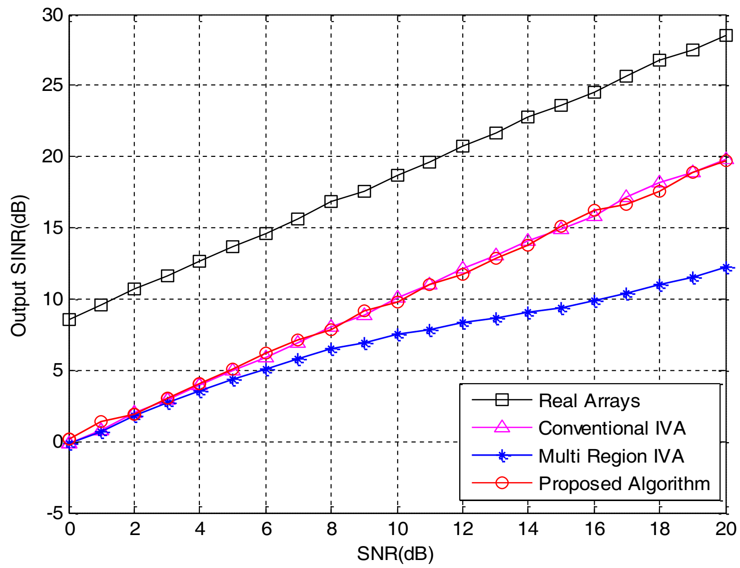

Figure 3. Analysis output SINR under small transformation angle.

4.1.2. Large Transformation Angle

Simulation 3: analysis beamforming performance of extended arrays. The desired signal direction is set to $\theta_0 = 0°$, and add one independent interference from −20°. SNR = 0 dB, INR = 40 dB. [−90°, 90°] is divided evenly into 6 sections as the transformation area of Multi-Region IVA [7]. The conventional IVA [2] transformation area is defined as [−90°, 90°], step size selected as 0.1°, and snapshots are 200.

As shown in Fig. 4, in the case of large transformation angle, the influence of transformation error cannot be ignored. The interference coming from −20° cannot be suppressed when conventional IVA [2] is adopted. The Multi-Region IVA [7] can form a shallow null (about −26 dB). In the same situation, a deep null (about −42 dB) in the −20° is formed by the proposed algorithm. That is to say, this beamforming performance is the better than others.

Simulation 4: analysis output SINR for different input SNRs. Set input SNR range from 0 dB to 20 dB. Other conditions are such as simulation 3.

It can be known from Fig. 5 that the output SINR of conventional IVA [2] is the worst, since interference cannot be suppressed. The proposed algorithm output SINR is the same as Multi-Region IVA [7] when the SNR is small. Yet, with the increase of the input SNR, the proposed algorithm has obvious advantage.

4.2. The Situation of Burst Interference

Simulation 5: analysis beamforming performance of extended arrays. On the basis of simulation 1, add another burst interference incident from 40°, INR = 40 dB. [−90°, 90°] is divided evenly into 6 sections

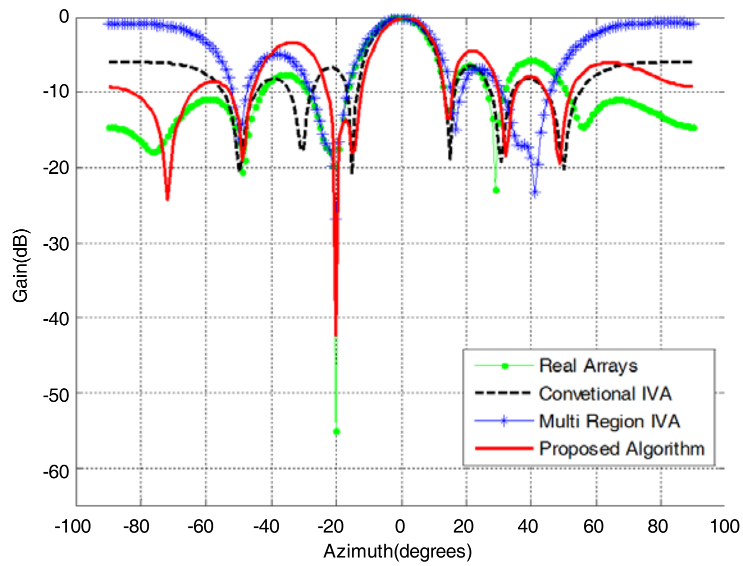

Figure 4. Analysis beamforming under large transformation angle.

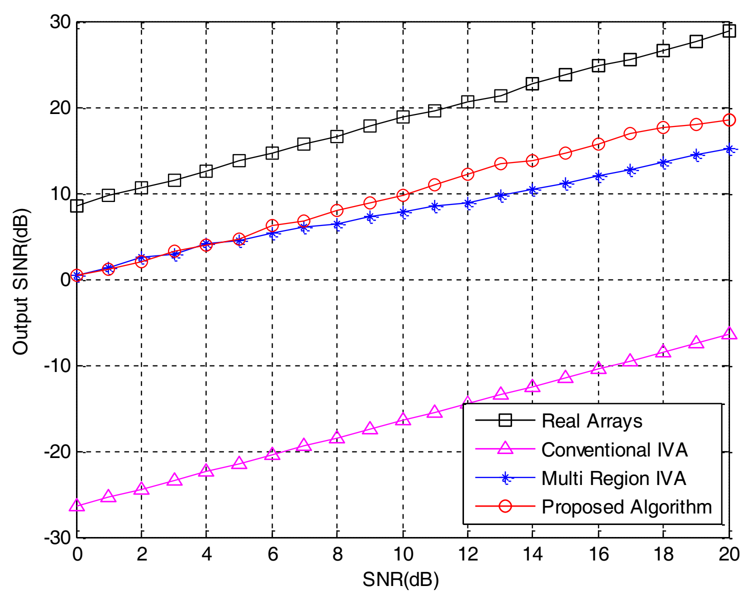

Figure 5. Analysis output SINR under large transformation angle.

as the transformation area of Multi-Region IVA [7]. The conventional IVA [2] transformation area is selected as $[-30°, 0°]$.

Figure 6 shows that burst interference cannot be suppressed by conventional IVA [2]. Multi-Region IVA [7] and the proposed algorithm can suppress any interference. However, the proposed algorithm has better suppression performance.

Simulation 6: analysis output SINR for different input SNR. Set input SNR range from 0 dB to 20 dB. Other conditions are such as simulation 5.

Figure 7 shows that burst interference cannot be suppressed by conventional IVA [2], so the output SINR is the worst. The output SINR of Multi-Region IVA [7] and the proposed algorithm are high, and the output SINR of the proposed algorithm is higher when the SNR is bigger.

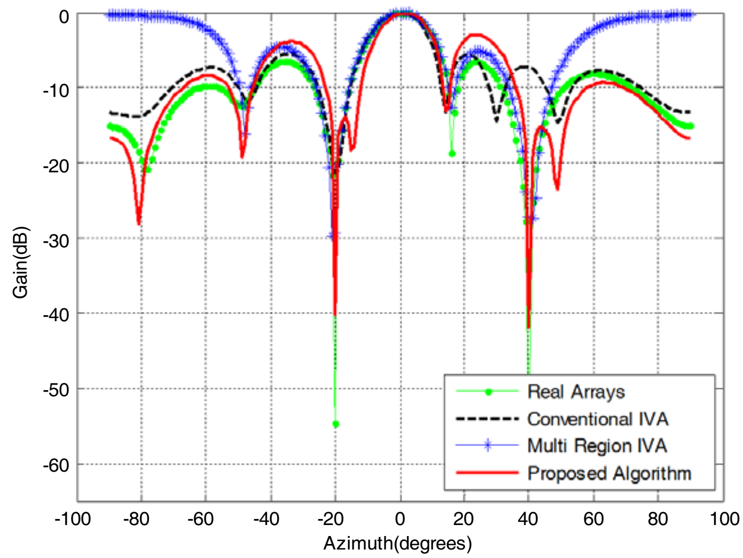

Figure 6. Analysis beamforming under burst interference.

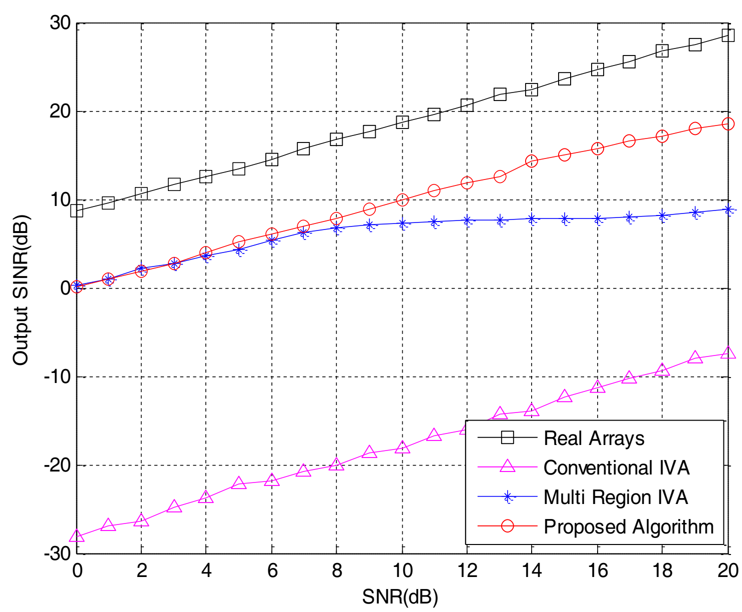

Figure 7. Analysis output SINR under burst interference.

4.3. Analysis of Beamforming Performance of Extended Arrays under Large Interpolated Step

Simulation 7: On the basis of simulation 1. Interpolated step is selected as 1°. [−90°, 90°] is divided evenly into 6 sections as the transformation area of Multi-Region IVA [7], and snapshots are 200.

As shown in Fig. 8, conventional IVA [2] and Multi-Region IVA [7] have poor suppression performance when the interpolated step is 1°. However, the proposed algorithms can still achieve beamforming well because it is not affected by interpolated step.

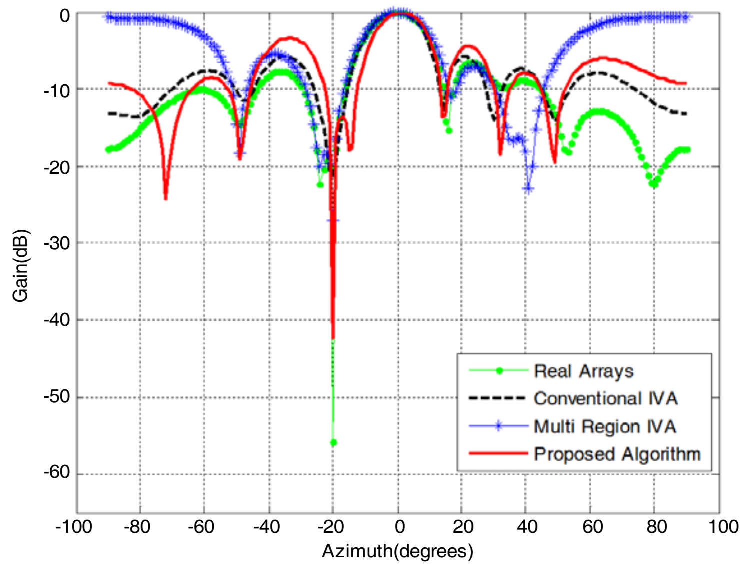

Figure 8. Analysis beamforming with large interpolated step.

4.4. Analysis of Complexity in Array Extension

Simulation 8: analysis of the number of multiplications under different elements. On the basis of simulation 1, set the elements of real arrays from 4 to 16.

Simulation 9: Analysis of the number of multiplications under different snapshots. On the basis of simulation 1, set the snapshot from 200 to 1400.

It can be seen from Fig. 9 and Fig. 10 that as the numbers of snapshots and elements increase, the number of multiplications of both algorithms increases, but the proposed algorithm is much lower than conventional IVA [2].

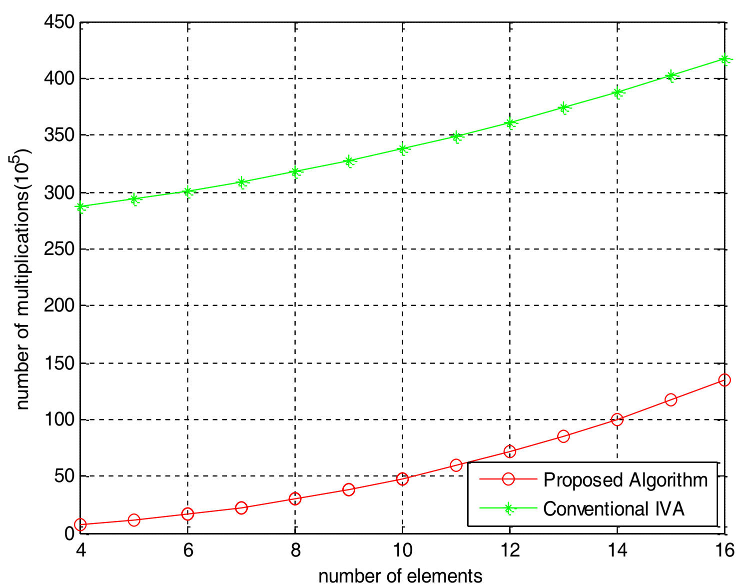

Figure 9. Analysis the number of multiplications with different elements.

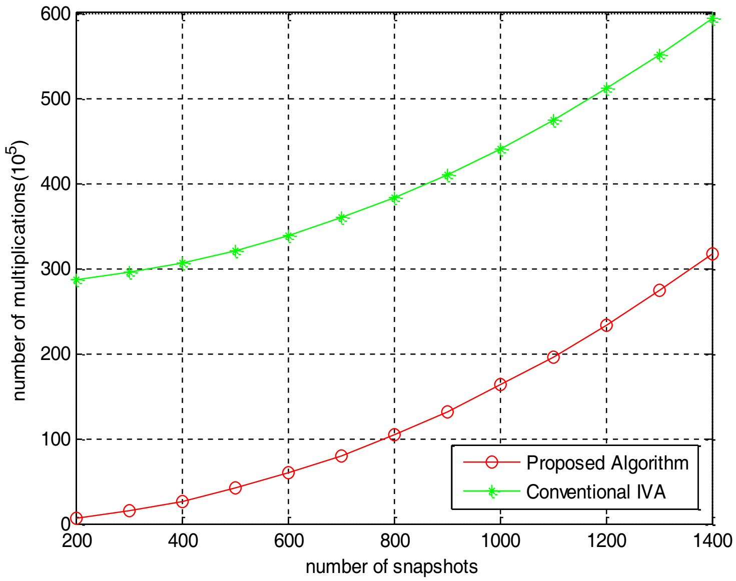

Figure 10. Analysis the number of multiplications with different snapshots.

4.5. Analysis of the Complexity of Array Extension

Simulation 10: On the basis of simulation 1, selectting the computer whose CPU is Inter Celeron G1840 and main frequency 2.8 GHz, operating system is 64-bit, and then combined with MATLAB to simulate the running time of algorithm. The results are shown in Table 3.

Table 3. The complexity of array extension.

Algorithm	Conventional IVA	Proposed Algorithm
Angle range	30°	180°
Complexity	0.122705 s	0.009868 s

It can be seen from Table 3 that the proposed algorithm not only improves the angle range, but also has much lower complexity than the conventional IVA [2].

5. CONCLUSION

Although the interpolated transformation can increase the freedom of array, it has 'angle-sensitive' and 'interpolated step-sensitive'. Theoretically, the more interpolated points, the better performance of extended arrays, but the complexity will be greater. In this paper, the new extension algorithm is proposed by using the conjugate information of received data, and the source covariance matrix is a real diagonal matrix. The high complexity of array extension and low beamforming performance caused by 'angle sensitivity' and 'interpolated step sensitivity' are solved.

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