Progress In Electromagnetics Research, PIER 35, 115–125, 2002

DYADIC GREEN'S FUNCTION FOR AN UNBOUNDED ANISOTROPIC MEDIUM IN CYLINDRICAL COORDINATES

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Abstract—The dyadic Green's function for an unbounded anisotropic medium is treated analytically in the Fourier domain. The Green's function, which is expressed as a triple Fourier integral, can be next reduced to a double integral by performing the integration over the longitudinal Fourier variable or the transverse Fourier variable. The singular behavior of Green's dyadic is discussed for the general anisotropic case.

1. INTRODUCTION

Dyadic Green's function technique has long proved to be a valuable tool in the representation of electromagnetic fields. It is known that an explicit closed-form expression can be written for the Green's dyadic for the uniaxial anisotropic medium with either electric or magnetic anisotropy [1], or both [2]. But for more general anisotropic media (e.g., the anisotropic plasma and the biaxial medium), such closed-form expression does not seem to be feasible [3–6]. A three-dimensional Fourier transformation is employed to treat analytically the dyadic Green's function for an infinite unbounded triaxial anisotropic medium [3]. Dyadic Green's function in an infinite anisotropic plasma medium was derived and computed [5]. In [6], the dyadic Green's function in a biaxial anisotropic medium is treated by means of a Fourier transform and the delta-type source singularity is examined carefully.

In this paper, the dyadic Green's function for the more general anisotropic medium is carried out with the extension of work in [6]. The medium is characterized by a tensor relative dielectric permittivity $\widehat{\varepsilon}$ of the form

$$\widehat{\varepsilon} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & 0 \\ \epsilon_{yx} & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix}. \tag{1}$$

where ϵ_{xx}, ϵ_{yy} and ϵ_{zz} are real positive quantities. The anisotropic medium described by (1) includes various materials of physically realizable media, such as plasma, uniaxial material, and biaxial material, *etc.*. The medium is magnetically isotropic with the scalar relative magnetic permeability $\mu_0 = 4\pi \times 10^{-7} H/m$.

The dyadic Green's function is firstly expressed as a triple inverse Fourier integral. Then, the integration over the longitudinal Fourier variable or transverse Fourier variable is performed. The singular behavior of Green's dyadic is discussed for the anisotropic case. When $\epsilon_{xy} = \epsilon_{yx} = 0$ in (1), the dyadic Green's function and the singular behavior coincide with those of Cottis, Vazonouras, and Spyrou [6].

An $\exp(-j\omega t)$ time dependence is assumed and suppressed throughout the text.

2. FORMULATION OF THE PROBLEM

The dyadic Green's function due to a point source excitation located at $\mathbf{r}'$ inside an anisotropic medium is defined as the solution of the vector wave equation

$$\nabla \times \nabla \times \widehat{\mathbf{G}}(\mathbf{r}, \mathbf{r}') - k_0^2 \widehat{\varepsilon} \cdot \widehat{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = \widehat{\mathbf{I}} \delta(\mathbf{r} - \mathbf{r}') \tag{2}$$

where k_0 is the free-space wave number, and $\widehat{\mathbf{I}}$ is the unit dyadic. Using Fourier transform, $\widehat{\mathbf{G}}(\mathbf{r},\mathbf{r}')$ is represented as

$$\widehat{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \int\int\int \widehat{\mathbf{g}}(\mathbf{k}) \exp[j\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')]d\mathbf{k} \tag{3}$$

where $\widehat{\mathbf{g}}(\mathbf{k})$ is Fourier transform of $\widehat{\mathbf{G}}(\mathbf{r},\mathbf{r}')$, and $\mathbf{k} = k\hat{k} = p\hat{p} + \lambda\hat{z}$ is the Fourier transform variable in cylindrical coordinates (p, φ_p, λ). The limits of integration in (3) run from $-\infty$ to $+\infty$ and are omitted for simplicity; this convention will be maintained in the following. The dielectric tensor of (1) is written in cylindrical coordinates as

$$\widehat{\varepsilon} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ \epsilon_{21} & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix}, \tag{4}$$

where

$$\epsilon_{11} = \left(\epsilon_{xx}\cos^2\varphi_p + \epsilon_{yy}\sin^2\varphi_p\right) + (\epsilon_{xy}+\epsilon_{yx})\sin\varphi_p\cos\varphi_p, \tag{5}$$

$$\epsilon_{12} = \left(\epsilon_{xy}\cos^2\varphi_p - \epsilon_{yx}\sin^2\varphi_p\right) + (\epsilon_{yy}-\epsilon_{xx})\sin\varphi_p\cos\varphi_p, \tag{6}$$

$$\epsilon_{21} = \left(\epsilon_{yx}\cos^2\varphi_p - \epsilon_{xy}\sin^2\varphi_p\right) + (\epsilon_{yy}-\epsilon_{xx})\sin\varphi_p\cos\varphi_p, \tag{7}$$

$$\epsilon_{22} = \left(\epsilon_{xx}\sin^2\varphi_p + \epsilon_{yy}\cos^2\varphi_p\right) + (\epsilon_{xy}+\epsilon_{yx})\sin\varphi_p\cos\varphi_p, \tag{8}$$

$$\epsilon_{33} = \epsilon_{zz}. \tag{9}$$

Substituting (3) and (4) into (2), with corresponding Fourier integral expression for the delta function and applying the operator $\nabla\times\nabla\times$ with respect to the ρ, φ, z spatial cylindrical coordinates, the matrix equation for the elements of $\mathbf{g}(\mathbf{k})$ results as follows:

$$\widehat{\mathbf{A}}(\mathbf{k})\cdot\widehat{\mathbf{g}}(\mathbf{k}) = \left(\frac{1}{2\pi}\right)^3 \widehat{\mathbf{I}}, \tag{10}$$

where

$$\widehat{\mathbf{A}}(\mathbf{k}) = k^2\left(\widehat{\mathbf{I}} - \hat{k}\hat{k}\right) - k_0^2\widehat{\varepsilon}. \tag{11}$$

$\widehat{\mathbf{G}}(\mathbf{r},\mathbf{r}')$ can be obtained in the form of a triple Fourier integral

$$\widehat{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \frac{1}{(2\pi)^3}\int\int\int \frac{\widehat{\mathbf{a}}^{(p)}(\mathbf{k})}{D(\mathbf{k})} \exp[j\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')]d\mathbf{k} \tag{12}$$

where $D(\mathbf{k})$ is the determinant of $\widehat{\mathbf{A}}(\mathbf{k})$, and $\widehat{\mathbf{a}}^{(p)}(\mathbf{k})$ is adjoint matrix given by

$$\widehat{\mathbf{a}}^{(p)}(\mathbf{k}) = \begin{bmatrix} a_{pp} & a_{p\varphi_p} & a_{p\lambda} \\ a_{\varphi_p p} & a_{\varphi_p\varphi_p} & a_{\varphi_p\lambda} \\ a_{\lambda p} & a_{\lambda\varphi_p} & a_{\lambda\lambda} \end{bmatrix}. \tag{13}$$

The elements of $\widehat{\mathbf{a}}^{(p)}(\mathbf{k})$ are given as follows:

$$\begin{aligned}
a_{pp} &= p^4 + \left[\lambda^2 - k_0^2(\epsilon_{22}+\epsilon_{33})\right]p^2 + k_0^2\epsilon_{33}\left(k_0^2\epsilon_{22}-\lambda^2\right), && (14)\\
a_{p\varphi_p} &= k_0^2\epsilon_{12}p^2 - k_0^4\epsilon_{12}\epsilon_{33}, && (15)\\
a_{p\lambda} &= a_{\lambda p} = \lambda p^3 - \lambda\left(k_0^2\epsilon_{22}-\lambda^2\right)p, && (16)\\
a_{\varphi_p p} &= k_0^2\epsilon_{21}p^2 - k_0^4\epsilon_{21}\epsilon_{33}, && (17)\\
a_{\varphi_p\varphi_p} &= -k_0^2\epsilon_{11}p^2 - k_0^2\epsilon_{33}\left(\lambda^2 - k_0^2\epsilon_{11}\right), && (18)\\
a_{\varphi_p\lambda} &= k_0^2\epsilon_{21}\lambda p, && (19)\\
a_{\lambda\varphi_p} &= k_0^2\epsilon_{12}\lambda p, && (20)\\
a_{\lambda\lambda} &= \left(\lambda^2 - k_0^2\epsilon_{11}\right)p^2 + \left(k_0^2\epsilon_{11}-\lambda^2\right)\left(k_0^2\epsilon_{22}-\lambda^2\right) - k_0^4\epsilon_{12}\epsilon_{21}. && (21)
\end{aligned}$$

To study the Green's function, one may regard $\widehat{\mathbf{G}}(\mathbf{r},\mathbf{r}')$ as a wave propagating either along the z axis or away from the z axis. Accordingly, the above integral should be written in a suitable form by expressing $D(\mathbf{k})$ as a biquadratic polynomial in either the λ Fourier variable or the p Fourier variable.

In the first case, one writes $D(\mathbf{k})$ as

$$D(\mathbf{k}) = -k_0^2\epsilon_{zz}\left(\lambda^2-\lambda_1^2\right)\left(\lambda^2-\lambda_2^2\right), \qquad (22)$$

where $\pm\lambda_i = \pm\lambda_i(p,\varphi_p), i = 1,2$ are the four roots of the biquadratic in λ as

$$-k_0^2\epsilon_{zz}\left(\lambda^4 + b_\lambda\lambda^2 + c_\lambda\right) = 0, \qquad (23)$$

where

$$\begin{aligned}
b_\lambda &= -k_0^2(\epsilon_{xx}+\epsilon_{yy}) + \left[1+\xi(\varphi_p)/\epsilon_{zz}\right]p^2, && (24)\\
c_\lambda &= -k_0^4\zeta(\varphi_p) - k_0^2\left[\xi(\varphi_p)+\zeta(\varphi_p)/\epsilon_{zz}\right]p^2 + \left[\xi(\varphi_p)/\epsilon_{zz}\right]p^4, && (25)\\
\xi(\varphi_p) &= \epsilon_{11}, && (26)\\
\zeta(\varphi_p) &= \epsilon_{11}\epsilon_{22} - \epsilon_{12}\epsilon_{21}. && (27)
\end{aligned}$$

In the second case, $D(\mathbf{k})$ is written as

$$D(\mathbf{k}) = -k_0^2\xi(\varphi_p)\left(p^2-p_1^2\right)\left(p^2-p_2^2\right), \qquad (28)$$

where $\pm p_i = \pm p_i(p,\varphi_p), i = 1,2$ are the four roots of the biquadratic in p as

$$-k_0^2\xi(\varphi_p)\left(p^4 + b_p p^2 + c_p\right) = 0, \qquad (29)$$

where

$$b_p = -k_0^2\left[\epsilon_{zz} + \zeta(\varphi_p)/\xi(\varphi_p)\right] + \left[1 + \epsilon_{zz}/\xi(\varphi_p)\right]\lambda^2, \tag{30}$$

$$c_p = \epsilon_{zz}/\xi(\varphi_p)\left[\lambda^4 - k_0^2(\epsilon_{xx} + \epsilon_{yy})\lambda^2 + k_0^4\zeta(\varphi_p)\right]. \tag{31}$$

Using the well-known expansion of a plane wave

$$\exp(j\mathbf{k}\cdot\mathbf{r}) = \exp(j\lambda z)\sum_{m=-\infty}^{\infty} j^m J_m(p\rho)\exp[jm(\varphi - \varphi_p)], \tag{32}$$

$\widehat{\mathbf{G}}(\mathbf{r}, \mathbf{r}')$ can be rewritten as

$$\begin{aligned}\widehat{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = & -\frac{1}{(2\pi)^3 k_0^2}\int_0^\infty p dp \int_0^{2\pi} d\varphi_p \int_{-\infty}^{\infty} d\lambda \\ & \times \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} j^{m-n}\exp\left[j\lambda(z - z')\right]\exp\left[j(m\varphi + n\varphi')\right] \\ & \cdot \exp\left[-j(m+n)\varphi_p\right] \times \frac{J_m(p\rho)J_n(p\rho')}{D(\mathbf{k})}\cdot\widehat{\mathbf{a}}(p, \varphi_p, \lambda)\end{aligned} \tag{33}$$

where $\widehat{\mathbf{a}}(\mathbf{k})$ is the matrix resulting after the translation of $\widehat{\mathbf{a}}^{(p)}(\mathbf{k})$ from the (p, φ_p, λ) cylindrical coordinate system to the (ρ, φ, z) cylindrical coordinate system. This is accomplished via the translation

$$\widehat{\mathbf{a}}(\mathbf{k}) = \widehat{\mathbf{T}}^{-1}\cdot\widehat{\mathbf{a}}^{(p)}(\mathbf{k})\cdot\widehat{\mathbf{T}} \tag{34}$$

and

$$\widehat{\mathbf{T}} = \begin{bmatrix} \hat{p}\cdot\hat{\rho} & \hat{p}\cdot\hat{\varphi} & \hat{p}\cdot\hat{z} \\ \hat{\varphi}_p\cdot\hat{\rho} & \hat{\varphi}_p\cdot\hat{\varphi} & \hat{\varphi}_p\cdot\hat{z} \\ \hat{z}\cdot\hat{\rho} & \hat{z}\cdot\hat{\varphi} & \hat{z}\cdot\hat{z} \end{bmatrix}. \tag{35}$$

3. INTEGRATION OVER THE LONGITUDINAL AND TRANSVERSE FOURIER VARIABLES

In this section, the case corresponding to $\mathbf{r} \neq \mathbf{r}'$ will be examined, while the case $\mathbf{r} \longrightarrow \mathbf{r}'$ will be examined in Section 4. Studies of the integrand function (33) are performed over the λ and p Fourier variables, respectively.

In the case of performing the integration over the λ variable, we write $D(\mathbf{k})$ as in (22) and four poles appear in the integrand function, namely, $\pm\lambda_i = \pm\lambda_i(p, \varphi_p), i = 1, 2$. The integrations to be performed are of the form

$$\widehat{\mathbf{I}}_\lambda = \int_{-\infty}^{\infty}\frac{\widehat{\mathbf{F}}(p, \varphi_p, \lambda)\cdot\exp\left[j\lambda(z - z')\right]}{(\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_2^2)}d\lambda. \tag{36}$$

The integrals given by (36) are broken into integrals of the form as

$$I_{\lambda}^{(i)}=\int_{-\infty}^{\infty}\frac{\lambda^{i}\cdot\exp\left[j\lambda(z-z')\right]}{\left(\lambda^{2}-\lambda_{1}^{2}\right)\left(\lambda^{2}-\lambda_{2}^{2}\right)}d\lambda, i=0,1,2,3. \tag{37}$$

These integrals are evaluated by appropriate contour integration to yield the final result [6]

$$I_{\lambda}^{(i)}=\frac{j\pi s}{\lambda_{1}^{2}-\lambda_{2}^{2}}\left\{\lambda_{1}^{i-1}\exp\left[j\lambda_{1}(z-z')\right]-\lambda_{2}^{i-1}\exp\left[j\lambda_{2}(z-z')\right]\right\} \tag{38}$$

where

$$s=\begin{cases}1, & z>z',\\ (-1)^{i}, & z<z'.\end{cases} \tag{39}$$

In the case of performing the integration over the p variable, we write $D(\mathbf{k})$ as in (28). Thus, four poles appear in the integrand function, namely, $\pm p_j=\pm p_j(\lambda,\varphi_p), j=1,2$. The integrations to be performed are of the form

$$\widehat{\mathbf{I}}_{p}=\int_{-\infty}^{\infty}\frac{\widehat{\mathbf{F}}(p,\varphi_{p},\lambda)\cdot J_{m}(p\rho)J_{n}(p\rho')}{\left(p^{2}-p_{1}^{2}\right)\left(p^{2}-p_{2}^{2}\right)}pdp. \tag{40}$$

It may easily be seen that both the $\hat{z}\hat{z}$ term containing a λ^4 term of $\widehat{\mathbf{F}}(p,\varphi_p,\lambda)$ when $z=z'$ and the $\hat{\rho}\hat{\rho}$ element containing a p^4 term of $\widehat{\mathbf{F}}(p,\varphi_p,\lambda)$when $\rho=\rho'$ correspond to the singularity source term, respectively. They are examined in section 4. The integrals given by (33) are broken into integrals of the form as

$$I_{p}^{(j)}=\int_{-\infty}^{\infty}\frac{p^{j}\cdot J_{m}(p\rho)J_{n}(p\rho')}{\left(p^{2}-p_{1}^{2}\right)\left(p^{2}-p_{2}^{2}\right)}dp, j=1,2,3,4. \tag{41}$$

Using the following relations

$$\frac{p}{\left(p^{2}-p_{1}^{2}\right)\left(p^{2}-p_{2}^{2}\right)}=\frac{1}{p_{1}^{2}-p_{2}^{2}}\cdot\left(\frac{p}{p^{2}-p_{1}^{2}}-\frac{p}{p^{2}-p_{2}^{2}}\right), \tag{42}$$

$$\frac{p^{2}}{\left(p^{2}-p_{1}^{2}\right)\left(p^{2}-p_{2}^{2}\right)}=\frac{1}{p_{1}^{2}-p_{2}^{2}}\cdot\left(\frac{p^{2}}{p^{2}-p_{1}^{2}}-\frac{p^{2}}{p^{2}-p_{2}^{2}}\right), \tag{43}$$

$$\frac{p^{3}}{\left(p^{2}-p_{1}^{2}\right)\left(p^{2}-p_{2}^{2}\right)}=\frac{1}{p_{1}^{2}-p_{2}^{2}}\cdot\left(\frac{pp_{1}^{2}}{p^{2}-p_{1}^{2}}-\frac{pp_{2}^{2}}{p^{2}-p_{2}^{2}}\right), \tag{44}$$

the integrals $I_p^{(j)}$ with respect to p can be reduced to

$$\int_{-\infty}^{\infty}\frac{p^{j}\cdot J_{m}(p\rho)J_{n}(p\rho')}{p^{2}-p_{1}^{2}}dp,\quad j=1,2 \tag{45}$$

$$\int_{-\infty}^{\infty} \frac{p^j \cdot J_m(p\rho) J_n(p\rho')}{p^2 - p_2^2} dp, \quad j = 1, 2. \tag{46}$$

Taking into account the following relations

$$\begin{aligned} J_m(p\rho) &= \frac{1}{2}\left[H_m^{(1)}(p\rho) + H_m^{(2)}(p\rho)\right], & (47)\\ H_m^{(1)}(-p\rho) &= (-1)^{n+1} H_m^{(2)}(p\rho), & (48) \end{aligned}$$

the integrals (45) can be easily computed. In the case of $\rho > \rho'$, we get

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{p^j \cdot J_m(p\rho) J_n(p\rho')}{p^2 - p_1^2} dp \\ &= \begin{cases} \frac{i\pi}{2} p_1^{j-1} H_m^{(1)}(p_1\rho)\left[H_m^{(1)}(p_1\rho') + H_m^{(2)}(p_1\rho')\right], & m+n = \text{even} \\ 0, & m+n = \text{odd}. \end{cases} \end{aligned} \tag{49}$$

In the case of $\rho < \rho'$, we get

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{p^j \cdot J_m(p\rho) J_n(p\rho')}{p^2 - p_1^2} dp \\ &= \begin{cases} \frac{i\pi}{2} p_1^{j-1} H_m^{(1)}(p_1\rho')\left[H_m^{(1)}(p_1\rho) + H_m^{(2)}(p_1\rho)\right], & m+n = \text{even} \\ 0, & m+n = \text{odd}. \end{cases} \end{aligned} \tag{50}$$

Using the equations (49) and (50), the integrals (46) can be obtained in the similar method.

4. DERIVATION OF THE DELTA-TYPE SOURCE SINGULARITY

The singular behavior of Green's dyadic has been extensively discussed over the past 30 years for the isotropic case [9–15]. In recent years, the singular behavior of Green's dyadic for the biaxial anisotropic case was discussed in [6]. In this paper, the singular behavior of the Green's dyadic for the more general anisotropic case is discussed.

Upon inspection of (12)–(13), taking into account (22) and (28), it may be seen that the λ^4 term in $a_{\lambda\lambda}$ and the p^4 term in a_{pp}, i.e., the following terms in $g_{\lambda\lambda}$ and g_{pp} are as follows:

$$\frac{\lambda^4}{k_0^2 \epsilon_{zz}\left(\lambda^2 - \lambda_1^2\right)\left(\lambda^2 - \lambda_2^2\right)}, \tag{51}$$

$$\frac{p^4}{k_0^2\xi(\varphi_p)\left(p^2-p_1^2\right)\left(p^2-p_2^2\right)}. \tag{52}$$

With the similar method in [6], one can easily get the delta-type terms. The first delta-type term in the case of performing the integration over λ in (51) corresponds to a pillbox principle volume. The second one in the case of performing the integration over p in (52) corresponds to a needle-shaped principal volume. It is readily seen that the delta-type term is $k_0^{-2}\epsilon_{zz}\delta(\mathbf{r}-\mathbf{r}')\hat{z}\hat{z}$, which is the same with that of [6], in the case of the pillbox principal volume.

Particular attention will be paid on the inspection on the delta-term corresponding to the needle-shaped principal volume.

In the case of a needle-shaped principal volume, the (52) term contributes to the delta-type term through

$$\frac{1}{k_0^2\xi(\varphi_p)}=\frac{1}{k_0^2\left[\left(\epsilon_{xx}\cos^2\varphi_p+\epsilon_{yy}\sin^2\varphi_p\right)+\left(\epsilon_{xy}+\epsilon_{yx}\right)\sin\varphi_p\cos\varphi_p\right]}. \tag{53}$$

Upon the translation in the cylindrical coordinate system via (34)–(35), this term appears in $g_{\rho\rho}$, $g_{\rho\varphi}$, $g_{\varphi\rho}$, $g_{\varphi\varphi}$, multiplied by

$$\begin{aligned}(\hat{p}\cdot\hat{\rho})^2 &= \cos^2(\varphi_p-\varphi), &(54)\\ (\hat{p}\cdot\hat{\varphi})^2 &= \sin^2(\varphi_p-\varphi), &(55)\\ (\hat{p}\cdot\hat{\rho})(\hat{p}\cdot\hat{\varphi}) &= \sin(\varphi_p-\varphi)\cos(\varphi_p-\varphi). &(56)\end{aligned}$$

Hence, the following term emerge:

$$\begin{aligned}&\frac{\cos^2\varphi_p}{\left[\left(\epsilon_{xx}\cos^2\varphi_p+\epsilon_{yy}\sin^2\varphi_p\right)+\left(\epsilon_{xy}+\epsilon_{yx}\right)\sin\varphi_p\cos\varphi_p\right]}\\ =\;&\frac{1+\cos 2\varphi_p}{\alpha+\beta_1\cos 2\varphi_p+\beta_2\sin 2\varphi_p}, &(57)\\ &\frac{\sin^2\varphi_p}{\left[\left(\epsilon_{xx}\cos^2\varphi_p+\epsilon_{yy}\sin^2\varphi_p\right)+\left(\epsilon_{xy}+\epsilon_{yx}\right)\sin\varphi_p\cos\varphi_p\right]}\\ =\;&\frac{1-\cos 2\varphi_p}{\alpha+\beta_1\cos 2\varphi_p+\beta_2\sin 2\varphi_p}, &(58)\\ &\frac{\sin\varphi_p\cos\varphi_p}{\left[\left(\epsilon_{xx}\cos^2\varphi_p+\epsilon_{yy}\sin^2\varphi_p\right)+\left(\epsilon_{xy}+\epsilon_{yx}\right)\sin\varphi_p\cos\varphi_p\right]}\\ =\;&\frac{\sin 2\varphi_p}{\alpha+\beta_1\cos 2\varphi_p+\beta_2\sin 2\varphi_p}, &(59)\end{aligned}$$

where

$$\alpha=\epsilon_{xx}+\epsilon_{yy},\beta_1=\epsilon_{xx}-\epsilon_{yy},\beta_2=\epsilon_{xy}+\epsilon_{yx} \tag{60}$$

and

$$\beta = \sqrt{\beta_1^2 + \beta_2^2}, \cos\varphi_0 = \frac{\beta_1}{\beta}, \sin\varphi_0 = \frac{\beta_2}{\beta}. \tag{61}$$

The constant parts contributed by the above terms may be found as the zero-order coefficients in their Fourier series expansion, as follows:

$$C_0^{(a)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1+\cos 2\varphi_p}{\alpha + \beta\cos(2\varphi_p - \varphi_0)} d\varphi_p, \tag{62}$$

$$C_0^{(b)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-\cos 2\varphi_p}{\alpha + \beta\cos(2\varphi_p - \varphi_0)} d\varphi_p, \tag{63}$$

$$C_0^{(c)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\sin 2\varphi_p}{\alpha + \beta\cos(2\varphi_p - \varphi_0)} d\varphi_p. \tag{64}$$

The above-mentioned three coefficients can be evaluated by means of the integrals

$$\int_0^{2\pi} \frac{1}{\alpha + \beta\cos(2\varphi_p - \varphi_0)} d\varphi_p = \frac{2\pi}{\sqrt{\alpha^2 - \beta^2}}, \tag{65}$$

$$\int_0^{2\pi} \frac{\cos 2\varphi_p}{\alpha + \beta\cos(2\varphi_p - \varphi_0)} d\varphi_p = \frac{\beta_1}{\beta}\left(\frac{2\pi}{\beta} - \frac{2\pi\alpha}{\beta\sqrt{\alpha^2 - \beta^2}}\right), \tag{66}$$

$$\int_0^{2\pi} \frac{\sin 2\varphi_p}{\alpha + \beta\cos(2\varphi_p - \varphi_0)} d\varphi_p = \frac{\beta_2}{\beta}\left(\frac{2\pi}{\beta} - \frac{2\pi\alpha}{\beta\sqrt{\alpha^2 - \beta^2}}\right), \tag{67}$$

which eventually yield

$$\begin{aligned} C_0^{(a)} &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1+\cos 2\varphi_p}{\alpha + \beta\cos(2\varphi_p - \varphi_0)} d\varphi_p \\ &= \frac{1}{\sqrt{\alpha^2 - \beta^2}}\left(1 - \frac{\alpha\beta_1}{\beta^2}\right) + \frac{\beta_1}{\beta^2}, \end{aligned} \tag{68}$$

$$\begin{aligned} C_0^{(b)} &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1-\cos 2\varphi_p}{\alpha + \beta\cos(2\varphi_p - \varphi_0)} d\varphi_p \\ &= \frac{1}{\sqrt{\alpha^2 - \beta^2}}\left(1 + \frac{\alpha\beta_1}{\beta^2}\right) - \frac{\beta_1}{\beta^2}, \end{aligned} \tag{69}$$

$$\begin{aligned} C_0^{(c)} &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\sin 2\varphi_p}{\alpha + \beta\cos(2\varphi_p - \varphi_0)} d\varphi_p \\ &= \frac{\beta_2}{\beta^2}\left(1 - \frac{\alpha}{\sqrt{\alpha^2 - \beta^2}}\right). \end{aligned} \tag{70}$$

From (33) and (34) and taking (68)–(70) into account, one concludes that the delta-type term is

$$
\begin{aligned}
&\Big\{\Big(C_0^{(a)}\cos^2\varphi + C_0^{(b)}\sin^2\varphi + C_0^{(c)}\sin\varphi\cos\varphi\Big)\,\hat\rho\hat\rho \\
+\;&\Big(C_0^{(a)}\sin^2\varphi + C_0^{(b)}\cos^2\varphi - C_0^{(c)}\sin\varphi\cos\varphi\Big)\,\hat\varphi\hat\varphi \\
+\;&\Big[\Big(C_0^{(a)} - C_0^{(b)}\Big)\sin\varphi cos\varphi + C_0^{(c)}\Big(\cos^2\varphi - \sin^2\varphi\Big)\Big] \\
\cdot\;&(\hat\rho\hat\varphi + \hat\varphi\hat\rho)\Big\}\delta(\mathbf{r}-\mathbf{r}'). \qquad (71)
\end{aligned}
$$

In the biaxial case $\epsilon_{xx} = \epsilon_1$, $\epsilon_{yy} = \epsilon_2$, $\epsilon_{zz} = \epsilon_3$, and $\epsilon_{xy} = \epsilon_{yx} = 0$, the above terms reduce to

$$
\begin{aligned}
&\left[\left(\frac{\cos^2\varphi}{\sqrt{\epsilon_1}} + \frac{\sin^2\varphi}{\sqrt{\epsilon_2}}\right)\hat\rho\hat\rho + \left(\frac{\sin^2\varphi}{\sqrt{\epsilon_1}} + \frac{\cos^2\varphi}{\sqrt{\epsilon_2}}\right)\hat\varphi\hat\varphi\right. \\
+\;&\left.\frac{\sqrt{\epsilon_1}-\sqrt{\epsilon_2}}{\sqrt{\epsilon_1\epsilon_2}}\sin\varphi\cos\varphi\,(\hat\rho\hat\varphi + \hat\varphi\hat\rho)\right] \times \frac{1}{\sqrt{\epsilon_1}+\sqrt{\epsilon_2}}\delta(\mathbf{r}-\mathbf{r}'). \qquad (72)
\end{aligned}
$$

The expression of (72) coincides to that given in the literature [6]. In the isotropic case $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon$, the above terms reduce to $(2\epsilon)^{-1}\delta(\mathbf{r}-\mathbf{r}')(\hat\rho\hat\rho + \hat\varphi\hat\varphi)$ for a needle-shaped principal volume or $\epsilon^{-1}\delta(\mathbf{r}-\mathbf{r}')\hat{\mathbf{z}}\hat{\mathbf{z}}$ for a pillbox-shaped principal volume. Both expressions coincide to those given in literature [9].

5. CONCLUSION

In this paper, the dyadic Green's function for an unbounded anisotropic medium is treated analytically in the Fourier domain. The Green's function, which is expressed as a triple Fourier integral, can be reduced to a double integral by performing the integration over the longitudinal Fourier variable or the transverse Fourier variable. The singular behavior of Green's dyadic is discussed for the general anisotropic case. The delta-type source term has been extracted for both the pillbox principal volume and the needle-shaped principal volume. In the limits $\epsilon_{xx} \neq \epsilon_{xy} \neq \epsilon_{yx}$ and $\epsilon_{xy} = \epsilon_{yx} = 0$, the term reduces to the corresponding biaxial case.

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