Progress In Electromagnetics Research Letters, Vol. 24, 69–76, 2011

EIGENVALUE ANALYSIS OF SPHERICAL RESONANT CAVITY USING RADIAL BASIS FUNCTIONS

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Abstract—This paper applies a meshless method based on radial basis function (RBF) collocation to solve three-dimensional scalar Helmholtz equation in rectangular coordinates and analyze the eigenvalues of spherical resonant cavity. The boundary conditions of spherical cavity are deduced. The RBF interpolation method and the collocation procedure are applied to the Helmholtz and boundary condition equations, and their discretization matrix formulations are obtained. The eigenvalues of spherical resonant cavity with natural conformal node distribution are computed by the proposed method. Their results are agreement with the analytic solution.

1. INTRODUCTION

In recent years, meshless methods (MLM) based on a set of independent points have been applied to the area of computational electromagnetics [1–4]. MLM based on radial basis functions (RBFs) is a space dimension independent method and a powerful interpolation technique. It was first proposed by Kansa [5] in 1990 to solve partial difference equations. There are some works based on radial basis functions analyzing the waveguide or resonant problems. For example, elliptical waveguides are computed by meshless collocation method with Wendland RBFs [6]. Zhao et al. [7] proposed a novel conformal meshless method based on RBF coupled with coordinate transformation technique to analyze arbitrary waveguide problems. Lai et al. [8] used meshless RBF method to solving Helmholtz equation and computed various waveguide problems. Waveguide eigenvalue

Received 9 April 2011, Accepted 19 May 2011, Scheduled 3 June 2011

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problem in cylindrical system is also solved by using RBFs [9]. In [8], radial point interpolation method is applied to the Maxwell's curl equations to analyze the eigenvalues of resonant structures. However, the spherical resonant cavity problems have never been computed using RBFs.

The focus of this paper is on the eigenvalues analysis of spherical resonant cavity using MLM based on RBF collocation. The boundary conditions of transverse electric (TE) and transverse magnetic (TM) modes in spherical cavity are deduced in Section 2. The RBF interpolation method and the collocation procedure are also presented in this section. In Section 3, the eigenvalues of spherical resonant cavity with natural conformal node distribution are computed. Conclusion is given in Section 4.

2. FORMULATION

For a spherical cavity enclosed by a perfect electrically conducting surface at $\mathbf{r} = R_0$ (see Figure 1), in the spherical coordinates, its Borgnis' functions $U(\mathbf{r})$ or $V(\mathbf{r})$ do not satisfy scalar Helmholtz equation [10]. But with the following transform,

$$U(\mathbf{r}) = \mathbf{r}F(\mathbf{r}) \quad \text{or} \quad V(\mathbf{r}) = \mathbf{r}F(\mathbf{r}) \tag{1}$$

$F(\mathbf{r})$ satisfies the scalar Helmholtz equation [10]

$$\nabla^2 F(\mathbf{r}) + k^2 F(\mathbf{r}) = 0 \tag{2}$$

The boundary condition of TE modes in the spherical coordinates is $V|_{\mathbf{r}=R_0} = 0$ [11]. From (1), we get

$$\mathbf{r}F|_{\mathbf{r}=R_0} = 0, \quad \text{i.e.,} \quad F|_{\mathbf{r}=R_0} = 0, \tag{3}$$

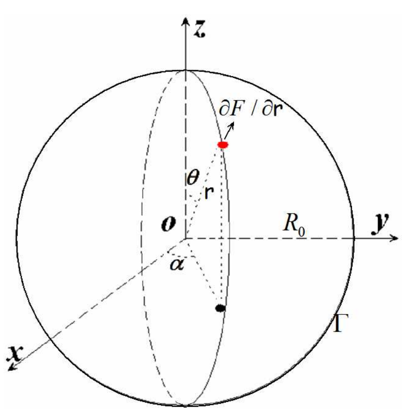


Figure 1. Structure of the spherical cavity.

For TM modes, the boundary condition is $\partial U/\partial \mathbf{r}|_{\mathbf{r}=R_0} = 0$. From (1), we get

$$\left.\frac{\partial(\mathbf{r}F)}{\partial \mathbf{r}}\right|_{\mathbf{r}=R_0} = \left.\left[F + \mathbf{r}\frac{\partial F}{\partial \mathbf{r}}\right]\right|_{\mathbf{r}=R_0} = 0. \tag{4}$$

In rectangular coordinates the three-dimension scalar Helmholtz Equation (2) can be expressed as

$$\frac{\partial^2 F(\mathbf{r})}{\partial x^2} + \frac{\partial^2 F(\mathbf{r})}{\partial y^2} + \frac{\partial^2 F(\mathbf{r})}{\partial z^2} + k^2 F(\mathbf{r}) = 0, \quad \mathbf{r} \in \Omega \tag{5}$$

The unknown function $F(\mathbf{r})$ in the computation domain Ω can be interpolated approximately by a series of RBF:

$$F(\mathbf{r}) \approx F^h(\mathbf{r}) = \sum_{I=1}^{N} \phi_I(\mathbf{r}) a_I, \quad \mathbf{r} \in \Omega \tag{6}$$

where $\phi_I(\mathbf{r}) = \phi(\|\mathbf{r} - \mathbf{r}_I\|)$, is the radial basis function centered at a set of independent points $\mathbf{r}_1, \ldots, \mathbf{r}_I, \ldots, \mathbf{r}_N \in \Omega$ (also called center nodes), a_I are unknown coefficients to be computed and $\|\mathbf{r} - \mathbf{r}_I\| = d_I$ represents the Euclidean distance between test node $\mathbf{r}$ and collocation node $\mathbf{r}_I$.

Substituting (6) into (5), we get

$$\sum_{I=1}^{N} a_I \left[\partial_{xx}\phi_I(\mathbf{r}) + \partial_{yy}\phi_I(\mathbf{r}) + \partial_{zz}\phi_I(\mathbf{r})\right] = -k^2 \sum_{I=1}^{N} a_I \phi_I(\mathbf{r}), \; \mathbf{r} \in \Omega \tag{7}$$

where

$$\partial_{pp}\phi_I(\mathbf{r}) = \frac{\partial^2 \phi_I(\mathbf{r})}{\partial d_I^2}\left(\frac{\partial d_I}{\partial p}\right)^2 + \frac{\partial \phi_I(\mathbf{r})}{\partial d_I}\frac{\partial d_I^2}{\partial p^2} \tag{8}$$

in which p represents x, y or z in rectangular coordinates.

Substituting (6) into (3), we get the RBF interpolation formulation of the boundary condition of TE modes

$$\sum_{I=1}^{N} a_I \phi_I(\mathbf{r})|_{\mathbf{r}=R_0} = 0, \tag{9}$$

For TM modes, substituting (6) into (4) and projecting the normal derivative $\partial F/\partial \mathbf{r}$ on the rectangular coordinates (see Figure 1), we have

$$\sum_{I=1}^{N} a_I \left[\phi_I(\mathbf{r}) + R_0 \left(\partial_x \phi_I(\mathbf{r}) \cos\alpha + \partial_y \phi_I(\mathbf{r}) \sin\alpha\right) \sin\theta \right. \\ \left. + R_0 \partial_z \phi_I(\mathbf{r}) \cos\theta\right]_{\mathbf{r}=R_0} = 0, \tag{10}$$

where

$$\partial_p \phi_I(\mathbf{r}) = \frac{\partial \phi_I(\mathbf{r})}{\partial d_I} \frac{\partial d_I}{\partial p} \tag{11}$$

Then, the Equations (7), (9), and (10) at a set of collocation nodes $\mathbf{r}_1, \ldots, \mathbf{r}_J, \ldots, \mathbf{r}_M \in \Omega$ ($M \geq N$) are considered. We get the discretization formulations of Helmholtz equation of spherical resonant cavity by MLM based on RBF collocation. They could be written as the matrix formulation:

$$\mathbf{Aa} = -k^2 \mathbf{Ba} \tag{12}$$

where $\mathbf{a} = [a_1, a_2, \ldots, a_N]^T$, for TE case, the individual elements of matrix $\mathbf{A}$ can be written as

$$A_{IJ} = \begin{cases} \partial_{xx}\phi_I(\mathbf{r}_J) + \partial_{yy}\phi_I(\mathbf{r}_J) + \partial_{zz}\phi_I(\mathbf{r}_J), & \mathbf{r}_J \text{ in } \Omega \\ \phi_I(\mathbf{r}_J), & \mathbf{r}_J \text{ on } \Gamma \end{cases}, \tag{13}$$

and for TM case, the individual elements of matrix $\mathbf{A}$ are

$$A_{IJ} = \begin{cases} \partial_{xx}\phi_I(\mathbf{r}_J) + \partial_{yy}\phi_I(\mathbf{r}_J) + \partial_{zz}\phi_I(\mathbf{r}_J), & \mathbf{r}_J \text{ in } \Omega \\ \phi_I(\mathbf{r}_J) + R_0 \partial_z \phi_I(\mathbf{r}_J) \cos\theta_J + R_0 \sin\theta_J & \\ \times \left[\partial_x \phi_I(\mathbf{r}_J) \cos\alpha_J + R_0 \partial_y \phi_I(\mathbf{r}_J) \sin\alpha_J\right], & \mathbf{r}_J \text{ on } \Gamma \end{cases} \tag{14}$$

The elements of matrix $\mathbf{B}$ for both TM and TE cases are

$$B_{IJ} = \begin{cases} \phi_I(\mathbf{r}_J), & \mathbf{r}_J \text{ in } \Omega \\ 0, & \mathbf{r}_J \text{ on } \Gamma \end{cases} \tag{15}$$

Equation (12) is the generalized eigenvalue equation. In this study, suppose collocation nodes located at the center nodes, i.e., $M = N$. The eigenvalue k_j and eigenvector $\mathbf{a}_j$ could be computed from (12), and then substituting $\mathbf{a}_j$ into (6), the field distribution of the jth-eigenmode can be obtained.

3. NUMERICAL RESULTS

In order to validate the present method, the eigenvalues of TM and TE modes in a spherical resonant cavity are calculated by MLM based on RBF collocation. The radius of spherical cavity is set $R_0 = 1\,\text{m}$. A conformal node distribution fitting naturally to the spherical surface of cavity wall is adopted, as shown in Figure 2. Let the number of node layer from the central point to the outer cavity wall be 9, and node distance on each surface layer be equal to layer distance h.

There are many different types of RBFs. Here, the quintic RBF is chosen:

$$\phi_I(\mathbf{r}) = r^5 \tag{16}$$

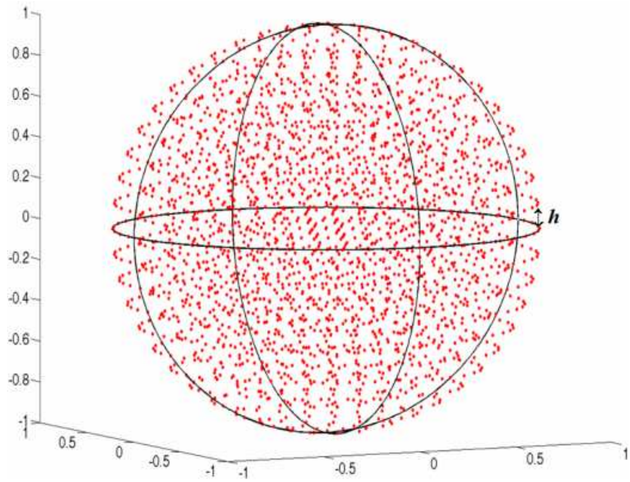


Figure 2. Natural conformal node distribution of the spherical cavity.

Table 1. Resonant wavenumbers of degenerate modes in TM case.

np	TM_{n0p} degenerate modes
11	2.7276, 2.7276, 2.7276
21	3.8474, 3.8474, 3.8474, 3.8474, 3.8474
$\times$	**4.6603**
31	4.9835, 4.9835, 4.9835, 4.9835, 4.9835, 4.9835, 4.9835
41	5.9902, 5.9902, 5.9902
12	6.1308, 6.1308, 6.1308, 6.1308, 6.1308, 6.1308, 6.1308, 6.1308, 6.1308
51	7.2815, 7.2815, 7.2816, 7.2816, 7.2816, 7.2816, 7.2816, 7.2816, 7.2816, 7.2816, 7.2817
22	7.3406, 7.3406, 7.3407, 7.3407, 7.3407

which is the globally supported RBF with no shape parameter to influence the calculation accuracy. In order to analyze the accuracy of the proposed method, the relative error is defined as follows:

$$Err = \frac{|k_i - k_{i0}|}{k_{i0}} \times 100\% \tag{17}$$

where k_i is the numerical eigenvalue (i.e., resonant wavenumber) of the i-th mode and k_{i0} is the analytic solution in the spherical cavity.

Because of the highly symmetrical configuration of the cavity, the TE_{nmp} and TM_{nmp} modes in spherical cavity are independent of m and highly degenerate including nth-order degenerate and polarization degenerate [11], where subscript "n, m, p" represent $\mathbf{r}$, α, and θ directions, respectively. Table 1 shows the resonant wavenumbers of degenerate TM_{n0p} modes computed by MLM based on RBF collocation. From the table, we can see that there are many repeated wavenumbers in most TM modes (their eigenvectors are not the same), which means the computed modes are degenerate. However, a single wavenumber 4.6603 appeared in results. This means that the solution is not a degenerate mode, but a pseudo solution.

The resonant wavenumbers of the ten lowest-order TM modes computed by MLM based on RBF collocation and their relative errors with the exact analytic solutions are shown in Table 2. From the table, we can see that there exist two pseudo solutions, and the maximum relative error is less than 2.7%.

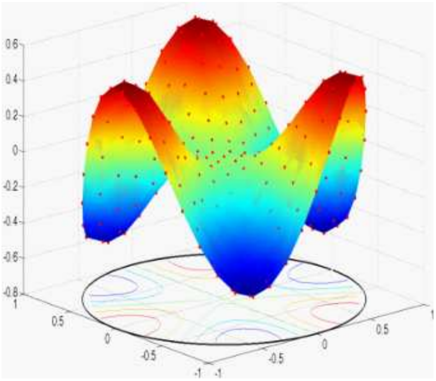
Table 3 shows the results of the ten lowest-order solutions of TE modes computed by MLM based on RBF collocation. Like the TM case, the degenerate modes of each TE mode are calculated and there exist two pseudo solutions too. But the relative errors of TE case are much less than those of TM case. The maximum relative error is less than 1%.

Table 2. Resonant wavenumbers and relative errors in TM case.

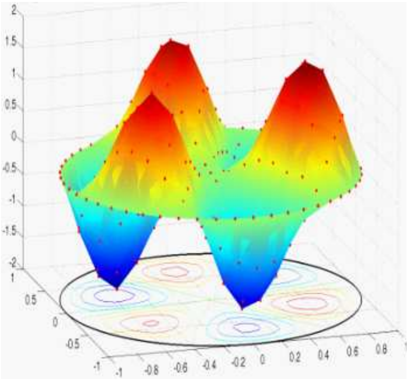
TM_{nmp}	TM_{101}	TM_{201}		TM_{301}	TM_{401}	TM_{102}	TM_{501}
Exact [11]	2.7437	3.8702		4.9734	6.0619	6.1167	7.1398
RBF-MLM	2.7276	3.8474	**4.6603**	4.9835	5.9902	6.1308	7.2817
Rela. Err.%	0.59	0.59		0.20	1.18	0.23	1.99
TM_{nmp}	TM_{202}		TM_{601}	TM_{302}	TM_{701}		
Exact [11]	7.4431		8.2108	8.7217	9.2754		
RBF-MLM	7.3406	**7.5159**	8.4312	8.6958	9.0282		
Rela. Err.%	1.38		2.68	0.30	2.67		

Table 3. Resonant wavenumbers and relative errors in TE case.

TM_{nmp}	TM_{101}	TM_{201}		TM_{301}	TM_{102}	TM_{401}	TM_{202}
Exact [11]	4.4934	5.7635		6.9879	7.7253	8.1826	9.0950
RBF-MLM	4.4940	5.7573	**6.2933**	6.9727	7.7129	8.1590	9.0495
Rela. Err.%	0.013	0.11		0.22	0.16	0.29	0.50
TM_{nmp}	TM_{501}		TM_{302}	TM_{602}	TM_{103}		
Exact [11]	9.3558		10.4171	10.5128	10.9041		
RBF-MLM	9.3270	**9.4102**	10.3394	10.4831	10.7984		
Rela. Err.%	0.31		0.75	0.28	0.97		



(a)



(b)

Figure 3. Field distributions of (a) TM_{301} and (b) TE_{301} modes on the equator section.

The field distributions of the TM_{301} and TE_{301} modes by MLM based on RBF collocation are shown in Figure 3. The field distributions perfectly represent the physical modes on the equator section.

4. CONCLUSION

In this letter, the scalar Helmholtz equation and boundary conditions in spherical cavity are analyzed. MLM based on RBF collocation is applied to solve the scalar Helmholtz equation in rectangular coordinates and compute the resonant wavenumber of spherical cavity. Under the natural conformal node distribution, the results of TE and TM modes of spherical cavity are in agreement with the analytic solutions. The abundant degenerate modes in spherical cavity are also calculated by the proposed method. But there exist some pseudo solutions in the results.

ACKNOWLEDGMENT

This work was supported by the High-Tech Research and Development Program of China (No. 2008AA01Z206) and the National Natural Science Foundation of China (No. 10926190, No. 60872029 and No. 60872034), and Project 9140A01020110DZ0211.

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