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Reflections on Maxwell's Treatise

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(*Invited Paper*)

Abstract—Over a period of about twenty years, Maxwell's determination and unification of the equations of electricity and magnetism evolved from his first paper on the subject in 1855–56, "On Faraday's Lines of Force," to the publication of his *Treatise on Electricity and Magnetism* in 1873. Notwithstanding the many historical accounts and textbooks devoted to Maxwell's work, I have not been able to find a reasonably concise, yet definitive summary of the fundamentals of exactly what Maxwell did in his *Treatise* and how he did it. This paper represents my own attempt to provide such a summary.

1. INTRODUCTION

On numerous occasions during the past forty years, I have turned to Maxwell's *Treatise on Electricity and Magnetism* [1] to learn how he approached various topics in electromagnetics ranging from the definition of electric and magnetic fields to the determination of the speed of light. I, like countless others, have continually benefited from the depth and rigor of some of his derivations, and, despite some important differences, have been appreciative of how little difference (except for notation) there is between many of his basic definitions and equations and those found in present-day textbooks on electromagnetism. However, I have been especially drawn to some salient features of Maxwell's theory of electricity and magnetism that appear to have been ignored, underemphasized, or possibly overlooked in later textbooks and historical accounts of his work — at least those with which I am familiar. For example, I, and presumably many others, have been led by commentaries on the *Treatise* to believe that Maxwell did not write down Faraday's law of induction (induced electromotive force) explicitly in his *Treatise* and that it was Heaviside and Hertz who first expressed "Maxwell's equations" in their more familiar contemporary form. Therefore, I was surprised to learn that, as documented in more detail below in Section 6.1, Maxwell wrote down the integral form of Faraday's law explicitly in his *Treatise* and stated it clearly in words a number of times, even though he did not include either the integral or differential form of Faraday's law explicitly in the summary of his equations in Art. 619. Moreover, he obtained the general form of Faraday's law for moving circuits and this allowed him to derive the electromotive force ($\mathbf{E}+\mathbf{v}\times\mathbf{B}$) per unit moving electric charge and thus the "Lorentz force" equation for moving electric charge.

Consequently, I will try to explain in the present paper, which originates from two conference talks [2], this and other topics in Maxwell's *Treatise* that have especially caught my attention, in hopes that these topics might also be of interest to other readers who have worked in the area of electromagnetics. The paper concentrates on some of the more important fundamentals of what Maxwell did and how he did it in the third edition of his *Treatise* [1] and not necessarily in his earlier publications on electricity and magnetism. Any additions, except for typographical corrections, to the second and third editions of the *Treatise* by their editors, W. D. Niven and J. J. Thomson, are ignored

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so as to concentrate on Maxwell's original contributions. (Without the curly bracketed additions of J. J. Thomson, the third edition is the same as the second edition. The first nine chapters of the first edition, all on electrostatics, were revised by Maxwell before his death to form the second edition, which contains a few footnotes in square brackets added to the first edition by W. D. Niven.)

The paper does not outline the history of the development of electromagnetics, which has been well-documented in several excellent books, for example, [3–6], nor does it trace the genesis and evolution of Maxwell's equations from Maxwell's original papers to the present time, as is done in the recent informative articles by O. M. Bucci [7] and J. W. Arthur [8]. In the present paper, I have intentionally avoided secondary sources by limiting myself to what Maxwell did in his *Treatise* and especially how he did it.

Most people, including myself, do not find Maxwell's *Treatise* an "easy read". The reasons for this are not that Maxwell's writing style is overly complicated or his English difficult to understand. To the contrary, Maxwell writes, as one might expect, very logically, simply, and clearly. The *Treatise* is not very easy to read mainly because of the relatively unfamiliar nineteenth century notation and terminology, the extensive cross referencing in the 866 "Articles" (sections) of the *Treatise*, and the nineteenth century physical explanations by a scientist with an extraordinary physical intuition. As his tutor, William Hopkins, at Cambridge University said in describing Maxwell, "...he is unquestionably the most extraordinary man I have met within the whole range of my experience; it appears impossible for Maxwell to think incorrectly on physical subjects ..." [9, p. 88]. Notwithstanding Maxwell's phenomenal powers of thought and physical intuition, he and other scientists were handicapped by the limited knowledge they had of the interaction between charge, current, and matter compared to what we have today.

In regard to Maxwell's extraordinary physical insight and ability to devise mechanical models of electromagnetic phenomena, it should be pointed out that Maxwell's *Treatise* does not contain for the most part his "theory of molecular vortices" published previously as a four-part paper entitled "On Physical Lines of Force" in the 1861–62 issues of the *Philosophical Magazine*. Maxwell mentions only briefly this molecular vortices model, stating that it was just one of an "infinite number" of possible "demonstrations that a mechanism may be imagined capable of producing a connection of the parts of the electromagnetic field" [1, Art. 831].

In regard to the notation in the *Treatise*, it should be emphasized that Maxwell did not use Hamilton's quaternion calculus. He used only the "language" of quaternions, namely scalars, vectors, and nablas, as we do today, except for a difference in notation for the divergence and curl of a vector $\mathbf{A}$:

$$S\nabla\mathbf{A} \text{ or } S.\nabla\mathbf{A} \text{ for } -\nabla\cdot\mathbf{A} \text{ and } V\nabla\mathbf{A} \text{ or } V.\nabla\mathbf{A} \text{ for } \nabla\times\mathbf{A} \tag{1}$$

with the V or $V.$ omitted in the curl of a vector if the divergence of the vector is zero. Maxwell uses the Hamiltonian terms "scalar" (S) and "vector" (V) and introduces both the terms "rotation" and "curl" for today's $\nabla\times\mathbf{A}$ and "convergence" (instead of "divergence" because of the minus sign in (1)) for today's $-\nabla\cdot\mathbf{A}$ [1, Arts. 11, 25, 409, 619]. He states that, "We shall not, however, expressly introduce quaternion notation. In this treatise we have endeavored to avoid any process demanding of the reader a knowledge of the calculus of quaternions" [1, Arts. 10, 303, 618]. Although Maxwell uses vectors, the majority of his equations are initially written in scalar form. Maxwell uses German letters for vectors [1, Art. 618], some of which look confusingly similar to the uninitiated.

Maxwell defines the ∇ operating on a scalar $a(x, y, z)$ as

$$\nabla a = \hat{\mathbf{x}}\frac{\partial a}{\partial x} + \hat{\mathbf{y}}\frac{\partial a}{\partial y} + \hat{\mathbf{z}}\frac{\partial a}{\partial z} \tag{2}$$

but he does not use a name such as gradient, del, or nabla for this operator in his *Treatise*, even though he corresponds with his lifelong friend P. G. Tait about using the term nabla (a triangular Hebrew harp) after the term was suggested to Tait by his colleague W. R. Smith (who later became a biblical scholar) [10]. Also, Maxwell uses the derivative symbol "d" instead of the partial derivative symbol "∂" (as is done in (2)), which was not in wide use during his lifetime, but with the usual partial-derivative definition of da/dx as the change in $a(x, y, z)$ divided by the change in x holding y and z fixed, and similarly for the partial derivatives with respect to y and z [1, Art. 17]. Because of the negative sign in his definition of "convergence," he also defines the Laplacian operator with a negative sign, $-(\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2)$, and calls it the "concentration" [1, Arts. 26, 617]. For the partial derivative with respect to time, he sometimes uses the overdot notation, for example, $\dot{\mathbf{A}}$ for $\partial\mathbf{A}/\partial t$ [1, Art. 599], as we do today (with multiple overdots indicating multiple partial time derivatives [1, Art. 101e]). Maxwell refers to the overdots as "Newton's notation" [1, Art. 556].

Throughout this paper, we use modern mathematical nomenclature and the SI (rationalized mksA) system of units. The "electromagnetic units" used in the electromagnetic equations in Maxwell's *Treatise* are in accord with an unrationalized mks system and look identical to the modern SI form of the equations except for factors of 4π that the rationalized mksA system eliminates, the value of the permeability of free space equal to unity rather than μ_0, and the value of the permittivity of free space equal to $1/(4\pi c^2)$ rather than ϵ_0, where c is the free-space speed of light [1, Arts. 428, 619, 620, 786]. In Art. 625 Maxwell states that, "The only systems of scientific value are the electrostatic and electromagnetic systems." He compares the dimensions of different quantities in both systems in Arts. 626–628 (see also Chapter 19 of Part IV and Art. 786).

In Art. 623, Maxwell notes that the dimensions of electromagnetic quantities become independent of units if charge (or a similar electromagnetic quantity) is taken as a fourth independent unit (in addition to the three mks units), and it was this idea that eventually led Giorgi to propose the Ampere as the fourth unit [11]. In general, Maxwell was also ahead of his time when it came to recommendations for the units of length, time, and mass. In Art. 3 he said that, "A more universal unit of time might be found by taking the periodic time of vibration of the particular kind of light whose wavelength is the unit of length." In Art. 5, he suggests that with these universal units of length and time, the unit of mass can be deduced from Newton's second law of motion, or, alternatively, from "the mass of a single molecule of a standard substance."

In the preface of his *Treatise*, Maxwell states its central purpose: "I have therefore thought that a treatise would be useful which should have for its principal object to take up the whole subject [of "electrical and magnetic phenomena"] in a methodical manner, and which should also indicate how each part of the subject is brought within the reach of methods of verification by actual measurement." Another principal aim of Maxwell in his *Treatise* is to put the experimental results of Faraday in "ordinary mathematical forms" (see also Art. 528). Near the end of the preface he summarizes his gratitude to Faraday with the statement, "If by anything I have here written I may assist any student in understanding Faraday's modes of thought and expression, I shall regard it as the accomplishment of one of my principal aims — to communicate to others the same delight which I have found myself in reading Faraday's *Researches*." It is noteworthy that Faraday's *Experimental Researches in Electricity* did not contain a single equation. Yet, it was Faraday's observations that influenced Maxwell to devise a mathematical theory of electromagnetics that created a revolutionary paradigm shift in physics from the concept of direct action at a distance to the idea of fields acting in space and matter. In the words of Einstein [12, p. 71], "...before Maxwell, physical reality, in so far as it has to represent the processes of nature, was thought of as consisting in material particles, whose variations consist only in movements governed by partial differential equations. Since Maxwell's time, physical reality has been thought of as represented by continuous fields, governed by partial differential equations, and not capable of mechanical interpretation."

2. CHARGE AND CURRENT

Maxwell's two-volume *Treatise* is divided into four Parts. Part I deals mainly with electrostatic fields produced by static electric charge and static electric displacement $\mathbf{D}$ caused by electric polarization. Part II is devoted mainly to steady electric current but does not consider displacement current $\dot{\mathbf{D}}$ or the magnetostatic fields produced by steady electric currents. Part III is devoted to magnetostatic fields produced by magnetic polarization (magnetization) and Part IV, entitled "Electromagnetism," treats the magnetostatic fields of electric conduction currents as well as time-varying electromagnetic fields and sources including displacement current $\dot{\mathbf{D}}$. Maxwell informs students of his *Treatise* near the end of the preface that, "The description of the phenomena, and the elementary parts of the theory of each subject, will be found in the earlier chapters of each of the four Parts into which this treatise is divided. The student will find in these chapters enough to give him an elementary acquaintance with the whole science."

2.1. Electric Charge, Current, and Polarization

In Maxwell's *Treatise*, electric charge [1, Art. 31], which he also calls "electrification," or "free electricity," or just "electricity" is a fluid substance, that is, a continuum [1, Art. 36], and electric conduction current is the "transference of electrification" [1, Art. 231]. Maxwell explains that either a two-fluid model of positive and negative charge, or a one-fluid model of a single (positive or negative) charge with the opposite charge fixed in the material [1, Arts. 36, 37] can equally well explain most of the observations to date, although he says there are difficulties with both theories and more experimentation is needed to better understand the nature of electricity and electrified bodies.

Maxwell wrote his *Treatise* before the discovery of the electron and thought it improbable that electric charge came in discrete indivisible units (molecules). He says in Art. 260 that "for convenience" one can explain electrolysis by assuming "*one molecule of electricity* ...[†] gross as it is, and out of harmony with the rest of the treatise .... It is extremely improbable however that when we come to understand the true nature of electrolysis we shall retain in any form the theory of molecular charges, for then we shall have obtained a secure basis on which to form a true theory of electric currents, and so become independent of these provisional theories." Nonetheless, for the purpose of defining and mathematically deriving electric fields, Maxwell assumes hypothetical charge-carrying particles that can become very small, ultimately approaching a single point [1, Arts. 44, 73].

Maxwell assumes conservation of charge as demonstrated by the experiments of Faraday [1, Art. 34]: "The total electrification [charge] of a body, or system of bodies, remains always the same, except in so far as it receives electrification [charge] from or gives electrification [charge] to other bodies." Because charge is conserved, he reasons in Art. 35 that charge and current must obey an "equation of continuity" similar to the one for "matter ... in hydrodynamics." He writes this hydrodynamic equation of continuity in Art. 295 and the equation of continuity for electric charge-current in Art. 325. Emphatically, he does not write the conduction current $\mathbf{J}$ as $\rho\mathbf{v}$ because, "In the case of the flow of electricity we do not know anything of its density or its velocity in the conductor, we only know the value of what, on the fluid theory, would correspond to the product of the density and the velocity. Hence in all such cases we must apply the more general method of measurement of the flux across an area" [1, Art. 12].

2.1.1. Electric Polarization as Charge Displacement

The experiments of Faraday also led Maxwell to define electric polarization in terms of electric charge displacement observed on the surface of a dielectric body placed in an electric field, "the extent of this displacement depending on the magnitude of the electromotive intensity [electric field]" [1, Art. 60]. He does not assume that this induced charge separation in a dielectric is caused by electric dipoles within the dielectric that give rise to a polarization vector $\mathbf{P}$ or polarization charge density $-\nabla\cdot\mathbf{P}$. He limits his description of dielectrics to the linear constitutive relation $\mathbf{D}=\epsilon\mathbf{E}$ [1, Art. 68] with $\nabla\cdot\mathbf{D}=\rho_e$ [1, Art. 83a, 612] (equations that he infers from Faraday's experimental results) but without suggesting the possibility of the more general constitutive relation $\mathbf{D}=\epsilon_0\mathbf{E}+\mathbf{P}$, where $\mathbf{E}$ is the electric field, $\mathbf{D}$ is the displacement, ϵ is the permittivity of the dielectric, ϵ_0 is the free-space permittivity, and ρ_e is the electric charge density.

Nowhere in his *Treatise* does he define an electric polarization vector[‡] (see the next subsection and Section 4.1 for more discussion of electric polarization). Maxwell summarizes his view of electric polarization by stating, "That whatever electricity may be, and whatever we may understand by the movement of electricity, the phenomenon which we have called electric displacement is a movement of electricity in the same sense as the transference of a definite quantity of electricity through a wire is a movement of electricity, the only difference being that in the dielectric there is a force which we have called electric elasticity which acts against the electric displacement, and forces the electricity back when the electromotive force is removed" [1, Art. 62]. The closest he seems to approach the idea of dielectrics containing electric dipoles is 1) in discussing the supposition of W. Thomson that

[†] Maxwell uses the word "molecule" as he and other scientists did at the time to describe the smallest possible portion of a substance that retains the properties of the substance. For example, they knew that a water molecule is comprised of two atoms of hydrogen and one atom of oxygen [13] but Maxwell and the scientific community at his time did not know that most of the volume of a molecule or atom consists of free space and that the charge in a molecule comes in integer multiples of the fixed charge of the electron.

[‡] Apparently, it was Larmor [14], Leathem [15], and Lorentz [16] who first introduced an electric polarization vector ($\mathbf{P}$).

the crystals of tourmaline "have a definite electric polarity" [1, Art. 60]; 2) in quoting Faraday's explanation of zero charge accumulation in dielectrics [1, Arts. 54, 109]; and 3) in explaining the theory of Mossotti that a dielectric contains small conducting elements insulated from one another and capable of charge separation (forming dipoles). Maxwell comments that, "This theory [Mossotti's] of dielectrics is consistent with the laws of electricity, and may be actually true" [1, Art. 62].

Nevertheless, Maxwell maintains his uncertainty about the nature of electric polarization throughout the rest of his *Treatise*. This is not surprising if we realize that, unlike with permanent magnets, there had been very little experimentation with permanently polarized dielectrics (electrets) to indicate that electric polarization was produced by molecular electric dipoles, even though some of the properties of tourmaline were known to Maxwell [1, Art. 58]. The uncertainty in the physics community about the origin of dielectric polarization is reiterated six years after Maxwell's death by Oliver Heaviside, who wrote in the paper in which he coined the term "electrets," that "The physical explanation of electrization [process of electric polarization in dielectrics] is of course a matter of speculation, as it depends not only on the nature of molecules, but also on their relation to the ether" [17]. (In this article by Heaviside, he suggests that electric polarization could even be caused by circulating magnetic current.) By 1900 however, Larmor, as well as many other physicists, favored the idea of a "discrete distribution of electricity [positive and negative "electrons"] among the molecules of matter" and Larmor would note in his book that Maxwell's "model of the electrodynamic field did not suggest to him [Maxwell] any means of representing the structure of permanently existing poles [discrete electric charge separation forming dipoles]" [18, pp. 19, 28].

2.1.2. Displacement Current

In the words of Maxwell [1, Art. 610], "One of the peculiarities of this treatise is the doctrine which it asserts, that the true electric current $\mathbf{J}_T$, that on which the electromagnetic phenomena depend, is not the same thing as $\mathbf{J}$, the current of conduction, but that the time-variation of $\mathbf{D}$, the electric displacement, must be taken into account in estimating the total movement of electricity, so that we must write $\mathbf{J}_T = \mathbf{J} + \dot{\mathbf{D}}$." It seems ironic that Maxwell would describe the essence of one of the most profound discoveries in physics since Newton as a "peculiarity". It led to a unified electromagnetic theory about which Freeman Dyson would write, "It is the prototype for Einstein's theories of relativity, for quantum mechanics, for the Yang-Mills theory of generalized gauge invariance, and for the unified theory of fields and particles known as the Standard Model of particle physics" [19].

Yet, there is indeed a peculiarity in Maxwell's claim that $\mathbf{J} + \dot{\mathbf{D}}$ is the total current density. Most treatments of electromagnetic theory today would refer to $\mathbf{J} + \dot{\mathbf{P}}$ as the total equivalent current density (plus $\nabla \times \mathbf{M}$ if Amperian magnetization is present). Probably, the main reason, as mentioned in the previous subsection, that Maxwell did not define an electric polarization vector $\mathbf{P}$ analogous to his magnetization vector $\mathbf{M}$ was the lack of experimentation with electrets (compared to the massive amount of experimentation with magnets) to strongly indicate the dipolar nature of electric polarization. Moreover, Maxwell did not have the advantage of knowing beforehand the form of the dynamical electromagnetic equations, in particular, he did not have the $\epsilon_0\dot{\mathbf{E}}$ term in these equations when he inserted the effects of electric polarization.

His classic derivation in Art. 607 (repeated in many textbooks) of the dynamic current equation with the $\mathbf{J} + \dot{\mathbf{D}}$ term begins with "Ampère's law" (see Art. 498)

$$\mathbf{J} = \nabla \times \mathbf{H}. \tag{3}$$

He then notes that taking the divergence of (3) gives $\nabla \cdot \mathbf{J} = 0$, which is incompatible with the continuity equation of Art. 325, $\nabla \cdot \mathbf{J} = -\partial\rho_e/\partial t$, and his generalized equation of Poisson in Art. 83a, $\nabla \cdot \mathbf{D} = \rho_e$. He therefore concludes that the $\mathbf{J}$ in (3) must be replaced by $\mathbf{J}_T = \mathbf{J} + \dot{\mathbf{D}}$, so that Ampère's law is generalized to

$$\mathbf{J}_T = \mathbf{J} + \dot{\mathbf{D}} = \nabla \times \mathbf{H}. \tag{4}$$

In this derivation, there is no requirement to introduce a separate electric polarization vector and thus it was natural for Maxwell to consider $\mathbf{J} + \dot{\mathbf{D}}$ rather than $\mathbf{J} + \dot{\mathbf{P}}$ as the total current density. Furthermore, from this perspective, polarization charge density ($-\nabla \cdot \mathbf{P}$) does not arise, just free charge density as the divergence of the displacement vector.

Maxwell's referring to $\mathbf{J}_T = \mathbf{J} + \dot{\mathbf{D}}$ as the total current density throughout his *Treatise* remains a subject of discussion to this day [20, 21]. From a purely mathematical viewpoint $\mathbf{J}_T$ in (4) can certainly be considered a solenoidal total current density since it has the dimensions of a current density and $\nabla \cdot \mathbf{J}_T$ equals 0 just as $\nabla \cdot \mathbf{J}$ equaled 0 before the displacement current density $\dot{\mathbf{D}}$ was inferred by Maxwell. In the words of Maxwell from Art. 607, "We have very little experimental evidence relating to the direct electromagnetic action of currents due to the variation of electric displacement in dielectrics, but the extreme difficulty of reconciling the laws of electromagnetism with the existence of electric currents which are not closed is one reason among many why we must admit the existence of transient currents due to the variation of displacement. Their importance will be seen when we come to the electromagnetic theory of light." However, Maxwell also seems to consider the displacement current on a physical par with conduction current when he concludes from Faraday's observations of charge induced by voltages applied to conductors separated by dielectrics that "The variations of electric displacement evidently constitute electric currents" [1, Arts. 83a, 60]. In addition, he assumes these displacement electric currents flow in a vacuum dielectric as well because "the energy of electrification resides in the dielectric medium, whether that medium be solid, liquid, or gaseous, dense or rare, or even what is called a vacuum, provided it be still capable of transmitting electrical action" [1, Art. 62]; see Section 5 below.

2.2. Magnetic Charge and Polarization

Maxwell's systematic development of a mathematical theory of magnetism combines the ideas of Poisson, Weber, and W. Thomson to explain the large body of experimental results known at the time. It is one of the highlights of his *Treatise* and, in combination with "Faraday's law" and his extraordinary deduction of the displacement current $\dot{\mathbf{D}}$, it led to the equations of electromagnetism, one of the monumental achievements in the history of mathematical physics. Commenting on Maxwell's accomplishment, Feynman, who was not prone to exaggeration, said "From a long view of the history of mankind — seen from, say, ten thousand years from now — there can be little doubt that the most significant event of the 19th century will be judged as Maxwell's discovery of the laws of electrodynamics" [22, Vol. II, Sec. 1–6].[§]

To explain mathematically the experiments of Örsted, Coulomb, Ampère, Gauss, and Faraday, Maxwell "regarded the magnet as a continuous and homogeneous body, the minutest part of which has magnetic properties of the same kind as the whole" [1, Arts. 382, 638]. Each of these minute parts, which he considers small enough to mathematically treat as a differential volume element of the continuum, produce the scalar potential (and magnetic fields) of a magnetic dipole [1, Art. 383], although Maxwell does not use the term "dipole" in his *Treatise* but rather calls each of these minute parts a magnetic "molecule or particle" [1, Arts. 380, 387], presumably because they have the same electromagnetic properties as the continuum. In order to derive the magnetic-dipole potential, Maxwell considers each minute particle as consisting of equal-magnitude positive and negative magnetic charge separated by a differential distance. Maxwell refers to the positive and negative magnetic charges as positive and negative magnetic "poles" [1, Art. 383], which Coulomb had experimentally discovered, using long thin magnetized rods, satisfied the inverse square law of attraction and repulsion [1, Arts. 373–374]. The equal and opposite poles produce a magnetic dipole moment with magnitude equal to the product of the separation distance and the value of the positive pole [1, Art. 384].

He defines a magnetization vector $\mathbf{M}$ having the direction of the magnetic moment of the minute particle and a magnitude equal to the dipole moment divided by the volume of the particle [1, Arts. 384, 385]. He then shows by means of an integration by parts that the scalar magnetic potential ψ_m can be rewritten as the potential of a magnetic charge density $\rho_m = -\mu_0 \nabla \cdot \mathbf{M}$ plus the potential of the corresponding surface magnetic charge density $\sigma_m = \mu_0 \hat{\mathbf{n}} \cdot \mathbf{M}$ (he doesn't use subscripts m and, as mentioned above, I am using more modern notation in SI equations, which include the free-space permeability μ_0), such that $\mathbf{H} = -\nabla \psi_m$ is Maxwell's primary mathematically defined magnetic field (which he appropriately calls the "magnetic force" [1, Art. 395]) since the source of his magnetization is magnetic charge and not circulating electric current.

[§] A good argument can be made that Darwin's formulation of a scientific argument for the theory of evolution by means of natural selection is an equally significant 19th century event. Interestingly, Maxwell expressed reservations about Darwin's theory of evolution [13].

The vector $\mathbf{B}$, which Maxwell calls the "magnetic induction" [1, Art. 400], is the secondary field defined in terms of $\mathbf{H}$ and $\mathbf{M}$, namely $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$.‖ Maxwell concludes this clever analysis by saying "Hence, if we assume σ_m and ρ_m to be the surface- and volume-densities of the distribution of an imaginary substance, which we have called 'magnetic matter,' the potential due to this imaginary distribution will be identical with that due to the actual magnetization of every element of the magnet" [1, 385, 386]. He further explains that "This method [using magnetic charge] of representing the action of a magnet as due to a distribution of 'magnetic matter' is very convenient, but we must always remember that it is only an artificial method of representing the action of a system of polarized particles."

The uncertainty expressed by Maxwell in the foregoing quote can be understood in terms of his discussions of magnetic matter in preceding and succeeding articles of his *Treatise*. Although he is quite certain that magnetic matter can be described mathematically by an electromagnetic continuum of magnetic "particles or molecules" having magnetic dipole moments produced by magnetic charge separation, he is uncertain as to whether this is indeed the fundamental physical "nature of a magnetic molecule" [1, 832]. In Art. 380 he explains that Poisson has postulated a two-fluid model of magnetic charge such "that the process of magnetization consists in separating to a certain extent the two fluids within each particle, and causing one fluid [of positive magnetic charge] to be more concentrated at one end, and the other fluid [of negative magnetic charge] concentrated at the other end of the particle [with the sum of the magnetic charges always zero]" [1, Art. 380]. However, he decides against Poisson's theory on both theoretical and experimental grounds (his theory and the experiments of Beetz) [1, Arts. 430, 442].

Maxwell prefers Weber's theory of ferromagnetism (and paramagnetism), about which theory he says that "We have seen (Art. 442 [Weber's theory]) that there are strong reasons for believing that the act of magnetizing iron or steel does not consist in imparting magnetization to the molecules of which it is composed, but that these molecules are already magnetic, even in unmagnetized iron, but with their axes placed indifferently in all directions, and that the act of magnetization consists in turning the molecules so that their axes are either rendered all parallel to one direction, or at least are deflected toward that direction" [1, Art. 832].

He goes on to say in Art. 833 that Weber's theory still does not decide how the dipole moments are produced and this leads him to seriously consider Ampère's suggestion that the dipole moments are actually created by circulating currents within the molecules: In Maxwell's words, "Still, however, we have arrived at no explanation of the nature of a magnetic molecule, that is, we have not recognized its likeness to any other thing of which we know more. We have therefore to consider the hypothesis of Ampère, that the magnetism of the molecule is due to an electric current constantly circulating in some closed path within it." (Here Maxwell is acknowledging that his introduction of magnetic charge, "convenient" as it is, has no basis in experimental observations of isolated magnetic charge (monopoles).) He assumes that such a circulating current meets with no resistance and performs a beautifully simple analysis in Arts. 836 and 837 using the equation of self and mutual inductance of a closed zero-resistance circuit to show that "primitive" current must exist on the circuits of molecules that produce ferromagnetism (and paramagnetism), whereas circuits with zero primitive currents can produce diamagnetism — a result he says agrees with another of Weber's ideas that "there exist in the molecules of diamagnetic substances certain channels round which an electric current can circulate [be induced] without resistance" [1, Art. 838]. Also, as explained in Section 4.2.2 below, Maxwell makes use of Amperian dipole moments by dividing a closed current-carrying circuit into elementary current loops in order to represent the circuit by a magnetic shell [1, Ch. I of Pt. IV].

In Art. 638, Maxwell says that if we adopt the Ampère-Weber circulating-electric-current model of magnetism, we can "arrive at results similar to those" he gets using the model of magnetic dipoles as separated magnetic charge, provided "we suppose our mathematical machinery to be so coarse that our line of integration cannot thread a molecular circuit, and that an immense number of magnetic molecules are contained in our element of volume." Maxwell does not provide us with the mathematical derivation of these "similar results" obtained from an electric-current (Amperian) model of magnetic dipoles, a derivation that can be found in later textbooks that derive magnetic polarization beginning

‖ If Maxwell had chosen $\mathbf{B}$ instead of $\mathbf{H}$ as the symbol for his primary magnetic field, what we call Maxwell's equations today would have $\mathbf{B}$ and $\mathbf{H}$ interchanged and the magnetic constitutive relation would be $\mathbf{H} = \mu_0(\mathbf{B} + \mathbf{M})$.

with **B** as the primary magnetic vector and Amperian magnetic dipoles as producing the magnetic dipole moments [23, Sec. 5.8]. In fact, nowhere in the *Treatise* does Maxwell average microscopic fields to get macroscopic fields. Nonetheless, he sets the stage for later derivations of the continuum Maxwellian equations from averaging sources and fields of the microscopic Maxwellian equations over macroscopic volume elements that contain many molecules or inclusions. The interested reader only laments that Maxwell did not include in his *Treatise* his derivation of the magnetic equations from Amperian magnetic dipoles.

3. DEFINITION AND MEASUREMENT OF THE PRIMARY ELECTRIC AND MAGNETIC FIELDS IN FREE SPACE

Maxwell first defines the primary electric and magnetic fields in free space (as opposed to within source regions). For the electric field $\mathbf{E}(\mathbf{r},t)$, which he calls the electromotive intensity or electromotive force on a stationary electric charge, he gives the definition: "The resultant electric intensity [electromotive intensity or force on a stationary electric charge] at any point is the force which would be exerted on a small body charged with the unit of positive electricity [electric charge], if it were placed there without disturbing the actual distribution of electricity" [1, Arts. 44, 68]. Similarly, the magnetic field $\mathbf{H}(\mathbf{r},t)$, which Maxwell called the magnetic force, is the force exerted on a nondisturbing unit magnetic charge (pole) [1, Art. 395]. Expressed in contemporary mathematical language

$$\mathbf{E}(\mathbf{r},t) = \lim_{q_e\to 0} \frac{\mathbf{f}_e(\mathbf{r},t)}{q_e} \tag{5}$$

and

$$\mathbf{H}(\mathbf{r},t) = \lim_{q_m\to 0} \frac{\mathbf{f}_m(\mathbf{r},t)}{q_m}. \tag{6}$$

In (5) and (6), the charges are not moving with respect to the laboratory in which the forces are measured. In principle, the electric and magnetic fields in free space could be determined by measuring the mechanical force on small bodies with known electric and magnetic charges. (Although magnetic charges do not exist, Coulomb had showed that the pole at each end of a very thin magnetic rod acts locally as a monopole and could, in principle, be used to measure the magnetic field.) In general, these direct methods for determining electric and magnetic fields in free space are not very practical. Nonetheless, these mathematical definitions of the fields in terms of forces on charges enable the precise representation of Faraday's lines of force as the curves tracing the directions of the force fields in space such that the curves form tubes of variable cross section proportional to the magnitude of the fields.

Later in Part IV of the *Treatise*, Maxwell shows that the electric field in (5) can be expressed in terms of the vector and scalar potentials as

$$\mathbf{E} = -\dot{\mathbf{A}} - \nabla\psi_e. \tag{7}$$

In addition, he generalizes the definition of the electromotive intensity or electromotive force to the force on unit electric charges moving with velocity **v**. He eventually proves from Faraday's law that this force on moving unit charges is given by [1, Arts. 598, 599]

$$\mathbf{E} = \mathbf{v}\times\mathbf{B} - \dot{\mathbf{A}} - \nabla\psi_e \tag{8}$$

where **B** is the magnetic induction.

Maxwell's use of the same symbol **E** (German letter $\mathfrak{E}$ in his *Treatise*) in (8) as in (7) can be confusing to the reader when comparing results in Parts I and II with those in Part IV of the *Treatise*. Therefore, to try to avoid such confusion in this paper, I will replace the symbol **E** for the electromotive intensity or force exerted on a moving electric charge in (8) with the symbol $\mathbf{E}_\mathrm{v}$, so that (8) is replaced by

$$\mathbf{E}_\mathrm{v} = \mathbf{v}\times\mathbf{B} - \dot{\mathbf{A}} - \nabla\psi_e = \mathbf{v}\times\mathbf{B} + \mathbf{E} \tag{9}$$

where the symbol **E** is reserved for the force on a *stationary* unit electric charge given in (5) and (7). In other words, Maxwell's notation in (8) is revised in the present paper to the more acceptable present-day notation in (9). I will refer to **E** as the electric field but use Maxwell's term "electromotive force" for

$\mathbf{E}_{\mathrm{v}}$, the force on a moving unit electric charge. *The evaluation of the electromotive force* $\mathbf{E}_{\mathrm{v}}$ *in (9) is commonly referred to today as the "Lorentz force" on a unit charge, even though it was derived explicitly by Maxwell by means of a brilliant deduction in Art. 598 from Faraday's law*; see Section 6.1 below. However, Maxwell does not use the $\mathbf{v} \times \mathbf{B}$ part of the "Lorentz force" in (9) to define $\mathbf{B}$, Maxwell's definition of which is given below in Section 4.2.

Another possible source of confusion to the reader of Maxwell's *Treatise* is that Maxwell uses the term "*total* electromagnetic force," which he first introduces for electrostatic fields as the work that would be done by the electric field on a unit of positive charge carried along a curve between two points in space (yielding a potential difference) [1, Art. 69]. In his articles on electrostatics, he is fairly consistent in his use of the modifier "*total*" and it is usually obvious from the context as to whether he is dealing with the "electromotive force" or the "total electromotive force," the potential difference between two points in space due to the electric field. However, later in dealing with time-varying fields in Part IV of the *Treatise*, Maxwell generalizes the use of the term "total electromagnetic force" (and often omits the word "total" or replaces it with the word "impressed" where appropriate, as in Art. 579) to mean the integral of the total force ($\mathbf{E}_{\mathrm{v}}$) per unit charge moving with each point of a closed circuit all or part of which is in motion [1, Arts. 595–596].

4. DEFINITION AND MEASUREMENT OF ELECTRIC AND MAGNETIC FIELDS IN SOURCE REGIONS

The force definitions of $\mathbf{E}$ and $\mathbf{H}$ in (5) and (6) hold in free space where the test charges do not "disturb" the other sources and thus are not applicable in source regions of electric and magnetic charge, current, and polarization. Therefore, Maxwell begins with integral expressions for static potentials of $\mathbf{E}$ and $\mathbf{H}$ that hold outside the source regions and defines mathematical potentials and fields inside source regions by using the same integral expressions for the potentials within the source regions. He does this implicitly without explanation for the electric fields in continuous electric charge, current, and polarization, whereas he explicitly explains this approach when he gets to defining magnetic fields in source regions of magnetization. Specifically, he says in Art. 395 that "The magnetic potential, as thus defined [by its integral expression], is found by the same mathematical process, whether the given point is outside the magnet or within it." He then has the problem of relating this "mathematically defined" continuum field to something that can be measured experimentally (at least, in principle).

Maxwell resolves this problem by defining "cavity fields" within the magnetization and relating the cavity fields to the mathematically defined fields. It is enlightening to quote the full paragraph from Art. 395 in which Maxwell explains this procedure, "To determine by experiment the magnetic force at a point within the magnet we must begin by removing part of the magnetized substance, so as to form a cavity [in principle, infinitesimally small] within which we are to place the magnetic pole [test magnetic charge]. The force acting on the pole will depend, in general, on the form of this cavity, and on the inclination of the walls of the cavity to the direction of magnetization. Hence it is necessary, in order to avoid ambiguity in speaking of the magnetic force within a magnet, to specify the form and position of the cavity within which the force is to be measured. It is manifest that when the form and position of the cavity is specified, the point within it at which the magnetic pole is placed must be regarded as no longer within the substance of the magnet, and therefore the ordinary methods [using the definition in (6)] of determining the force become at once applicable."

In the next two subsections, we outline in more detail how Maxwell defines and measures (in principle) electric and magnetic fields in source regions.

4.1. Definition and Measurement of Electric Fields in Source Regions

For the electrostatic field, Maxwell begins with the inverse square law for the force between electric charges as determined experimentally by Coulomb and others [1, Art. 66]. He derives the potential of a point charge and a sum of point charges in free space for the electric field defined in (5); then generalizes this potential to an integral over a volume V of electric charge density ρ_e and defines the potential for the point of observation $\mathbf{r}$ everywhere including points within the charge density of V to get [1, Arts. 69–71, 73]

$$\psi_e(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\limits_V \frac{\rho_e(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV' \tag{10a}$$

with

$$\mathbf{E}(\mathbf{r}) = -\nabla\psi_e(\mathbf{r}). \tag{10b}$$

Here, there is no concern about cavity fields explicitly because the values of ψ_e and $\mathbf{E}$ do not change if an infinitesimally small volume of charge density $\rho_e(\mathbf{r})$ is removed about an observation point $\mathbf{r}$ in the charge density and, in principle, the electric field $\mathbf{E}$ within charge density could be measured within such a cavity. In other words, $\mathbf{E}$ in (10) is identical to $\mathbf{E}$ in (5) measured in a cavity of the electric charge density (keeping the surrounding charge density unaltered).

Once he has the electric field defined everywhere by means of (10), he invokes the divergence theorem with $\mathbf{E}$ [1, Art. 75]

$$\oint\limits_S \mathbf{E}\cdot\hat{\mathbf{n}}dS = \int\limits_V \nabla\cdot\mathbf{E}dV \tag{11}$$

and proves Gauss's law from Coulomb's law and (5) [1, Art. 76], namely

$$\epsilon_0\oint\limits_S \mathbf{E}\cdot\hat{\mathbf{n}}dS = \int\limits_V \rho_e dV \tag{12}$$

so that (11) and (12) imply that [1, Art. 77]

$$\epsilon_0\nabla\cdot\mathbf{E} = \rho_e \tag{13}$$

as well as

$$\nabla^2\psi_e = -\rho_e/\epsilon_0. \tag{14}$$

Of course, (10b) implies that $\mathbf{E}$ is irrotational, that is

$$\nabla\times\mathbf{E} = 0. \tag{15}$$

The equations in (13) and (15) mathematically define the electric field in an electrostatic charge distribution.

As mentioned above in Section 2.1.1, Maxwell does not introduce an electric polarization vector $\mathbf{P}$ corresponding to a magnetic polarization (magnetization) vector $\mathbf{M}$ because, unlike magnetization, which he assumes is produced by magnetic dipoles, he does not assume that electric polarization is produced by electric dipoles. He simply interprets Faraday's experiments with dielectrics by assuming, in accordance with Faraday, that "electricity is displaced within it [a dielectric], so that a quantity of electricity which is forced in the direction of $\mathbf{E}$ across [a] unit of area fixed perpendicular to $\mathbf{E}$ is

$$\mathbf{D} = \epsilon\mathbf{E} \tag{16}$$

where $\mathbf{D}$ is the displacement, $\mathbf{E}$ the resultant intensity [electric field], and ϵ the specific inductive capacity [permittivity] of the dielectric" [1, Art. 68].

He also says that Faraday's experiments with displacement and resulting theory can be expressed in mathematical language as [1, Arts. 83a, 612]

$$\nabla\cdot(\epsilon\mathbf{E}) = \rho_e \tag{17a}$$

or from (16)

$$\nabla\cdot\mathbf{D} = \rho_e \tag{17b}$$

with $\mathbf{E}$ in a dielectric still given by the gradient of a potential

$$\mathbf{E}(\mathbf{r}) = -\nabla\psi_e(\mathbf{r}) \tag{18}$$

so that $\mathbf{E}$ is still irrotational in a dielectric

$$\nabla\times\mathbf{E} = \nabla\times(\mathbf{D}/\epsilon) = 0. \tag{19}$$

Between two dielectrics, Maxwell determines that the normal components of the displacement vector satisfy the boundary condition

$$D_{2n} - D_{1n} = \sigma_e \tag{20}$$

where σ_e is the electric surface-charge density. The equations in (17) and (19) mathematically define the static **E** and **D** fields in dielectric continua. Of course, the scalar potential ψ_e will no longer be given by (10a).

The only prescription in Maxwell's *Treatise* for measuring **D** (or "electric polarization") and **E** in dielectrics is given in Art. 60, "The amount of the displacement is measured by the quantity of electricity [charge] which crosses a unit of area, while the displacement increases from zero to its actual amount. This, therefore, is the measure of the electric polarization." We can formulate Maxwell's prescription for measuring $\mathbf{D}(\mathbf{r})$ in greater detail by imagining a thin, tiny parallel-plate capacitor inserted into the dielectric at the point of interest $\mathbf{r}$. With a voltmeter attached across the plates of the capacitor, rotate the capacitor to obtain the highest value of voltage. This will occur when the normal $\hat{\mathbf{n}}$ to the plates is in the direction of $\mathbf{D}(\mathbf{r})$. This thin parallel-plate capacitor will hardly disturb the fields of the dielectric except inside the metal plates where the fields are zero. Now short the plates of the capacitor so that they slowly discharge through an ammeter that measures the total discharge to bring the plates to zero voltage and thus zero fields between them. The surface charge densities $\pm\sigma_e(\mathbf{r})$ now on the shorted plates can be found by dividing the total discharge by the area of the plates. Thus, from (20), and the fields being zero within the plates, we have

$$\mathbf{D}(\mathbf{r}) = \sigma_e(\mathbf{r})\hat{\mathbf{n}}. \tag{21}$$

Once **D** is determined in this manner, $\mathbf{E} = \mathbf{D}/\epsilon$ follows from a measurement of the permittivity ϵ. In principle, $\sigma_e(\mathbf{r})$ could be measured in this manner to determine $\mathbf{D}(\mathbf{r})$ and $\mathbf{E}(\mathbf{r})$ in a dielectric but, in general, it would not be a practical measurement scheme.

One of the shortcomings of the *Treatise* is that it does not relate the **E** and **D** fields defined mathematically by (17) and (19), and experimentally determined in principle by Maxwell's prescription, implemented, for example, by the foregoing measurement scheme, to the primary free-space definition of **E** given in (5). In particular, Maxwell does not define cavity fields for dielectrics as he does for magnets, evidently because he is not convinced that the displacement vector **D** is produced by electric dipoles and thus he does not define a polarization vector **P**, which was not introduced until more than 15 years after his death; see Footnote 2.

4.2. Definition and Measurement of Magnetic Fields in Source Regions

For the magnetostatic field, Maxwell also begins, as mentioned above in Section 2.2, with Coulomb's experimental result showing that if the poles (magnetic charges) of a magnet could be separated they would obey an inverse square law similar to that of electric charges [1, Arts. 373–374]. Then, under the assumption indicated by experiments that magnets and magnetic material consist of magnetic dipoles, he writes an expression for the magnetic potential ψ_m outside a point magnetic-charge dipole in terms of its magnetic dipole moment $\mathbf{m}$ [1, Arts. 383–384]. He replaces $\mathbf{m}$ with a volume element of magnetization $\mathbf{M}dV$, integrates over V, and lets the same integral define the magnetic potential inside as well as outside the magnetization to get in Art. 385

$$\psi_m(\mathbf{r}) = \frac{1}{4\pi}\int\limits_V \mathbf{M}(\mathbf{r}')\cdot\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)dV'. \tag{22}$$

Lastly, integrating by parts (equivalent to using the divergence theorem), he converts this integral to [1, Arts. 385, 395]

$$\psi_m(\mathbf{r}) = -\frac{1}{4\pi}\int\limits_V \frac{\nabla'\cdot\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV' + \frac{1}{4\pi}\oint\limits_S \frac{\mathbf{M}(\mathbf{r}')\cdot\hat{\mathbf{n}}'}{|\mathbf{r}-\mathbf{r}'|}dS' \tag{23a}$$

with

$$\mathbf{H}(\mathbf{r}) = -\nabla\psi_m(\mathbf{r}) \tag{23b}$$

where S is the closed surface of the volume V of the magnetization. These equations in (23) were first obtained by Poisson [3, Vol. I, pp. 62–64].

From (23a) Maxwell notes in Arts. 385–386 that the volume and surface magnetic charge densities are

$$\rho_m = -\mu_0 \nabla \cdot \mathbf{M}(\mathbf{r}), \quad \sigma_m = \mu_0 \mathbf{M}(\mathbf{r}) \cdot \hat{\mathbf{n}}. \tag{23c}$$

If the volume V extends into free space beyond the surface of the magnetization, the surface integral in (23a) vanishes but is taken into account implicitly through the delta function in $\nabla \cdot \mathbf{M}$ at the surface of the magnetization that contributes if V extends into free space beyond the magnetization. Designating this extended volume by V_+, we can rewrite (23a) as simply

$$\psi_m(\mathbf{r}) = -\frac{1}{4\pi} \int\limits_{V_+} \frac{\nabla' \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV' \tag{24}$$

although Maxwell does not include this equation in his *Treatise*.

With a magnetostatic vector $\mathbf{H}$ defined mathematically by (23), Maxwell proceeds to explain how to experimentally measure, at least in principal, this mathematically defined magnetic field. Since the primary definition (6) of the magnetic field $\mathbf{H}$ must be carried out in free space, he uses (23) to calculate the magnetic field in small (ideally infinitesimal) free space cavities in the magnetization formed by the removal of a small volume of magnetization holding the remaining magnetization fixed. He finds for a circular-cylindrical cavity, whose axis is aligned with the magnetization vector $\mathbf{M}(\mathbf{r})$, and whose center is at the point $\mathbf{r}$, that the magnetic field in the free space at the center of the cylinder *differs* from the mathematically defined $\mathbf{H}$ by the amount [1, Art. 396]

$$\Delta \mathbf{H} = \mathbf{M} \left(1 - \frac{b}{\sqrt{a^2 + b^2}}\right) \tag{25}$$

where $2b$ is the length of the axis of the cylinder and a is its radius.

For an infinitesimally narrow cylinder ($a/b \to 0$), we have $\Delta \mathbf{H} = 0$ and the narrow-cylinder cavity magnetic field $\mathbf{H}_c^{\text{nc}}$ equals $\mathbf{H}$, the mathematically defined magnetic field, that is [1, Arts. 396–397]

$$\mathbf{H}_c^{\text{nc}} = \mathbf{H} \tag{26}$$

where $\mathbf{H}_c^{\text{nc}}$ is measurable (in principle) in the free space of the cavity and thus provides a method for measuring $\mathbf{H}$.

If "the length of the cylinder is small compared with its diameter, so that the cylinder becomes a thin disk" ($b/a \to 0$) perpendicular to $\mathbf{M}$, we have $\Delta \mathbf{H} = \mathbf{M}$ and the thin-disk cavity magnetic field $\mathbf{H}_c^{\text{td}}$ equals $\mathbf{H} + \mathbf{M}$, that is

$$\mathbf{H}_c^{\text{td}} = \mathbf{H} + \mathbf{M}. \tag{27}$$

Maxwell now says that, since $\mathbf{H} + \mathbf{M}$ in the thin disk perpendicular to $\mathbf{M}$ is an actual free-space force (measurable, in principle), he defines a new magnetic vector $\mathbf{B}$ (equal to $\mu_0 \mathbf{H}_c^{\text{td}}$ in (27)) as [1, Arts. 399–400]

$$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}). \tag{28}$$

He names this cavity-defined, measurable (in principle) magnetic vector $\mathbf{B}$ the "magnetic induction." He states that William Thomson has called $\mathbf{H}$ and $\mathbf{B}$ within cavities of magnetization the "polar" and "electromagnetic" definition, respectively, of magnetic force. In free space, $\mathbf{M}$ is zero and the general constitutive relation in (28) simplifies to $\mathbf{B} = \mu_0 \mathbf{H}$. In linear magnetic material, he introduces magnetic susceptibility χ_m and permeability $\mu = \mu_0(1 + \chi_m)$ such that [1, Arts 426–428]

$$\mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{B} = \mu \mathbf{H}. \tag{29}$$

Streamlining somewhat the next part of Maxwell's derivation by using (24) instead of (23a), take the divergence of (28) to get

$$\nabla \cdot \mathbf{B} = \mu_0 \left(\nabla \cdot \mathbf{M} + \nabla \cdot \mathbf{H}\right). \tag{30}$$

From (23b) and (24), we see that the magnetostatic fields obey the same equations as electrostatic fields with $\rho_m = -\mu_0 \nabla \cdot \mathbf{M}$ replacing ρ_e/ϵ_0, the magnetic potential $\mu_0 \psi_m$ replacing the electric potential ψ_e, and $\mu_0 \mathbf{H}$ replacing $\mathbf{E}$. Therefore, we can use the electrostatic results to prove (analogously to (13)) that

$$\nabla \cdot \mathbf{H} = \rho_m/\mu_0 = -\nabla \cdot \mathbf{M} \tag{31}$$

which when inserted into (30) yields

$$\nabla \cdot \mathbf{B} = 0 \tag{32}$$

everywhere (inside as well as outside magnetic material) [1, Arts. 401–404]. Eq. (32) is the mathematical statement of the empirical evidence that free magnetic charge does not exist; there are no magnetic monopoles. Maxwell concludes Art. 405 with the cogent statements, "The importance of the magnetic induction [**B**] as a physical quantity will be more clearly seen when we study electromagnetic phenomena. When the magnetic field is explored by a moving wire, as in Faraday's *Exp. Res. 3076*, it is the magnetic induction and not the magnetic force [**H**] which is directly measured."

In Art. 429 Maxwell uses (32) to derive the following boundary condition on the normal component of **B** across an interface of two magnetic media, one with permeability μ and the other with permeability μ'

$$\mu H_n = \mu' H_n'. \tag{33}$$

Because (32) holds everywhere, inside and outside magnetic material, **B** can be written as the curl of a vector potential **A** [1, Arts. 405–407]

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{34}$$

Inserting (24) into (23b), bringing the gradient operator under the integral sign, applying vector identities and integral formulas, and using the delta-function expression $\nabla^2(1/|\mathbf{r}-\mathbf{r}'|) = -\delta(\mathbf{r}-\mathbf{r}')/(4\pi)$, one finds that

$$\mathbf{H}(\mathbf{r}) = -\mathbf{M}(\mathbf{r}) + \nabla \times \frac{1}{4\pi} \int\limits_{V_+} \mathbf{M}(\mathbf{r}') \times \nabla' \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) dV' \tag{35}$$

which implies that

$$\begin{aligned} \mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \int\limits_{V_+} \mathbf{M}(\mathbf{r}') \times \nabla' \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) dV' = \frac{\mu_0}{4\pi} \int\limits_{V} \mathbf{M}(\mathbf{r}') \times \nabla' \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) dV' \\ &= \frac{\mu_0}{4\pi} \int\limits_{V_+} \frac{\nabla' \times \mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} dV' = \frac{\mu_0}{4\pi} \int\limits_{V} \frac{\nabla' \times \mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} dV' + \frac{\mu_0}{4\pi} \oint\limits_{S} \frac{\mathbf{M}(\mathbf{r}') \times \hat{\mathbf{n}}'}{|\mathbf{r}-\mathbf{r}'|} dS'. \end{aligned} \tag{36}$$

Maxwell obtains the first line of this Eq. (36) in Art. 405. He notes in Art. 406 that taking the divergence of (36) shows that

$$\nabla \cdot \mathbf{A} = 0. \tag{37}$$

Although he derives (37) for magnetostatics, he will later retain this "Coulomb gauge" for the vector potential in the time dependent electromagnetic equations; see Section 6.2.

One cannot help but admire the remarkably direct way in which Maxwell determines the magnetostatic scalar and vector potentials and their associated **H** and **B** fields produced by magnetization **M** directly from Coulomb's inverse square law for magnetic poles (charges). It may be worth repeating that he begins by mathematically defining the **H** field as that vector field given by the same potential integral for observation points inside the source region of **M** as it is for observation points outside **M**, and then relates this mathematically defined **H** field to fields that can, in principle, be measured within free-space cavities of the magnetization. In so doing, he finds it convenient to introduce the magnetic induction defined as $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$.

As we have seen, in Maxwell's treatment of electric and magnetic fields, he begins by defining **E** and **H** as the primary fields in free space, then introduces permittivity ϵ for dielectrics and magnetization **M** for magnetic material, and obtains secondary fields $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$.

4.2.1. Magnetostatic Fields Produced by Electric Conduction Current

Although most of Part II of the *Treatise* deals with electric conduction current, Maxwell does not formulate the results of the experiments of Ampère, Weber, and Faraday that determine the magnetostatic fields produced by current until Chapter I of Part IV. Specifically, he concludes in Art. 498 and repeats in the first half of Art. 607 that the "line integral of the magnetic force [field] round a closed curve is numerically equal to the electric current through the closed curve." In Art. 607 he also uses Stoke's theorem to convert this line-integral result to the partial differential equation

$$\mathbf{J} = \nabla \times \mathbf{H}. \tag{38}$$

In free space, $\mathbf{H} = \mathbf{B}/\mu_0$ and with $\mathbf{B} = \nabla \times \mathbf{A}$ along with $\nabla \cdot \mathbf{A} = 0$, the vector potential satisfies $\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$, which has the solution (see Art. 617)

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int\limits_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV' \tag{39}$$

Taking the curl of (39), it can be shown that the magnetic field in a small cavity made by removing a small volume of continuous current density $\mathbf{J}(\mathbf{r})$ is the same as the magnetic field mathematically defined by the integral without removing the small volume. Thus, in magnetostatic current density there is no need to distinguish between mathematically defined magnetic fields and magnetic fields that can be measured; and, indeed, Maxwell does not concern himself with cavity magnetic fields in conduction current density $\mathbf{J}$.

4.2.2. Force Produced by a Magnetic Field on Electric Conduction Current

Chapter I of Part IV also draws from the experimental results and analyses of Ampère, Weber, and Faraday to determine the "mechanical force" (call it $d\mathbf{f}$) exerted by an external magnetic induction $\mathbf{B}$ on a line element $d\mathbf{c}$ of an electric circuit carrying a line conduction current I. Maxwell does not write out the final equation explicitly but states it so clearly in Art. 490 that we can write

$$d\mathbf{f} = I d\mathbf{c} \times \mathbf{B}. \tag{40}$$

Later in Art. 603, he generalizes this result to volume conduction current density $\mathbf{J}$ and writes the equation for the volume force density $\mathbf{F}$ explicitly as

$$\mathbf{F} = \mathbf{J} \times \mathbf{B}. \tag{41}$$

He derives the result in (40), as we explain shortly, without ever considering $\mathbf{J}$ (or I) as a moving electric charge density $\rho\mathbf{v}$ and without knowledge of the magnetic force $\rho\mathbf{v} \times \mathbf{B}$ on a moving electric charge density, a force he determines much later as part of the derivation of the time-varying equations, in particular, Faraday's law (see Section 6.1).

In Art. 501 at the end of Chapter I of Part IV, Maxwell emphasizes that, "It must be carefully remembered, that the mechanical force which urges a conductor carrying a current across the lines of magnetic force [**B**], acts, not on the electric current, but on the conductor which carries it." The key words here are "mechanical force" as opposed to "electromotive force." Much of Maxwell's *Treatise* is based on electromotive forces acting on charges and currents (see, for example, Arts. 569–570 as well as the end of Art. 501),¶ so he is not saying here that forces do not act on charge and current. But what is he saying? Maxwell continues with, "If the conductor be a rotating disk or a fluid it will move in obedience to this force, and this motion may or may not be accompanied by a change of position of the electric current which it carries." In other words, if a moving conductor carrying current (presumably produced by a voltage source attached to the moving conductor; see Art. 579, for example) encounters an external magnetic field (**B**), it will not only experience a mechanical force but the currents may shift with respect to the moving conductor.

However, Maxwell continues next with the following statement that has perplexed readers ever since, starting with J. J. Thomson, the editor of the Third Edition of Maxwell's *Treatise* (see Thomson's footnote to Art. 501), and that led E. H. Hall to perform experiments to study the effect of magnetic fields on current carrying wires — an effect that is named after Hall [24]. "But if the current itself

¶ One of the important points that Maxwell makes in Art. 570 is that "the work done by an electromotive force is of exactly the same kind as the work done by an ordinary force .... Part of the work done by an electromotive force acting on a conducting circuit is spent in overcoming the resistance of the circuit, and this part of the work is thereby converted into heat. Another part of the work is spent in producing the electromagnetic phenomena observed by Ampère, in which conductors are made to move by electromagnetic forces. The rest of the work is spent in increasing the kinetic energy of the current, and the effects of this part of the action are shewn in the phenomena of the induction currents observed by Faraday." (He does not consider here the possibility of radiation loss but does later in Arts. 792–793.)

be free to choose any path through a fixed [as opposed to moving] solid conductor or a network of wires, then, when a constant magnetic force [**B**] is made to act on the system, the path of the current through the conductors is not permanently altered, but after certain transient phenomena [when **B** is first applied], called induction currents, have subsided, the distribution of the current will be found to be the same as if no magnetic force [**B**] were in action." In other words, Maxwell is saying that only electromotive forces change the distribution of current in a conductor.

We know from the experiments of Hall and others, as well as from our present-day understanding of the relation between charge carriers and the material of conductors, that this statement is not true. A magnetic field applied to an established current in a conductor can produce a permanent change in the distribution of the current in addition to the change in current distribution produced by the electromotive force induced by the intially time-varying applied magnetic field. But why did Maxwell believe otherwise? Given Maxwell's phenomenal powers of reason and physical intuition, it is hard to imagine that he erred in his physical reasoning. It must be then that the model of current in a conductor that Maxwell used to explain the experiments that showed that a magnetic field exerts a force on a current carrying conductor led him to the conclusion that an established conduction current in a fixed conductor would not have its distribution changed by an applied magnetic induction **B**. Indeed, one sees that this is the case by reading Maxwell's argument and analysis in Arts. 489–492 where he derives (40).

Maxwell says in Art. 489 that, "From this, applying the principal that action and reaction are equal and opposite, we conclude that the mechanical action [force] of the magnetic system [external source producing a **B** field] on a current carrying circuit is *identical* [emphasis mine] with its action [force] on a magnetic shell [thin surface layer of magnetization normal to the surface — see Art. 409] having the circuit for its edge."[+] With this identity as an hypothesis, which holds for quasi-static sources and fields where there is no time delay, Maxwell derives (40) in Arts. 489–490 by replacing the electric current I with a thin solid magnetic shell (open thin surface S with normal $\hat{\mathbf{n}}$ and magnetic moment per unit area equal to $I\hat{\mathbf{n}}$) extending to the limits of the current loop and evaluating the force exerted by the external magnetic induction **B** on the magnetic shell (or any portion of the magnetic shell) using the potential-energy W for magnetic-charge[♯] magnetic shells that he has obtained previously in Art. 410, namely

$$W = -I \int\limits_S \hat{\mathbf{n}} \cdot \mathbf{B} dS. \tag{42}$$

(If **B** is uniform, the potential energy of the entire shell can be written as $W = -\mathbf{m} \cdot \mathbf{B}$ where $\mathbf{m} = I \int_S \hat{\mathbf{n}} dS$; see also Art. 639.) He obtains the force f_x on the shell by displacing (translating without rotating) the entire shell an amount δx and noting that conservation of energy demands that

$$f_x \delta x = -I\delta W = I\delta \int\limits_S \hat{\mathbf{n}} \cdot \mathbf{B} dS. \tag{43}$$

Next he says to displace just one portion of the shell by moving an increment $d\mathbf{c}$ of its edge parallel to itself a distance δx. This slight deformation of the shell corresponds to deforming slightly the path of the original current-carrying circuit and changing the total dipole moment of the shell. (For simplicity, let the direction of δx be perpendicular to both the edge $d\mathbf{c}$ and the normal $\hat{\mathbf{n}}$.) Then $df_x \delta x = -I\delta W = I\hat{\mathbf{n}} \cdot \mathbf{B} dc \delta x$ or $df_x = I\hat{\mathbf{n}} \cdot \mathbf{B} dc$, which in vector form becomes

$$d\mathbf{f} = Id\mathbf{c} \times \mathbf{B} \tag{44}$$

[+] Maxwell has already explained in Arts. 483–484 that, in accordance with Ampère, any closed circuit of current can be divided into elementary current loops on a surface bounding the closed circuit, and that each of these elementary current loops produce an external magnetic field equal to that of a magnetic dipole moment $I\Delta S\hat{\mathbf{n}}$.

[♯] Maxwell has not yet determined the potential energy of an Amperian magnetic dipole in an external field but because he has argued that the force on the Amperian and magnetic-charge dipole in an external field has to be the same, he can use the magnetic-charge potential energy of which he is certain. Thus, Maxwell avoids the "hidden energy problem" of Amperian dipoles [25], [26, Sec. 7.4], [27, Sec. 2.1.10]. Later in Arts. 521–524 (see also Art. 637), when he determines the mutual potential energy between two current carrying circuits, he does not take into account the "hidden energy" needed to hold the current constant in each circuit and thus gets the negative of the mutual potential energy he has determined between two magnetic-charge dipoles. In Art. 637 he explains that the reason for this difference is that "the potential of the magnetic system [magnetic-charge dipoles] is single valued at every point of space, whereas that of the electric system [Amperian dipoles] is many-valued.

and which also holds (as Maxwell shows) for any other displacement direction of δx. He says this magnetic-shell force will be the same force experienced by the current-carrying conductor modeled by the magnetic shell. In other words, in order to evaluate the magnetic-induction force on an established current in a stationary circuit, Maxwell models the circuit by a thin, fixed, solid, magnetic shell and determines the force on this shell by evaluating its change in potential energy caused by an incremental displacement (local translation without rotation).* (The more general case of magnetic induction $\mathbf{B}$ applied to an established solenoidal conduction current density $\mathbf{J}$ can be modeled by a distribution of thin magnetic shells.)

Now we can understand why Maxwell would say that a magnetic force ($\mathbf{B}$) acting on previously established current in a fixed solid conductor would not change the distribution of current. It is because he assumes that the action of $\mathbf{B}$ on the fixed, solid current carrying conductor is identical to the action of $\mathbf{B}$ on a fixed, solid distribution of magnetic shells replacing the current carrying conductor. He assumes that once electric conduction current is induced on a stationary conductor by an initial electromotive force, it behaves as a distribution of spatially varying but fixed magnetic dipole moments with respect to its reaction to an applied magnetic field. Consequently, throughout the *Treatise*, he distinguishes the "mechanical force" that the magnetic induction exerts on a current carrying conductor (the $\mathbf{J}\times\mathbf{B}$ force) from the part of the electromotive force ($\mathbf{v}\times\mathbf{B}$) exerted by $\mathbf{B}$ on a moving unit of electric charge.

Recall, as mentioned above in this section and in Section 2.1, that Maxwell never expresses conduction current $\mathbf{J}$ as $\rho\mathbf{v}$ and he informs us in Art. 570 that the interaction of electricity and ordinary matter is not understood: "If we ever come to know the formal relation between electricity and ordinary matter, we shall probably also know the relation between electromotive force and ordinary [mechanical] force." Maxwell's clever idea of modeling established conduction current by magnetization allows him to determine the magnetic force on established conduction current, but that same model led him apparently to mistakenly decide that, for fixed conductors, it was more accurate to interpret the force as acting primarily on a composite of current and conductor rather than primarily on the established current which, in turn, could act on the conductor; and like the magnetization in a permanent magnet, the established current in a stationary current-carrying conductor would not change (except for transient induced currents) under the influence of an applied magnetic induction $\mathbf{B}$.

5. ENERGY AND STRESS IN ELECTROMAGNETIC FIELDS

Maxwell does not consider the energy or power supplied by time dependent fields to charge, current, and polarization. He does not derive what we refer to today as Poynting's theorem. However, he derives several expressions for electrostatic and magnetostatic energies. For example, he finds the potential energy of an electric charge in an external electric field in the first paragraph of Art. 84 and continues in Art. 84, and later in Arts. 630–631, to find the self energy (energy of formation) of a system of charges and electric polarization in terms of their potentials and electrostatic fields. In Arts. 389 and 632–633, he finds the potential energy of permanent magnetic-charge magnetization in an external magnetostatic field. For Amperian magnetic dipoles with their currents held fixed, he determines their potential energy in an external magnetostatic field in Arts. 521 and 637 (but without the energy of the electromotive force needed to hold the current fixed; see Footnote 7). The self energy of Amperian magnetic dipoles is found in Arts. 634–636.

Maxwell also obtains stress tensors (dyadics) for electrostatic and magnetostatic fields in Arts. 105–106 and 641–642, respectively.

Here in this section, we will concentrate on Maxwell's derivation for the self energies of distributions of electric current and Amperian magnetization as well as distributions of electric charge and

* By rotating (without translating) a magnetic dipole moment $\mathbf{m}$ in an external magnetic field $\mathbf{B}=\mu_0\mathbf{H}$, Maxwell found previously from the change in potential energy W in Art. 390 that the torque on the magnetic dipole was $\mathbf{m}\times\mathbf{B}$; see also Art. 639. One could also determine from this torque formula the force on an increment of the edge of a magnetic shell in a uniform field $\mathbf{B}$ by letting $\mathbf{m}=IA\hat{\mathbf{n}}$, the dipole moment of the entire planar current-carrying circuit of area A. Choosing a rectangular circuit with side lengths b and c, the torque is $Ibc\hat{\mathbf{n}}\times\mathbf{B}$. Then choosing $\mathbf{B}$ perpendicular to the sides of length c, and using symmetry to argue that the forces on the sides of length b produce no torque in the direction of $\hat{\mathbf{n}}\times\mathbf{B}$, one sees that the torque on the magnetic shell is equal to the torque exerted by the forces $\mathbf{f}$ on the sides of length c, forces that are also perpendicular to the sides of length c. This torque is equal to $b\hat{\mathbf{b}}\times\mathbf{f}$ and, thus, $\hat{\mathbf{b}}\times\mathbf{f}=Ic\hat{\mathbf{n}}\times\mathbf{B}=Ic\hat{\mathbf{b}}\times(\hat{\mathbf{c}}\times\mathbf{B})$, which implies $\mathbf{f}=Ic\,\hat{\mathbf{c}}\times\mathbf{B}$ or, with dc replacing c and $d\mathbf{f}$ replacing $\mathbf{f}$, we have the desired result $d\mathbf{f}=Id\mathbf{c}\times\mathbf{B}$.

polarization. His derivation of potential energy of permanent magnets in an external field will be reviewed and Maxwell's derivation of his stress dyadics will also be outlined.

5.1. Electrostatic and Magnetostatic Energies of Formation

The total potential energy $W_{\rm es}$ needed to bring a system of electrostatic charges from an infinite distance to their present given finite distance from the chosen origin is determined in Arts. 84–85 and Art. 630 as

$$W_{\rm es} = \frac{1}{2} \int\limits_{V_\infty} \rho_e \psi_e dV. \tag{45}$$

In Art. 630, Maxwell assumes that (45) will also hold in a dielectric continuum with $\nabla \cdot \mathbf{D} = \rho_e$ and thus

$$W_{\rm es} = \frac{1}{2} \int\limits_{V_\infty} \nabla \cdot \mathbf{D} \psi_e dV. \tag{46}$$

In Art. 631, Maxwell uses the divergence theorem and the facts that, for a finite system, ψ_e and $\mathbf{D}$ decay at least as fast as $1/r$ and $1/r^2$ as $r \to \infty$, respectively, to convert (46) to

$$W_{\rm es} = -\frac{1}{2} \int\limits_{V_\infty} \mathbf{D} \cdot \nabla \psi_e dV = \frac{1}{2} \int\limits_{V_\infty} \mathbf{D} \cdot \mathbf{E} dV. \tag{47}$$

Since Maxwell defines $\mathbf{D}$ as $\epsilon \mathbf{E}$, (47) can be rewritten as $W_{\rm es} = (1/2) \int_{V_\infty} \epsilon |\mathbf{E}|^2 dV$ (also see Art. 59), which is the correct result for the increase in electrostatic energy of an initially uncharged electrostatic system containing linear dielectrics [28, p. 111, Eq. (32)].

For nonlinear constitutive relations between $\mathbf{D}$ and $\mathbf{E}$, (47) generalizes to [28, p. 111, Eq. (31)]

$$W_{\rm es} = \int\limits_{V_\infty} \int_0^{q_e} \psi_e \delta q_e dV = \int\limits_{V_\infty} \int_0^{\mathbf{D}} \psi_e \nabla \cdot \delta \mathbf{D} dV \tag{48}$$

or

$$W_{\rm es} = \int\limits_{V_\infty} \int_0^{\mathbf{D}} \mathbf{E} \cdot \delta \mathbf{D} dV. \tag{49}$$

In nonlinear dielectric material the integral from 0 to $\mathbf{D}$ in (49) is irreversible if the dielectric displays an "hysteresis" heat loss. An easy way to verify (49) is to apply a quasi-static voltage $\mathcal{V}$ to a parallel-plate capacitor containing the local dielectric, then integrate $\mathcal{V}I = \mathcal{V}\delta q_e/\delta t$ from the initial time (when the charge q_e on the plates and all the fields are zero) to the present time t (to obtain the energy supplied to the capacitor) noting that $\mathcal{V} = Ed$ and $D = q_e A$ or $\delta D = A\delta q_e$, where E and D are the electric and displacement fields normal to the capacitor plates and d is the separation distance between the capacitor plates with area A (so that Ad is the volume of the dielectric).

If a dielectric with zero applied fields is comprised of permanent randomly oriented electric dipoles, the energy in (49) is the increase in energy over the initial electrostatic energy of the randomly oriented dipoles. The energy of formation that includes the initial energy of formation of the randomly oriented electric dipoles can be found by using Eq. (45) such that $\rho_e(\mathbf{r})$ includes the microscopic charge distribution that produces the macroscopic electric polarization. Then the microscopic displacement is given in terms of the microscopic electric field as $\mathbf{d} = \epsilon_0 \mathbf{e}$ and $W_{\rm es} = (\epsilon_0/2) \int_{V_\infty} |\mathbf{e}|^2 dV$, where $\mathbf{e}$ and $\mathbf{d}$ refer to the microscopic fields. This is the energy that can be extracted from the microscopic electric field if the extraction can be done by a process that does not dissipate some of the energy.

Note that the integration in (45) is only over the volume where $\rho_e(\mathbf{r})$ is nonzero, whereas the integration in (47) is over all space. As Maxwell states in Art. 631, "Hence, the electrostatic energy of the whole field will be the same if we suppose that it resides in every part of the field where electrical force [the electric field $\mathbf{E}$] and electrical displacement [$\mathbf{D}$] occur, instead of being confined to the places where free electricity [charge ρ_e] is found." This idea of an energy residing in the fields even if the medium is a vacuum further indicated to Maxwell that the vacuum was a medium (the ether) that physically supported electric polarization and electric current in the form of displacement current. "The energy in any part of the medium is stored up in the form of a state of constraint called electric polarization [$\mathbf{D} = \epsilon\mathbf{E}$, where $\epsilon = \epsilon_0$ in the vacuum ether], the amount of which depends on the resultant electromagnetic intensity [$\mathbf{E}$] at the place" [1, Art. 622].

To determine the magnetostatic energy of formation $W_{\rm ms}$ of a distribution of static current $\mathbf{J}$ with $\nabla \cdot \mathbf{J} = 0$, Maxwell begins by combining his expression in Art. 578 of the total potential energy of formation of a system of N closed circuits, each carrying a current I_n, in terms of the "momentum" Φ_n of the self and mutual inductances of each circuit with the expression of this momentum in Art. 590 in terms of the vector potential to obtain in Art. 634

$$W_{\rm ms} = \frac{1}{2}\sum_{n=1}^{N} I_n \Phi_n \tag{50}$$

with

$$\Phi_n = \oint\limits_{C_n} \mathbf{A} \cdot d\mathbf{c} = \int\limits_{S_n} \mathbf{B} \cdot \hat{\mathbf{n}} dS. \tag{51}$$

He continues in Art. 634 to generalize (50)–(51) to a continuous distribution of tubular circuits of continuous volume current density $\mathbf{J}$, namely

$$W_{\rm ms} = \frac{1}{2}\int\limits_{V_\infty} \mathbf{J} \cdot \mathbf{A} dV. \tag{52}$$

Since $\mathbf{J} = \nabla \times \mathbf{H}$ and $\mathbf{B} = \nabla \times \mathbf{A}$, Maxwell finds in Art. 635, with the help of the divergence theorem and the vector identity $\nabla \times (\mathbf{A} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{H})$, as well as $\mathbf{H}$ decaying as $1/r^3$ or faster as $r \to \infty$, that (52) can be rewritten as

$$W_{\rm ms} = \frac{1}{2}\int\limits_{V_\infty} \mathbf{B} \cdot \mathbf{H} dV. \tag{53}$$

It appears that in Art. 635, Maxwell assumes that (53) holds also when there is magnetization $\mathbf{M}$ present. However, one can demonstrate, by applying a quasi-static current to a permeable-core inductor, that the energy stored in the permeable material is $\int_0^t \mathcal{V} I \delta t = \int_0^{\mathcal{V}} I \delta\mathcal{V} = VH\delta B$, where H and B are the axial magnetic field and induction inside the solenoid of volume V. For arbitrary directions of $\mathbf{H}$ and $\mathbf{B}$, this expression for magnetostatic energy generalizes to [28, p. 124, Eq. (33)]

$$W_{\rm ms} = \int\limits_{V_\infty} \int_0^{\mathbf{B}} \mathbf{H} \cdot \delta\mathbf{B} dV. \tag{54}$$

This can be derived by assuming the magnetization is produced by current loops and any change in the current applies an electromotive force (voltage) to all the other current increasing the energy by $\sum_j I_j \delta\Psi_j$, where $\delta\Psi_j = \int_{S_j} \hat{\mathbf{n}} \cdot \delta\mathbf{B} dS = \oint_{C_j} \delta\mathbf{A} \cdot d\mathbf{c}$. For continuous tubes of current, the summation becomes the integral $\int_{V_\infty} \mathbf{J} \cdot \delta\mathbf{A} dV$ or because $\nabla \times \mathbf{H} = \mathbf{J}$ and $\delta\mathbf{B} = \nabla \times \delta\mathbf{A}$ we have (54) after integrating from $\mathbf{B} = 0$ to the existing value of $\mathbf{B}$. In nonlinear magnetic material the integral from 0 to $\mathbf{B}$ in (54) is irreversible if the magnetic material displays an "hysteresis" heat loss. Consequently, we see that the expression (53) for the magnetostatic energy derived by Maxwell is valid, in general, only if $\mathbf{B}$ is linearly related to $\mathbf{H}$, for example, if $\mathbf{B} = \mu\mathbf{H}$.

If permeable magnetic material with zero applied fields is comprised of permanent randomly oriented Amperian magnetic dipoles, the energy in (54) is the increase in energy over the initial magnetostatic energy of the randomly oriented dipoles. The energy of formation that includes the initial energy of formation of the randomly oriented Amperian magnetic dipoles can be found by using Eq. (52) such that $\mathbf{J}(\mathbf{r})$ includes the microscopic current distribution that produces the macroscopic magnetization. Then the microscopic induction is given in terms of the microscopic magnetic field as $\mathbf{b} = \mu_0\mathbf{h}$ and from (53) we have $W_{\rm ms} = (\mu_0/2)\int_{V_\infty} |\mathbf{h}|^2 dV$, where $\mathbf{h}$ and $\mathbf{b}$ refer to the microscopic fields.

This is the energy that can be extracted from the microscopic magnetic field if the extraction can be done by a process that does not dissipate some of the energy.

For a permanent magnetic-charge magnetic dipole m_0 placed in an external magnetic field $\mathbf{H}_{\rm ext} = -\nabla\psi_m^{\rm ext}$, Maxwell finds in Art. 389 that the energy needed to assemble the equal and opposite magnetic charges of the magnetic dipole from an infinite distance is $\mu_0\mathbf{m}_0\cdot\nabla\psi_m^{\rm ext} = -\mu_0\mathbf{m}_0\cdot\mathbf{H}_{\rm ext}$, which he rewrites for a volume element of permanent magnetization in an external magnetic field as $-\mu_0\mathbf{M}_0\cdot\mathbf{H}_{\rm ext}dV$. If the external magnetic field is that of other permanent magnets or other parts of the same permanent magnetic, this expression leads Maxwell in Art. 632 to the potential energy required to assemble a system of permanent magnetic-charge magnets, namely

$$W^0_{\rm msc} = -\frac{\mu_0}{2}\int\limits_{V_\infty}\mathbf{M}_0\cdot\mathbf{H}_0 dV. \tag{55}$$

Since $\mathbf{M}_0\cdot\mathbf{H}_0 = -\mathbf{M}_0\cdot\nabla\psi_{0m}$ and $\nabla\cdot\mathbf{B}_0 = 0 = \mu_0\nabla\cdot(\mathbf{H}_0+\mathbf{M}_0)$, Eq. (55) converts to

$$W^0_{\rm msc} = \frac{\mu_0}{2}\int\limits_{V_\infty}|\mathbf{H}_0|^2 dV \tag{56}$$

the result that Maxwell obtains in Art. 633.

To fully include the microscopic magnetostatic energy of formation of the magnetic dipoles that produce the magnetic-charge magnetization, the microscopic equations with free magnetic-charge density are required, namely $\mathbf{h} = -\nabla\psi_m$ and $\nabla\cdot\mathbf{b} = \rho_m$, so that the total magnetostatic energy of formation is (analogously to the electrostatic energy of formation)

$$W_{\rm msc} = \frac{1}{2}\int\limits_{V_\infty}\rho_m\psi_m dV = \frac{1}{2}\int\limits_{V_\infty}\nabla\cdot\mathbf{b}\psi_m dV = \frac{1}{2}\int\limits_{V_\infty}\mathbf{b}\cdot\mathbf{h}dV \tag{57}$$

or since the microscopic magnetic fields are related by $\mathbf{b} = \mu_0\mathbf{h}$, we have $W_{\rm msc} = (\mu_0/2)\int_{V_\infty}|\mathbf{h}|^2 dV$, the same form we get for microscopic Amperian magnetic dipoles, although the values of $\mathbf{h}$ for the Amperian and magnetic-charge dipoles are different. Maxwell does not include (57) or the free magnetic-charge equation $\nabla\cdot\mathbf{b} = \rho_m$ in his *Treatise*.

5.2. Electrostatic and Magnetostatic Stress Dyadics

In Art. 104 Maxwell expresses the force $\mathbf{F}_E$ on the electric charge density ρ_e in a volume V with surface S as the integral

$$\mathbf{F}_E = \int\limits_V \rho_e\mathbf{E}dV = \epsilon_0\int\limits_V\nabla^2\psi_e\nabla\psi_e dV \tag{58}$$

and then converts the second volume integral in (58) to a surface integral in order to get the electrostatic stress dyadic. Here we will shorten the derivation by using the first volume integral in (58) with $\epsilon_0\nabla\cdot\mathbf{E} = \rho_e$ and the vector-dyadic identities $(\nabla\cdot\mathbf{E})\mathbf{E} = \nabla\cdot(\mathbf{E}\mathbf{E}) - \mathbf{E}\cdot\nabla\mathbf{E}$ and $\mathbf{E}\cdot\nabla\mathbf{E} = \nabla(|\mathbf{E}|^2) = \nabla\cdot(|\mathbf{E}|^2\bar{\mathbf{I}})/2$ to convert (58) to

$$\mathbf{F}_E = \epsilon_0\int\limits_V\nabla\cdot\left(\mathbf{E}\mathbf{E} - \frac{1}{2}|\mathbf{E}|^2\bar{\mathbf{I}}\right)dV = \epsilon_0\oint\limits_S\hat{\mathbf{n}}\cdot\left(\mathbf{E}\mathbf{E} - \frac{1}{2}|\mathbf{E}|^2\bar{\mathbf{I}}\right)dS \tag{59}$$

so that $\epsilon_0(\mathbf{E}\mathbf{E} - |\mathbf{E}|^2\bar{\mathbf{I}}/2)$ is the Maxwell electrostatic stress dyadic.

Through a rather circuitous, though ingenious argument, Maxwell determines in Arts. 639–640 that the magnetostatic force $\mathbf{f}_H dV$ on a volume element of magnetic-charge magnetization $\mathbf{M}dV$ in a magnetized body that is also carrying conduction current $\mathbf{J}$ is given by

$$\mathbf{f}_H dV = [\mu_0(\nabla\mathbf{H})\cdot\mathbf{M} + \mathbf{J}\times\mathbf{B}]\,dV \tag{60}$$

or since $(\nabla\mathbf{H})\cdot\mathbf{M} = \mathbf{M}\cdot\nabla\mathbf{H} + \mathbf{M}\times(\nabla\times\mathbf{H}) = (\mathbf{M}\cdot\nabla)\mathbf{H} + \mathbf{M}\times\mathbf{J}$, it follows that

$$\mathbf{f}_H dV = \mu_0\left[(\mathbf{M}\cdot\nabla)\,\mathbf{H} + \mathbf{J}\times\mathbf{H}\right]dV. \tag{61}$$

This magnetostatic force also holds for Amperian magnetization in a magnetized body carrying conduction current because, as Maxwell states at the beginning of Art. 489, "From this, applying the principle that action and reaction are equal and opposite, we conclude that the mechanical action [force] of a magnetic system on the electric circuit is identical with its action [force] on a magnetic shell having the circuit for its edge." Since this application of Newton's law of action and reaction to magnetostatic forces on magnetic shells holds regardless of whether the magnetic dipoles are created by magnetic charge or circulating Amperian electric current, the force in (61) is the same for both magnetic-charge and Amperian magnetization.

Integration of (61) over the volume V of a current carrying magnetized body yields the total force $\mathbf{F}_H$ on the body, namely

$$\mathbf{F}_H = \mu_0 \int\limits_V [(\mathbf{M}\cdot\nabla)\mathbf{H} + \mathbf{J}\times\mathbf{H}]\, dV. \tag{62}$$

This expression is confirmed by the macroscopic and microscopic derivations of magnetostatic force given in [27, Sec. 2.1.10] provided the surface S of V lies in free space outside the magnetized body. The integrand of (62) converts to a divergence by rewriting the integrand as $[(\mathbf{B}/\mu_0 - \mathbf{H})\cdot\nabla]\mathbf{H} - \mathbf{H}\times(\nabla\times\mathbf{H}) = -\nabla\cdot(|\mathbf{H}|^2\bar{\mathbf{I}})/2 + (\mathbf{B}\cdot\nabla)\mathbf{H}/\mu_0$ and using the identity $(\mathbf{B}\cdot\nabla)\mathbf{H} = \nabla\cdot(\mathbf{BH})$ since $\nabla\cdot\mathbf{B} = 0$. Then (62) becomes

$$\mathbf{F}_H = \int\limits_V \nabla\cdot\left(\mathbf{BH} - \frac{\mu_0}{2}|\mathbf{H}|^2\bar{\mathbf{I}}\right) dV = \oint\limits_S \hat{\mathbf{n}}\cdot\left(\mathbf{BH} - \frac{\mu_0}{2}|\mathbf{H}|^2\bar{\mathbf{I}}\right) dS \tag{63}$$

which can be rewritten as simply

$$\mathbf{F}_H = \mu_0 \int\limits_V \nabla\cdot\left(\mathbf{HH} - \frac{1}{2}|\mathbf{H}|^2\bar{\mathbf{I}}\right) dV = \mu_0 \oint\limits_S \hat{\mathbf{n}}\cdot\left(\mathbf{HH} - \frac{1}{2}|\mathbf{H}|^2\bar{\mathbf{I}}\right) dS \tag{64}$$

because the surface S of V must, in general, be in free space for (63) to hold. We see from (64) that the magnetostatic stress dyadic is given by $\mu_0(\mathbf{HH} - |\mathbf{H}|^2\bar{\mathbf{I}}/2)$. Maxwell obtains the stress dyadic of Eq. (63) in Art. 641.

In view of (59) and (64), the Maxwell stress dyadic $\bar{\mathbf{T}}$ for static electromagnetic fields can be written as

$$\bar{\mathbf{T}} = \epsilon_0\left(\mathbf{EE} - \frac{1}{2}|\mathbf{E}|^2\bar{\mathbf{I}}\right) + \mu_0\left(\mathbf{HH} - \frac{1}{2}|\mathbf{H}|^2\bar{\mathbf{I}}\right). \tag{65}$$

Despite the fact that for time varying electromagnetic fields, the time rate of change of the electromagnetic momentum must be included to obtain the total force on polarized material, the static Maxwell stress dyadic in (65) remains valid for time varying electromagnetic fields as well [27, Sec. 2.1.10].

6. TIME DEPENDENT ELECTROMAGNETIC FIELDS

In Part IV of the *Treatise*, Maxwell formulates both the general expression of Faraday's law for moving circuits and the time-varying generalization of Ampère's law that includes displacement current. From Faraday's law for moving circuits, he derives what later became known as the Lorentz force on a moving electric charge. He then summarizes in vector form what he calls the general equations of the electromagnetic field. These equations are used in Ch. XX of Part IV to develop the "electromagnetic theory of light" where he obtains the wave equation and the speed of light in terms of the permittivity and permeability of free space (the ether) or of a magnetodielectric medium.

6.1. Faraday's Law

In Arts. 530–541 of Chapter III of Part IV, Maxwell explains some of Faraday's experiments by means of "primary" and "secondary" circuits that allow him to summarize in Art. 541 the "true law of magneto-electric induction [Faraday's law of induced electromotive force]" as follows: "The total electromagnetic force acting round a circuit at any instant is measured by the rate of decrease of the number of lines of magnetic force [field] which pass through it." In Chapter IV of Part IV, he explains that Faraday's experiments with a single solenoidal circuit also demonstrate a self-induced electromotive force. In Chapters V and VI of Part IV, he introduces Lagrange-Hamilton dynamics to further undergird the equations he obtains in Chapters VII–IX of Part IV for a single dynamical theory of electricity that includes self-induced and mutually induced electromotive forces. Maxwell also explains at the end of Art. 552 that he has chosen the Lagrange method because it is consistent with the methods of the rest of his *Treatise* that view electromagnetic interactions as occurring through the fields rather than by direct action at a distance. It should be emphasized, however, that Maxwell's use of Lagrange-Hamilton dynamics in his *Treatise* to further justify his mathematical formulation of Faraday's law is not essential to his determination of Faraday's law or the generalized Ampère's law containing the displacement current because he had deduced both of these laws from known experimental results in his 1861–62 four-part paper "On Physical Lines of Force" without the use of Lagrange-Hamilton dynamics.

Maxwell begins the formulation of time-varying electromagnetic-field equations per se with Chapter VII, "Theory of Electric Circuits," in Part IV of the *Treatise*. In this chapter as well as the following Chapter VIII, specifically in Arts. 578–592, he culminates a lengthy argument based on the experimental results of Ampère and Faraday with a mathematical formulation of these results in a form we recognize today as Faraday's law or Maxwell's first equation. It is most noteworthy that, although Maxwell does not include Faraday's law explicitly in his summary of equations in Art. 619 because, evidently, he decided finally to emphasize the vector and scalar potential representations of his equations,[&] he first wrote down the integral form of Faraday's law in Arts. 579 and 595 as

$$E(t) = -\frac{d}{dt}p(t) \tag{66}$$

where $E(t)$ is the line integral of the dynamic electromotive force per unit electric charge in a closed circuit that can be moving (or deforming). For a stationary circuit, E has been given in Art. 69 as $\oint_C \mathbf{E}\cdot d\mathbf{c}$. For a moving circuit, E can be written in terms of our notation $\mathbf{E}_\mathrm{v}$ as [1, Art. 598, Eq. (6)]

$$E(t) = \oint\limits_{C(t)} \mathbf{E}_\mathrm{v}(\mathbf{r},t)\cdot d\mathbf{c} \tag{67}$$

where $C(t)$ denotes the curve of the moving closed circuit and $\mathbf{E}_\mathrm{v}(\mathbf{r},t)$ is the force per unit electric charge moving with each point $\mathbf{r}$ of the circuit. (It should be noted that in Art. 579, E represents the "impressed" voltage produced by a battery in the circuit so that $E-IR$ in Art. 579 equals $-\oint_{C(t)}\mathbf{E}_\mathrm{v}\cdot d\mathbf{c}$ and thus in Art. 579 Maxwell correctly writes $E-IR = dp/dt$.)

The $p(t)$ in (66) is given in Arts. 590–591 as

$$p(t) = \oint\limits_{C(t)} \mathbf{A}(\mathbf{r},t)\cdot d\mathbf{c} = \int\limits_{S(t)} \mathbf{B}(\mathbf{r},t)\cdot\hat{\mathbf{n}}dS \tag{68}$$

with $S(t)$ any open surface bounding $C(t)$. Maxwell observes in Art. 590 that with (66) in the form of Newton's second law of motion, the vector potential $\mathbf{A}$ can be interpreted as an "electrokinetic momentum," which he later refers to as the "electromagnetic momentum." With (67) and (68) inserted into (66), we see that Maxwell has obtained the most general integral form of Faraday's law

$$\oint\limits_{C(t)} \mathbf{E}_\mathrm{v}(\mathbf{r},t)\cdot d\mathbf{c} = -\frac{d}{dt}\oint\limits_{C(t)} \mathbf{A}(\mathbf{r},t)\cdot d\mathbf{c} = -\frac{d}{dt}\int\limits_{S(t)} \mathbf{B}(\mathbf{r},t)\cdot\hat{\mathbf{n}}dS. \tag{69}$$

When he writes (66) in Art. 595, however, he has not yet shown for moving circuits that the electromotive intensity or force ($\mathbf{E}_\mathrm{v}$) on a moving unit charge is equal to what now is generally called the Lorentz force $\mathbf{E}+\mathbf{v}\times\mathbf{B}$ per moving unit electric charge, where $\mathbf{v}$ is the velocity of the moving charge.

For stationary circuits, he confirms toward the end of Art. 598 that, as in Art. 69, $E(t) = \oint_C \mathbf{E}(\mathbf{r},t)\cdot d\mathbf{c}$, where $\mathbf{E}$ is the electric field on a fixed unit electric charge defined in Art. 68 (Eq. (5)

[&] Physicists often note that quantum field theory is indebted to Maxwell's emphasis on the vector and scalar potentials in his final summary of equations in Art. 619. Also, nonlocal quantum phenomena like the Aharonov-Bohm effect are often made more palatable by invoking the electromagnetic potentials rather than the fields.

above). Consequently, Maxwell has obtained the integral form of Faraday's law for stationary circuits, namely

$$\oint_C \mathbf{E}(\mathbf{r},t)\cdot d\mathbf{c} = -\oint_C \frac{\partial}{\partial t}\mathbf{A}(\mathbf{r},t)\cdot d\mathbf{c} = -\int_S \frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t)\cdot\hat{\mathbf{n}}dS. \tag{70}$$

Application of Stoke's theorem to (70) yields the differential form of Faraday's law

$$\nabla\times\mathbf{E}(\mathbf{r},t) = -\nabla\times\frac{\partial}{\partial t}\mathbf{A}(\mathbf{r},t) = -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t) \tag{71}$$

although Maxwell does not write this differential form of Faraday's law in his *Treatise* (nor in his 1864–65 *Phil. Trans. Roy. Soc. London* paper entitled "A Dynamic Theory of the Electromagnetic Field," which contains only the integral form of Faraday's law).^ The first equation in (71) implies that

$$\mathbf{E}(\mathbf{r},t) = -\frac{\partial}{\partial t}\mathbf{A}(\mathbf{r},t) - \nabla\psi_e(\mathbf{r},t) \tag{72}$$

where $\psi_e(\mathbf{r},t)$ is a time-dependent as well as a spatially dependent scalar potential function. In Art. 598 Maxwell says that $\psi_e(\mathbf{r},t)$ "represents, according to a certain definition, the *electric potential*," which he later says in Art. 783 "is proportional to the volume charge density of free electricity [$\rho_e(\mathbf{r},t)$]." He does not assume that $\nabla\cdot\mathbf{A} = 0$ when he makes these statements in these articles (see Sections 6.2 and 6.4 below).

Returning to Maxwell's general integral form of Faraday's law for moving circuits in (66), expressed more fully in (69), we find in Art. 598 Maxwell's ingenious evaluation of $-(d/dt)\oint_{C(t)}\mathbf{A}(\mathbf{r},t)\cdot d\mathbf{c}$ to prove that $\mathbf{E}_{\mathrm{v}}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t)+\mathbf{v}(\mathbf{r},t)\times\mathbf{B}(\mathbf{r},t)$, where $\mathbf{v}(\mathbf{r},t)$ is the velocity at each point $\mathbf{r}$ of the moving circuit $C(t)$. He thus completes the formulation of Faraday's integral law for moving circuits, and in so doing has derived the force exerted on a moving unit electric charge by the magnetic induction $\mathbf{B}$. Maxwell accomplishes this feat as follows.

He writes $\mathbf{A}(\mathbf{r},t)\cdot d\mathbf{c}$ in rectangular coordinates as

$$\mathbf{A}(\mathbf{r},t)\cdot d\mathbf{c} = A_x\frac{dx}{dc}dc + A_y\frac{dy}{dc}dc + A_z\frac{dz}{dc}dc \tag{73}$$

with $dc = |d\mathbf{c}| = |d\mathbf{r}|$. In taking the time derivative of $\oint_{C(t)}\mathbf{A}(\mathbf{r},t)\cdot d\mathbf{c}$, the time dependence of the integration curve $C(t)$ can be ignored if the position vector $\mathbf{r} = x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}}$ is assumed to be a function of time such that x, y, and z are functions of c and t. (The scalar differential element dc of the scalar integration variable c cancels in each term of (73) so its time dependence can be ignored; to see this more clearly, one can change the integration variable from c to $c' = c/c_{\max}$ at each time t so that the integration over c' ranges from 0 to 1 at each time t.) Then the time derivative can be taken inside the integral to get

$$\begin{aligned}\frac{d}{dt}\oint_{C(t)}\mathbf{A}(\mathbf{r},t)\cdot d\mathbf{c} &= \oint_C \frac{d}{dt}\left[A_x\left(\mathbf{r}(t),t\right)\frac{\partial x}{\partial c}+\ldots\right]dc \\ &= \oint_C\left[\frac{\partial}{\partial t}A_x(\mathbf{r},t)\frac{\partial x}{\partial c}+A_x(\mathbf{r},t)\frac{\partial^2 x}{\partial c\partial t}+\frac{\partial A_x}{\partial c}\frac{\partial x}{\partial t}+\frac{\partial A_x}{\partial y}\frac{\partial y}{\partial t}\frac{\partial x}{\partial c}+\frac{\partial A_x}{\partial z}\frac{\partial z}{\partial t}\frac{\partial x}{\partial c}+\ldots\right]dc\end{aligned} \tag{74}$$

where $\frac{\partial A_x}{\partial x}\frac{\partial x}{\partial c}$ has been set equal to $\frac{\partial A_x}{\partial c}$. The partial derivatives with respect to t and (x,y,z,c) are taken holding c and t fixed, respectively. (Fixing c at any one time t also fixes x, y, and z.) Since

^ We know that Maxwell deliberately chose to emphasize the integral form of Faraday's law in his *Treatise* and 1864–65 *Phil. Trans. Roy. Soc. London* paper since he had deduced the differential form of this law from his "theory of molecular vortices" that he used in his 1861–62 four-part *Philosophical Magazine* paper entitled "On Physical Lines of Force" to explain Faraday's experimental results. In Part II (April 1861) of that paper, which contained no integrals, he wrote the scalar version of $\nabla\times\mathbf{E} = -\mu\partial\mathbf{H}/\partial t$ as Eq. (54). Faraday's law is not contained in Maxwell's earlier 1855–56 *Cambridge Phil. Trans.* paper entitled "On Faraday's Lines of Force" in either the integral or differential form.

$\frac{\partial A_x}{\partial c}\frac{\partial x}{\partial t} + A_x\frac{\partial^2 x}{\partial c \partial t} = \frac{\partial}{\partial c}(A_x\frac{\partial x}{\partial t})$ is a perfect differential, its integral around the closed curve C is zero, reducing (74) to

$$\begin{aligned}\frac{d}{dt}\oint\limits_{C(t)} \mathbf{A}(\mathbf{r},t)\cdot d\mathbf{c} &= \oint\limits_{C}\left[\frac{\partial}{\partial t}A_x(\mathbf{r},t)\frac{\partial x}{\partial c} + \frac{\partial A_x}{\partial y}\frac{\partial y}{\partial t}\frac{\partial x}{\partial c} + \frac{\partial A_x}{\partial z}\frac{\partial z}{\partial t}\frac{\partial x}{\partial c} + \ldots\right]dc \\ &= \oint\limits_{C}\left[\frac{\partial}{\partial t}\mathbf{A}(\mathbf{r},t) - \mathbf{v}(\mathbf{r},t)\times\big(\nabla\times\mathbf{A}(\mathbf{r},t)\big)\right]\cdot d\mathbf{c}. \end{aligned} \tag{75}$$

Inserting $\mathbf{B} = \nabla\times\mathbf{A}$ into (75), and the resulting equation into (69), Maxwell has proven that

$$\oint\limits_{C(t)} \mathbf{E}_{\mathrm{v}}(\mathbf{r},t)\cdot d\mathbf{c} = -\oint\limits_{C(t)}\left[\frac{\partial}{\partial t}\mathbf{A}(\mathbf{r},t) - \mathbf{v}(\mathbf{r},t)\times\mathbf{B}(\mathbf{r},t)\right]\cdot d\mathbf{c} \tag{76}$$

and, thus, he concludes that

$$\mathbf{E}_{\mathrm{v}}(\mathbf{r},t) = -\frac{\partial}{\partial t}\mathbf{A}(\mathbf{r},t) - \nabla\psi_e(\mathbf{r},t) + \mathbf{v}(\mathbf{r},t)\times\mathbf{B}(\mathbf{r},t) \tag{77}$$

or in our present-day notation, in accordance with (72)

$$\mathbf{E}_{\mathrm{v}}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t) + \mathbf{v}(\mathbf{r},t)\times\mathbf{B}(\mathbf{r},t) \tag{78}$$

since $\mathbf{E}_{\mathrm{v}}$ in (77) with $\mathbf{v} = 0$ has to equal $\mathbf{E}$ in (72). It should be emphasized that Maxwell uses Faraday's law to obtain (78) and not the current force $\mathbf{J}\times\mathbf{B}$ in (41) that he has found from the force on a magnetic-shell model of circulating current.

Thus, Maxwell has been able to represent Faraday's experimental results in the general form of "Faraday's law" given in (69) with $\mathbf{E}_{\mathrm{v}}$ given in (78) [1, Arts. 598–599]. *In one magnificent synthesis of mathematical and physical insight, he has not only put Faraday's law on a solid mathematical foundation but he has also derived the Lorentz force for a moving electric charge.* (Recall, as explained in Section 3, that Maxwell uses the same symbol $\mathfrak{E}$ for both the symbols $\mathbf{E}_{\mathrm{v}}$ and $\mathbf{E}$ that we use here, and when Maxwell denotes $\mathbf{E}_{\mathrm{v}}$ by $\mathfrak{E}$ he expects the reader to know from the context that $\mathfrak{E}$ with $\mathbf{v} = 0$ is the electric field defined in Arts. 44 and 68 and given here as $\mathbf{E}$ in Eq. (5).)

It is also possible to prove (69) and (77)–(78) from (70) using the Helmholtz transport theorem [29, Ch. 6] of vector calculus, but Maxwell does not do this even though he mentions Helmholtz's work with moving circuits in Art. 544. Effectively, he proves the Helmholtz transport theorem for the electromagnetic fields in Faraday's law as part of his derivation reproduced above in (74)–(78).

After deriving (78) from (69) in Art. 598, he says that $\mathbf{E}_{\mathrm{v}}$ in (78) is the most general form of the electromotive force on a moving particle of unit positive charge and that this is the force that produces the current in a *moving* conductor or dielectric, that is, the force that should be used in Ohm's law for a conductor moving with velocity $\mathbf{v}$ or in the constitutive relation involving permittivity in a dielectric moving with velocity $\mathbf{v}$. We shall further discuss these constitutive relations for moving media with regard to (86) and (87) below.

6.2. Generalized Ampère's Law and the Coulomb Gauge

Maxwell's extraordinary deduction in Art. 607 that generalizes Ampère's Law to time dependent fields is repeated above in Section 2.1.2 and the resulting "Maxwell second equation" is given in (4). Maxwell uses this equation to show in Art. 616 that it is permissible to choose what today is commonly called the Coulomb gauge, that is, $\nabla\cdot\mathbf{A} = 0$. His proof begins by inserting $\mathbf{B} = \nabla\times\mathbf{A}$ into (4) to obtain

$$-\nabla\times\nabla\times\mathbf{A} = \nabla^2\mathbf{A} - \nabla(\nabla\cdot\mathbf{A}) = -\mu_0(\mathbf{J}_T + \nabla\times\mathbf{M}) \equiv -\mu_0\mathbf{J}_0. \tag{79}$$

Note that $\nabla\cdot\mathbf{J}_0 = 0$. (Maxwell actually assumes in his proof that $\mathbf{B} = \mu\mathbf{H}$ in (4) but I have used the general constitutive relation $\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M})$ in (4) to obtain (79).) He then defines a vector $\mathbf{A}_0$ and a scalar ϕ as

$$\mathbf{A}_0(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}_0(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}dV' \tag{80a}$$

$$\phi(\mathbf{r},t) = \frac{1}{4\pi}\int_V \frac{\nabla'\cdot\mathbf{A}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}dV' \tag{80b}$$

and observes that $\mathbf{A} = \mathbf{A}_0 - \nabla\phi$ satisfies (79) because $\nabla^2\mathbf{A}_0 = -\mu_0\mathbf{J}_0$, $\nabla\cdot\mathbf{A}_0 = 0$ (since $\nabla\cdot\mathbf{J}_0 = 0$), and $\nabla^2\phi = -\nabla\cdot\mathbf{A}$. He completes the proof by explaining that since $\mathbf{B} = \nabla\times\mathbf{A} = \nabla\times\mathbf{A}_0$, the scalar ϕ "is not related to any physical phenomena" and can be omitted so that $\mathbf{A} = \mathbf{A}_0$ and thus $\nabla\cdot\mathbf{A} = \nabla\cdot\mathbf{A}_0 = 0$.

In the following Art. 617, Maxwell qualifies his proof of $\nabla\cdot\mathbf{A} = 0$ and $\nabla^2\mathbf{A} = -\mu_0\mathbf{J}_0$ by noting that it assumes $\mathbf{A}$ "vanishes at an infinite distance" (fast enough for the solution in (80a) to exist). For time dependent fields that are identically zero beyond a finite distance from the sources, $\mathbf{A}$ can always be made to vanish at an infinite distance. However, for infinite periodic waves, one cannot assume $\mathbf{A}$ vanishes at an infinite distance and, indeed when Maxwell uses his equations for infinite periodic disturbances to arrive at the speed of light in Arts. 783–784, he does not assume initially that $\nabla\cdot\mathbf{A} = 0$ (see Section 6.4 below).

6.3. Maxwell's Summary of the "General Equations of the Electromagnetic Field" in Vector Form

In Art. 619 Maxwell summarizes in vector form the major "general equations of the electromagnetic field" that he has determined in his *Treatise*. He begins in Art. 618 with a list of the "principal vectors and scalars" as well as their names that are required in these equations. I will list the symbols for these vectors and scalars as he does along with their more familiar present-day replacements and names (used in this paper):

		Vectors	
$\boldsymbol{\rho}$	$\rightarrow$	$\mathbf{r}$	**position vector of a point**
$\mathfrak{A}$	$\rightarrow$	$\mathbf{A}$	**vector potential**
$\mathfrak{B}$	$\rightarrow$	$\mathbf{B}$	**magnetic induction vector**
$\mathfrak{C}$	$\rightarrow$	$\mathbf{J}_T = \mathbf{J} + \dot{\mathbf{D}}$	**total (conduction + displacement) current**
$\mathfrak{D}$	$\rightarrow$	$\mathbf{D}$	**electric displacement vector**
$\mathfrak{E}$	$\rightarrow$	$\mathbf{E}_\mathrm{v} = \mathbf{E} + \mathbf{v}\times\mathbf{B}$	**electromotive force (electric field if v = 0)**
$\mathfrak{F}$	$\rightarrow$	$\mathbf{F}$	**force density on polarized material**
$\mathfrak{G}$	$\rightarrow$	$\mathbf{v}$	**velocity of a point**
$\mathfrak{H}$	$\rightarrow$	$\mathbf{H}$	**magnetic field**
$\mathfrak{I}$	$\rightarrow$	$\mathbf{M}$	**magnetization (magnetic polarization)**
$\mathfrak{K}$	$\rightarrow$	$\mathbf{J}$	**conduction current density**
		Scalars	
Ψ	$\rightarrow$	ψ_e	**electric potential**
Ω	$\rightarrow$	ψ_m	**magnetic potential**
e	$\rightarrow$	ρ_e	**electric charge density**
m	$\rightarrow$	ρ_m	**magnetic charge density**
C	$\rightarrow$	σ	**electric conductivity**
K	$\rightarrow$	ϵ	**permittivity**
μ	$\rightarrow$	μ	**permeability**

All of the vectors as well as the electric and magnetic potentials can be functions of position $\mathbf{r}$ and time t. Maxwell also says in Art. 618 that σ, ϵ, and μ can be functions of $\mathbf{r}$ and also linear vector operators (tensors or, equivalently, dyadics); see also Arts. 298, 101e, 428, 608, 609, and 794. In addition, he says that "ϵ and μ are certainly always self-conjugate [symmetric/reciprocal] and σ is probably so also," although he considers and writes out the completely general nonsymmetric dyadics for σ and ϵ in Arts. 298 and 101e, and discusses at length nonreciprocal Faraday rotation in certain media caused by a magnetic field applied in the direction of propagation [1, Ch. XXI, Pt. IV]. He informs the reader in Art. 101e that in some media such as glass, the permittivity ϵ can be temporally dispersive (or, equivalently, frequency dispersive). Maxwell does not consider spatial dispersion.

Once Maxwell has given this review list of his vector and scalar symbols, he proceeds in In Art. 619 to summarize what he considers the major equations that he has developed in the preceding articles of his *Treatise*, equations about which Einstein would write, "This change in conception of reality is the most profound and the most fruitful that physics has experienced since the time of Newton" [12, p. 71]. During one of Einstein's visits to Cambridge University in the 1920s, someone commented that he had done great work by standing on Newton's shoulders. Einstein replied, "No, I stand on the shoulders of Maxwell" [30, p. 24]. I will write these equations with today's more commonly used symbols defined in the above table in the order in which they appear in Art. 619, commenting on each of them sequentially.

He begins with the "equation of magnetic induction" expressing $\mathbf{B}$ as the curl of the vector potential $\mathbf{A}$

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{81}$$

and reminds the reader that $\mathbf{A}$ "is subject to the condition $\nabla \cdot \mathbf{A} = 0$." Earlier in Art. 617, as we have mentioned above, he qualifies the use of this Coulomb gauge to vector potentials that vanish at infinity. In different parts of the *Treatise*, Maxwell refers to $\mathbf{A}$ as the vector potential, the electrokinetic momentum, or the electromagnetic momentum.

The "equation of electromotive force" he writes as

$$\mathbf{E}_{\mathrm{v}} = \mathbf{v} \times \mathbf{B} - \dot{\mathbf{A}} - \nabla\psi_e \tag{82}$$

or, using our symbol $\mathbf{E}$ for what Maxwell has shown in (72) is equal to $-\dot{\mathbf{A}} - \nabla\psi_e$, the electric field on a stationary charge, we have $\mathbf{E}_{\mathrm{v}} = \mathbf{v} \times \mathbf{B} + \mathbf{E}$, the more familiar form of the Lorentz force on a moving unit electric charge. In other words, Maxwell's equation of electromotive force is identical to what has become known as the Lorentz force equation for a moving unit electric charge. Maxwell notes in Art. 601 that in the closed integral for the total electromotive force, $E = \oint_C \mathbf{E}_{\mathrm{v}} \cdot d\mathbf{c}$, the scalar potential function ψ_e disappears.

The "equation of mechanical force" that gives the force per unit volume on material carrying charge, current, and polarization is expressed by Maxwell as

$$\mathbf{F} = \mathbf{J}_T \times \mathbf{B} - \rho_e \nabla\psi_e - \rho_m \nabla\psi_m \tag{83a}$$

where we recall that $\mathbf{J}_T = \mathbf{J} + \dot{\mathbf{D}}$. Maxwell's use of the scalar potentials for the electric and magnetic fields in (83a) is somewhat puzzling. Even if we assume that the electric charge density ρ_e and the equivalent magnetic charge density $\rho_m = -\mu_0 \nabla \cdot \mathbf{M}$ are fixed to a body with zero velocity, Maxwell generally allows $\mathbf{E}$ and $\mathbf{H}$ to be functions of time and says, for example, in Art. 604 that the displacement current $\dot{\mathbf{D}}$ can contribute to the mechanical force. Therefore, it seems it would have been more appropriate for Maxwell to have used the electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$ in (83a) instead of $-\nabla\psi_e$ and $-\nabla\psi_m$, respectively, which apply to irrotational (and usually static) fields.[≀]

These considerations led J. J. Thomson, in response to G. F. Fitzgerald, to change $-\nabla\psi_e$ to $\mathbf{E}_{\mathrm{v}}$ in the Third Edition of the *Treatise* [1, footnote in curly brackets, Art. 619]. Since Maxwell always uses $\mathbf{J}$ rather than $\rho_e \mathbf{v}$, I will change $-\nabla\psi_e$ to $\mathbf{E}$ (instead of $\mathbf{E}_{\mathrm{v}}$) and also $-\nabla\psi_m$ to $\mathbf{H}$ before discussing the mechanical force equation further. Then (83a) becomes

$$\mathbf{F} = \left(\mathbf{J} + \dot{\mathbf{D}}\right) \times \mathbf{B} + \rho_e \mathbf{E} - \mu_0 (\nabla \cdot \mathbf{M})\mathbf{H} \tag{83b}$$

where we have substituted $-\mu_o \nabla \cdot \mathbf{M}$ for ρ_m. The magnetic force on bodies carrying conduction current and magnetization that Maxwell uses in (83b) is $\mathbf{J} \times \mathbf{B} - \mu_0 (\nabla \cdot \mathbf{M})\mathbf{H}$. He modifies this magnetic force

[≀] Maxwell was certainly aware of the additional conditions imposed by the use of the scalar potentials in (83a) because he emphasizes in Art. 619 that the magnetic scalar potential Eq. (92) requires that "the magnetic force [field] can be derived from a potential," and states in Art. 646 in his discussion of mechanical force and stress that "In a field in which electrostatic as well a electromagnetic action is taking place, we must suppose the electrostatic stress [and mechanical force] described in Part I [which uses the electric scalar potential] to be superimposed on the electromagnetic stress [and mechanical force] which we have been considering." This would explain his use of the scalar potentials for $\mathbf{E}$ and $\mathbf{H}$ in (83a), but not entirely justify it.

later in Art. 640 to magnetized bodies that can also carry electric current and obtains $\mathbf{J}\times\mathbf{B}+\mu_0(\nabla\mathbf{H})\cdot\mathbf{M}$; see (60) above. Evidently, Maxwell is acknowledging that (83b) needs to be modified if the current and magnetization occur together in the same part of the body. Since $(\nabla\mathbf{H})\cdot\mathbf{M} = (\mathbf{M}\cdot\nabla)\mathbf{H} - (\nabla\times\mathbf{H})\times\mathbf{M}$ and $\nabla\times\mathbf{H} = \mathbf{J}$ in conduction current, this modification would change (83b) to

$$\mathbf{F} = \mu_0\left(\mathbf{J} + \dot{\mathbf{D}}\right)\times\mathbf{H} + \rho_e\mathbf{E} + \mu_0(\mathbf{M}\cdot\nabla)\mathbf{H}. \tag{83c}$$

To facilitate our discussion of (83a)–(83c), we can write down the generally accepted, present-day form of the mechanical force density in a nondiamagnetic polarizable continua with electric charge densities ρ_e and $\rho_m = -\mu_0\nabla\cdot\mathbf{M}$, namely [27, Sec. 2.1.10]

$$\mathbf{F} = \mu_0\left(\mathbf{J} + \dot{\mathbf{P}}\right)\times\mathbf{H} + \left[\rho_e + (\mathbf{P}\cdot\nabla)\right]\mathbf{E} + \mu_0(\mathbf{M}\cdot\nabla)\mathbf{H} - \mu_0\epsilon_0\dot{\mathbf{M}}\times\mathbf{E}. \tag{83d}$$

Comparing (83c) with (83d) helps us understand how Maxwell arrived at his "equation of mechanical force." As explained in Section 2.1.2, he did not define a polarization vector $\mathbf{P}$ and viewed $\mathbf{J}+\dot{\mathbf{D}}$ rather than $\mathbf{J}+\dot{\mathbf{P}}$ as the total equivalent electric current density in a dielectric. Without $\mathbf{P}$ he had no concept of $-\nabla\cdot\mathbf{P}$ or $\mathbf{P}\cdot\nabla$ as producing mechanical electric force apart from $\nabla\cdot\mathbf{D} = \rho_e$. Moreover, he did not consider equivalent magnetic current density $\dot{\mathbf{M}}$ in his *Treatise* and, thus, did not include this term in (83c). These three changes account for the differences between Maxwell's and the present-day "equation of mechanical force" in (83a)–(83c) and (83d), respectively.

The "equation of magnetization"

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) \tag{84}$$

is the defining equation for the secondary magnetic vector, the magnetic induction $\mathbf{B}$, in terms of the primary magnetic field $\mathbf{H}$ and the magnetization $\mathbf{M}$. In Maxwell's *Treatise*, the $\mathbf{H}$ field is primary because magnetization $\mathbf{M}$ is defined by magnetic charge separation; see Sections 2.2 and 4.2.

The "equation of electric currents"

$$\mathbf{J}_T = \nabla\times\mathbf{H} \tag{85a}$$

or with $\mathbf{J}_T$ written out from (88) below

$$\mathbf{J} + \dot{\mathbf{D}} = \nabla\times\mathbf{H} \tag{85b}$$

is the generalization of Ampère's law to include displacement current $\dot{\mathbf{D}}$; see Sections 2.1.2 and 6.2. It, along with Faraday's law, couples the electric and magnetic time varying fields and led to Maxwell's discovery of electromagnetic wave propagation; see Section 6.4 below.

Next Maxwell says that the "equation of current of conduction is by Ohm's law"

$$\mathbf{J} = \sigma\mathbf{E}_{\mathrm{v}} \tag{86}$$

and the "equation of electric displacement" is

$$\mathbf{D} = \epsilon\mathbf{E}_{\mathrm{v}}. \tag{87}$$

As mentioned above, Maxwell allows both the conductivity σ and the permittivity ϵ to be functions of position $\mathbf{r}$ and for them to be replaced by dyadics. The most distinguishing feature of the constitutive relations in (86) and (87) is the occurrence of $\mathbf{E}_{\mathrm{v}} = \mathbf{E} + \mathbf{v}\times\mathbf{B}$ where here $\mathbf{v}$ is the velocity of a moving conductor or dielectric, respectively. This is not a mistake. Maxwell clearly explains in Art. 598 that in a conductor or dielectric moving with velocity $\mathbf{v}$, the magnetic-field electromotive force $\mathbf{v}\times\mathbf{B}$ will also produce current and electric displacement, respectively. Today we know that these moving conductor and moving dielectric constitutive equations in (86) and (87) are invalid at relativistic velocities but hold to a good approximation for $(v/c)^2 \ll 1$. Maxwell assumed Galilean transformations for $(\mathbf{r}, t)$, and thus implicitly for $\mathbf{J}$ and $\mathbf{D}$, in his equations [1, Art. 600] because he assumed that his equations held with respect to an invisible pervasive medium which Faraday, Maxwell, and others called the ether [1, Arts. 782 and 865–866].

"The equation of the total current, arising from the variation of the electric displacement as well as from conduction, is

$$\mathbf{J}_T = \mathbf{J} + \dot{\mathbf{D}}.\text{''} \tag{88}$$

Maxwell emphasizes the importance of this result by listing it as a separate equation.

"When the magnetization [$\mathbf{M}$] arises from magnetic induction [that is, linearly induced by $\mathbf{H}$]

$$\mathbf{B} = \mu\mathbf{H}\text{"} \tag{89}$$

where, as mentioned above, Maxwell allows the permeability μ to be a function of position $\mathbf{r}$ and also to be replaced by a dyadic.

Maxwell lists his divergence equation for the electric charge density as

$$\rho_e = \nabla \cdot \mathbf{D}. \tag{90}$$

There is an erroneous minus sign in this equation in Art. 619 of the *Treatise* that is corrected in (90). (There is no doubt that it is an unintentional writing or typing error because the sign is correct in Art. 612 and everywhere else this equation appears in the *Treatise*.)

Maxwell writes the divergence equation for magnetic charge as

$$\rho_m = -\mu_0 \nabla \cdot \mathbf{M} \tag{91}$$

which can be rewritten as $\rho_m = \mu_0 \nabla \cdot \mathbf{H}$ because $\nabla \cdot \mathbf{B} = \mu_0 \nabla \cdot (\mathbf{H} + \mathbf{M}) = 0$. This divergence equation is actually a definition of ρ_m in terms of $\mathbf{M}$ because, unlike electric charge density, there is no free magnetic charge density (magnetic poles always come in equal and opposite pairs). If there were free magnetic charge density ρ_m, then $\nabla \cdot \mathbf{B}$ would equal ρ_m rather than 0 (see end of Section 5.1).

For Maxwell's last equation in his list of equations in Art. 619, he writes, "When the magnetic force [field] can be derived from a potential

$$\mathbf{H} = -\nabla\psi_m.\text{"} \tag{92}$$

In all, Maxwell lists 12 equations for his "general equations of the electromagnetic fields" in the summarizing Art. 619. Combining the definition of $\mathbf{J}_T$ in (88) with (85a) as we did above to get (85b), there are four major dynamic equations for the electromagnetic fields:

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{93a}$$

$$\mathbf{E}_{\mathrm{v}} = \mathbf{v} \times \mathbf{B} - \dot{\mathbf{A}} - \nabla\psi_e \quad (\mathbf{E} = -\dot{\mathbf{A}} - \nabla\psi_e) \tag{93b}$$

$$\mathbf{J} + \dot{\mathbf{D}} = \nabla \times \mathbf{H} \tag{93c}$$

$$\nabla \cdot \mathbf{D} = \rho_e. \tag{93d}$$

There are three definitions:

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) \tag{94a}$$

$$\mathbf{J}_T = \mathbf{J} + \dot{\mathbf{D}} \tag{94b}$$

$$\rho_m = -\mu_0 \nabla \cdot \mathbf{M} \tag{94c}$$

and three constitutive relations:

$$\mathbf{J} = \sigma\mathbf{E}_{\mathrm{v}} \tag{95a}$$

$$\mathbf{D} = \epsilon\mathbf{E}_{\mathrm{v}} \tag{95b}$$

$$\mathbf{B} = \mu\mathbf{H}. \tag{95c}$$

One can set the velocity of the conductor in (95a) and that of the dielectric in (95b) equal to zero to get the more familiar form of these constitutive relations for stationary bodies in terms of $\mathbf{E}$ rather than $\mathbf{E}_{\mathrm{v}}$.

The remaining two equations are the equation of the mechanical force in (83) and the equation for the magnetic field in (92) when it is irrotational so that it can be written as the gradient of a potential function.

Taking the curl of the equation for $\mathbf{E}$ in (93b) and inserting $\mathbf{B}$ from (93a) immediately recovers Faraday's law, the integral form of which Maxwell has deduced, written out, and discussed extensively in previous articles of his *Treatise* but does not include explicitly in this final list in Art. 619; see Section 6.1 above. Substitution of Faraday's law and $\nabla \cdot \mathbf{B} = 0$, derivable from (93a), for the Eq. (93a) and the $\mathbf{E}$ equation in (93b), one obtains the four equations that today are universally accepted as Maxwell's differential equations:

$$\nabla \times \mathbf{E} + \dot{\mathbf{B}} = 0 \tag{96a}$$
$$\nabla \times \mathbf{H} - \dot{\mathbf{D}} = \mathbf{J} \tag{96b}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{96c}$$
$$\nabla \cdot \mathbf{D} = \rho_e. \tag{96d}$$

The symmetry that can be extended to these Maxwellian equations by including a $-\mathbf{J}_m$ and ρ_m on the right-hand sides of (96a) and (96c), respectively, was not apparently considered by Maxwell but was later introduced by Heaviside [31].

6.4. Electromagnetic Theory of Light

Chapter XX of Part IV entitled "Electromagnetic Theory of Light" may be considered the culmination of Maxwell's *Treatise*. As Maxwell says in the first article [781] of this chapter, "If it should be found that the velocity of propagation of electromagnetic disturbances is the same as the velocity of light, and this not only in air, but in other transparent media, we shall have strong reasons for believing that light is an electromagnetic phenomenon, ..." He proves that this is indeed the case by beginning in Art. 783 with the total current in a uniform, stationary, electrically conductive (σ) magnetodielectric (μ, ϵ) medium, which can be free space (ether), namely

$$\mathbf{J}_T = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} = \left(\sigma + \epsilon\frac{\partial}{\partial t}\right)\mathbf{E} = -\left(\sigma + \epsilon\frac{\partial}{\partial t}\right)\left(\frac{\partial \mathbf{A}}{\partial t} + \nabla\psi_e\right) \tag{97}$$

since

$$\mathbf{E} = -\left(\frac{\partial \mathbf{A}}{\partial t} + \nabla\psi_e\right). \tag{98}$$

Using his generalization of Ampère's law, he then also writes $\mathbf{J}_T$ as

$$\mathbf{J}_T = \nabla \times \mathbf{H} = \nabla \times \mathbf{B}/\mu = \nabla \times \nabla \times \mathbf{A}/\mu \tag{99}$$

so that

$$\mu\mathbf{J}_T = -\nabla^2\mathbf{A} + \nabla(\nabla \cdot \mathbf{A}). \tag{100}$$

Note that in (97) and thus in (99)–(100) there are no applied currents, at least during the time, say $t > 0$, being considered. Also, at this point in the derivation, Maxwell does not assume the "Coulomb gauge," $\nabla \cdot \mathbf{A} = 0$, as he does elsewhere in the *Treatise* because he does not assume a priori that for periodic disturbances $\mathbf{A}$ will approach zero at an infinite distance. (See his derivation of $\nabla \cdot \mathbf{A} = 0$ for time varying fields in Art. 616, repeated above in Section 6.2.)

Equating $\mathbf{J}_T$ in (97) and (99), Maxwell finds

$$\mu\left(\sigma + \epsilon\frac{\partial}{\partial t}\right)\left(\frac{\partial \mathbf{A}}{\partial t} + \nabla\psi_e\right) - \nabla^2\mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) = 0. \tag{101}$$

To prove that he can let $\nabla \cdot \mathbf{A} = 0$, he takes the divergence of (101) and assumes the medium is now a nonconductor so that $\sigma = 0$, to get

$$\frac{\partial}{\partial t}\left[\frac{\partial(\nabla \cdot \mathbf{A})}{\partial t} + \nabla^2\psi_e\right] = 0. \tag{102}$$

He then says that with $\sigma = 0$ (that is, $\mathbf{J} = 0$), the volume charge density ρ_e is independent of time (because of the continuity equation) and thus $\nabla^2\psi_e = -\rho_e/\epsilon$ is independent of time, so that ψ_e is independent of time.$^\triangleright$ This implies from (102) that $\nabla \cdot \mathbf{A} = c_1 t + c_0$ where c_1 and c_2 are constants.

$^\triangleright$ Maxwell has been criticized, apparently first by G. F. FitzGerald [5, pp. 114–119], at this juncture in his derivation for assuming $\nabla^2\psi_e = -\rho_e/\epsilon$ before showing that $\nabla \cdot \mathbf{A} = 0$. This criticism is unwarranted, however, because from the divergence of (98), one obtains $-\rho_e/\epsilon = \partial(\nabla \cdot \mathbf{A})/\partial t + \nabla^2\psi_e$, and since the gradient $\nabla\phi$ of any function $\phi(\mathbf{r}, t)$ can be added to $\mathbf{A}(\mathbf{r}, t)$ without changing the fields, the new gradient function can be made to satisfy $\nabla^2(\partial\phi/\partial t + \psi_e) = -\rho_e/\epsilon$ by choosing $\phi(\mathbf{r}, t) = -\int^t \psi_e(\mathbf{r}, t')dt' + t\int_V [\rho_e(\mathbf{r}')/|\mathbf{r} - \mathbf{r}'|]dV'/(4\pi\epsilon)$.

Since Maxwell is considering only "periodic disturbances," he concludes that $\nabla \cdot \mathbf{A} = 0$ and (101) reduces to the homogeneous time-dependent equations

$$\nabla^2 \mathbf{A} - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = 0, \quad \nabla \cdot \mathbf{A} = 0 \tag{103}$$

with

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}. \tag{104}$$

The scalar potential ψ_e is omitted in the electric field equation of (104) because a time-independent ψ_e does not contribute to a periodic disturbance.

One obvious set of solutions to (103) are the plane waves

$$\mathbf{A} = \mathbf{A}_0 \mathrm{f}\left(t/\sqrt{\mu\epsilon} - \hat{\mathbf{n}} \cdot \mathbf{r}\right), \quad \hat{\mathbf{n}} \cdot \mathbf{A}_0 = 0 \tag{105}$$

with

$$\mathbf{B} = \nabla \times \mathbf{A} = -\hat{\mathbf{n}} \times \mathbf{A}_0 \mathrm{f}'\left(t/\sqrt{\mu\epsilon} - \hat{\mathbf{n}} \cdot \mathbf{r}\right) \tag{106a}$$

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = -\mathbf{A}_0 \mathrm{f}'\left(t/\sqrt{\mu\epsilon} - \hat{\mathbf{n}} \cdot \mathbf{r}\right)/\sqrt{\mu\epsilon} \tag{106b}$$

where $\mathbf{A}_0$ is a constant vector and $\mathrm{f}'(w) = d\mathrm{f}(w)/dw$.

In Arts. 784–786, Maxwell determines that the wave Equation in (103) implies that the velocity of an electromagnetic disturbance in a nonconducting medium is equal to $1/\sqrt{\mu\epsilon}$. Moreover, he says "On the theory that light is an electromagnetic disturbance, propagated in the same medium through which other electromagnetic actions are transmitted, $[1/\sqrt{\mu\epsilon}]$ must be the velocity of light, a quantity the value of which has been estimated by several methods." In Art. 787, he compares three directly measured values of the speed of light "through air or through the planetary spaces" with the results of measuring $\mu_0\epsilon_0$ to get the speed $1/\sqrt{\mu_0\epsilon_0}$. Although they are all within about 10% of each other, Maxwell hopes "that, by further experiment, the relation between the magnitudes of the two quantities may be more accurately determined." The propagated fields are no longer intimately connected with their sources and the forces that the propagated fields exert on charges do not obey Newton's third law of equal and opposite instantaneous action and reaction between the charges and the sources of the propagated fields.

In the remainder of Chapter XX, Maxwell briefly considers plane-wave light propagation, the energy in a propagating light wave, radiation pressure in a light wave, and light waves in conducting media. It appears that he does not consider in his *Treatise* the propagation of disturbances with frequencies other than light frequencies. Nonetheless, in his 1865 *Phil. Trans. Roy. Soc. London* paper entitled "A Dynamical Theory of the Electromagnetic Field," he states, "This velocity is so nearly that of light, that it seems we have strong reason to conclude that light itself (*including radiant heat, and other radiations if any*) [emphasis mine] is an electromagnetic disturbance in the form of waves propagated through the electromagnetic field according to electromagnetic laws." This propagation of electromagnetic energy from one body to another further convinced Maxwell of the existence of a vacuum medium that he and others called an ether because "we are unable to conceive of propagation in time, except either as the flight of a material substance through space, or as the propagation of a condition of motion or stress in a medium already existing in space" [1, Art. 866].

At the end of Chapter XX, Maxwell mentions the work of L. V. Lorenz published in *Annalen der Physik und Chemie* in July 1867 (pp. 243–263), with English translation in the 1867 *Philosophical Magazine* (pp. 287–301), two years after Maxwell published his "A Dynamical Theory of the Electromagnetic Field" paper in the *Phil. Trans. Roy. Soc. London* in 1865 (pp. 459–512). Lorenz, following the ideas of Riemann [3, Vol. I, pp. 267–270], postulated that the scalar and vector potentials were given by integrals of retarded-time distributions of charge and current, respectively, and that electrical waves consisting of transverse electric currents may be propagated with a velocity comparable to that of light. Lorenz showed that in order for the charge and current to satisfy the continuity equation, the scalar and vector potential had to satisfy what has come to be known as the Lorenz-Lorentz gauge. One should note, however, that the paper of Lorenz does not contain the magnetic field nor does it give a means for determining the correct dynamic field equations. It did not contain Faraday's law, the generalized Ampère's law, or equations equivalent to these laws. Moreover, the equations of Lorenz did not fundamentally relate the speed of propagation in his retarded time to $\mu\epsilon$. The importance of Lorenz's paper lies in its introduction of the retarded-time potentials and their associated "Lorenz gauge."

The prediction of the fundamental equations of Maxwell that $1/\sqrt{\mu_0\epsilon_0}$ equals the speed of light reconfirmed Maxwell's equations and their unification of electricity and magnetism as one of the most profound discoveries in human scientific history. It placed Maxwell, like Newton before him and Einstein after him, at the beginning of a new scientific era. As Einstein put it, "One scientific epoch ended and another began with James Clerk Maxwell" [32].

7. CONCLUDING SUMMARY

Unlike Maxwell's previous paper on electromagnetic fields ("On Physical Lines of Force," *Philosophical Magazine*, 1861–62), mechanical models for the connections between electromagnetic fields are largely absent from his *Treatise*, in which he uses scalars, vectors, and nablas as we do today except for a slight difference in notation for the divergence and curl of vectors. His electromagnetic equations are written in the form of an unrationalized mksA system of units with μ_0 equal to unity. Even though the Coulomb or Ampere had not been introduced as an independent unit during the lifetime of Maxwell, he suggested that charge become a fourth independent unit in addition to length, mass, and time. He also had the foresight of suggesting the period and wavelength of light as the future standards of time and length, and that with these standards, Newton's second law of motion could be used to define a standard of mass, or, alternatively, the unit of mass could be defined in terms of a molecule of a standard substance.

Maxwell's central purpose of his *Treatise* was to present the subject of electrical and magnetic phenomena in a methodical manner. Another primary aim was to place Faraday's experiments and modes of thought and expression on a firm mathematical basis and to communicate to others Faraday's *Researches*. His mathematical formulation of Faraday's ideas of lines of force eventually created a revolutionary change in the emphasis of physics from direct forces between particles at a distance to force fields in space and matter.

Maxwell believed that charge is a fluid continuum and expressly predicted that the future would show that charge, including that involved in electrolysis, does *not* come in discrete elementary units; in particular, he did not anticipate the discovery of the electron. Similarly, he believed that current is a continuum produced by the transference of charge, but says that the fundamental nature, including the density or velocity of the charge that makes up the fluid, is completely unknown. He says that it is uncertain what in conduction current $\mathbf{J}$ would correspond to the product of a charge density and velocity even though he assumes that conduction current and charge obey the continuity equation, $\nabla \cdot \mathbf{J} + \partial\rho_e/\partial t = 0$, as demonstrated by Faraday's experiments.

The experiments of Faraday also led Maxwell to define an electric polarization displacement vector as $\mathbf{D} = \epsilon\mathbf{E}$ that satisfies $\nabla \cdot \mathbf{D} = \rho_e$, where ρ_e is the free electric charge. However, he does not introduce a polarization vector $\mathbf{P}$ or the equation $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$ in his *Treatise*. He mentions that electric polarization could well be produced by electric charge separation on small conducting elements (to form electric dipoles), but he retains his uncertainty about the nature of electric polarization throughout the *Treatise*.

In order to make Ampère's law consistent with the equation of continuity for charge and conduction current as well as with the divergence equation satisfied by $\mathbf{D}$, Maxwell infers the generalization of Ampère's law containing the displacement current $\dot{\mathbf{D}}$ and calls $\mathbf{J} + \dot{\mathbf{D}}$ the total (conduction plus displacement) current. In this derivation, there is no need to introduce a separate electric polarization vector $\mathbf{P}$ and thus polarization charge density $(-\nabla \cdot \mathbf{P})$ does not arise. Also, since Maxwell did not have the advantage of knowing beforehand the form of his generalization of Ampère's law without electric polarization, that is, with just the $\epsilon_0\dot{\mathbf{E}}$ term, he considered $\mathbf{J} + \dot{\mathbf{D}}$ as the total equivalent current and $\nabla \cdot \mathbf{D}$ as the total equivalent charge in a dielectric rather than $\mathbf{J} + \dot{\mathbf{P}}$ and $\rho_e - \nabla \cdot \mathbf{P}$, respectively, as we would today. Also, Maxwell seemed to view the free-space vacuum "ether" no differently than polarized material and thus he concluded that displacement current flows like electric current in the vacuum ether as well as in material dielectrics.

Maxwell also considered magnetized bodies as continua with their minutest parts having the same magnetic properties as the whole. However, the minutest parts of magnetic polarization (magnetization), unlike electric polarization, he assumed were magnetic dipoles comprised of magnetic charge separation. Although he says that Ampère's hypothesis that magnetic dipole moments are produced by circulating currents within the molecules of the magnetic material is not inconsistent with observations, he assumes that the magnetic dipole moments are caused by charge separation to simplify the derivations of the magnetic fields in terms of their sources and to maintain a convenient parallelism between electric and magnetic charge satisfying Coulomb's law. This means that in Maxwell's *Treatise*, the magnetic field $\mathbf{H}$ is the primary magnetic vector and that the magnetic induction $\mathbf{B}$, which he defines as $\mu_0(\mathbf{H}+\mathbf{M})$, is the secondary magnetic vector. He finds that $\nabla\cdot\mathbf{B}=0$ because unlike free electric charge, free magnetic charge (magnetic monopoles) has not been experimentally observed.

Maxwell defines the primary fields, $\mathbf{E}$ and $\mathbf{H}$, in terms of the force on a stationary unit electric and magnetic charged particle in free space (or ether, as Maxwell believed), respectively, with the charged particles small enough not to disturb the sources of the fields. One of the confusing difficulties in reading Maxwell's *Treatise* is his use in Part IV of the same symbol (German vector $\mathfrak{E}$) for the force on a moving unit electric charge as for the electric field $\mathbf{E}$ in Parts I and II. I have tried to eliminate this confusion in the present paper by using a different symbol $\mathbf{E}_{\rm v}$ for the force on a moving unit electric charge.

Maxwell defines primary fields in free space only. Moreover, nowhere in Maxwell's *Treatise* does he average microscopic fields to get the macroscopic fields in a continuum since he begins by assuming only continuum charge, current, and polarization. Thus, we asked how Maxwell obtained equations that hold within a continuum containing continuous distributions of charge, current, and polarization. What he does he explains in detail in his development of the magnetostatic fields. From Coulomb's law (for magnetic or electric charges) between charges in free space, he derives integral expressions for the potentials outside source regions in terms of the sources that produce these fields. He then defines these potentials inside source regions by using the same integral expressions for the potentials inside the source regions. He relates these *mathematically defined fields* to the fields that can be measured inside free-space cavities formed by removing the source densities from an infinitesimally small volume within the source region holding all the other sources fixed. Inside electric charge and current densities, the cavity electric field is identical to the mathematically defined electric field. Inside electric polarization, he does not relate the mathematically defined fields to cavity fields because, as mentioned above, he does not define a polarization vector $\mathbf{P}$. In electric polarization, he simply says that the $\mathbf{D}$ can be measured in a dielectric by measuring the charge separation (for example, by means of a capacitor) and then $\mathbf{E}$ is determined from $\mathbf{D}=\epsilon\mathbf{E}$ assuming that ϵ has been measured. One of the shortcomings of Maxwell's *Treatise* is that it does not relate this measured $\mathbf{E}$ or $\mathbf{D}$ field to free-space cavity electric fields within the dielectric.

Using cavity-field concepts in magnetic material that originated with W. Thomson and Poisson, Maxwell finds the fields along the axis of a free-space circular-cylindrical cavity whose axis is parallel to the removed magnetization $\mathbf{M}$. For thin needle shaped cylinders and thin disk cylinders, he determines that the magnetic field $\mathbf{H}_c$ in the cavity equals $\mathbf{H}$ and $\mathbf{H}+\mathbf{M}$, respectively, where $\mathbf{H}$ is the mathematically defined magnetic field in the continuum. He then defines the new measurable vector $\mathbf{B}=\mu_0(\mathbf{H}+\mathbf{M})$ called the magnetic induction that he says becomes important in studying time varying electromagnetic fields (where the time variation of $\mathbf{B}$ "induces" an electromotive force).

Maxwell determines that the force exerted by the magnetic induction $\mathbf{B}$ on a magnetostatic current density $\mathbf{J}$ is equal to $\mathbf{J}\times\mathbf{B}$ without knowledge of the "Lorentz force" or using the relation $\mathbf{J}=\rho\mathbf{v}$ between current and moving charge density. Remarkably, he accomplishes this by assuming that, with regard to force, circulating current is equivalent to a distribution of magnetic shells with magnetic moments normal to the surfaces of the shells and that the force on the magnetic shells will be the same as on the current. Because he is uncertain about the details of the interaction between current and matter, he assumes a literal equivalence of the actions on existing current and magnetic shells in stationary matter by external magnetic induction. This leads him to the erroneous conclusion that the distribution of established current in a stationary material would not be altered by the application of a magnetic induction $\mathbf{B}$. That is, he assumes that only an electromotive force changes the current distribution in conductors, and thus, he does not anticipate the Hall effect.

Maxwell derives the electrostatic and magnetostatic energy densities of formation for linear material as $\mathbf{E}\cdot\mathbf{D}/2$ and $\mathbf{H}\cdot\mathbf{B}/2$, respectively. He finds that the electric and magnetic stress dyadics that can be used to determine the total static force on a polarized body in free space are given by

$\epsilon_0(\mathbf{EE} - |\mathbf{E}|^2\bar{\mathbf{I}}/2)$ and $\mu_0(\mathbf{HH} - |\mathbf{H}|^2\bar{\mathbf{I}}/2)$, respectively. The potential energy density of assembling permanent magnetization $\mathbf{M}_0$ producing a magnetic field $\mathbf{H}_0$ is given by $\mu_0|\mathbf{H}_0|^2/2$. The idea of electromagnetic energy density existing throughout space and matter further convinced Maxwell of the existence of a vacuum medium that he referred to as the ether.

A centerpiece of Maxwell's dynamical theory of electromagnetism is his mathematical representation of the experimental results of Faraday in terms of what we refer to today as the integral form of Faraday's law. It equates the negative of the time rate of change of magnetic flux through an open surface to the integral of the force per unit electric charge integrated around the closed curve bounding the open surface. Maxwell expresses this law in its most general form with the time derivative outside the magnetic-induction integral over an open surface (or the equivalent vector-potential integral over the closed curve bounding the open surface) that can be in motion. Then, in one of the tours de force of his *Treatise*, he evaluates the moving integral to prove that the electromotive force $\mathbf{E}_\mathrm{v}$ exerted on a moving unit electric charge is $\mathbf{E} + \mathbf{v} \times \mathbf{B}$. In other words, Maxwell derives what has become known as the "Lorentz force" on a moving charge from his general mathematical form of Faraday's law without recourse to the $\mathbf{J} \times \mathbf{B}$ force that he has derived using a magnetic-shell model for circulating current. He also uses this $\mathbf{E} + \mathbf{v} \times \mathbf{B}$ force to generalize Ohm's law, as well as the linear constitutive relation for electric displacement, to moving conductors and moving delectrics, respectively.

Despite Maxwell's emphasis on Faraday's law in Chapters VII and VIII of Part IV of his *Treatise*, he does not include Faraday's law explicitly in his list of general equations of the electromagnetic field in Chapter IX of Part IV, but includes the differential form of Faraday's law implicitly in the two equations $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\partial\mathbf{A}/\partial t - \nabla\psi_e$. Maxwell apparently emphasized the importance of the potentials because they were the mortar in the electrostatic and magnetostatic foundations on which he constructed the dynamic electromagnetic field equations. Maxwell's emphasis on the electromagnetic potentials seemed to anticipate the advent of quantum physics.

The summary list of 12 equations in Art. 619 consist of four main equations (three vector and one scalar), three definitions (two vector and one scalar), three vector constitutive relations, the vector equation for mechanical force density, and the vector magnetic field equation in terms of the gradient of a potential when the magnetic field is irrotational. The four main equations are trivially converted to what today are usually considered the four Maxwell equations.

Using his four main equations, Maxwell deduces the homogeneous vector Helmholtz equation for the vector potential satisfying the Coulomb gauge and determines that the speed of propagation of electromagnetic disturbances in a nonconducting magnetodielectric medium is equal to $1/\sqrt{\mu\epsilon}$, which, he finds, is close to the speed of light in free space, a result that leads Maxwell to conclude that light is an electromagnetic phenomenon and to later end his *Treatise* encouraging investigations of the medium (ether) in which this propagation of energy takes place.

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