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FIRST-ORDER MATERIAL EFFECTS IN GYROMAGNETIC SYSTEMS

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Abstract—In an attempt to bridge the gap between theory and applications, this paper brings together a few diverse subjects, and presents them as much as possible in self-contained form.

A general perturbation method is developed for calculating the first order effects in quite general bi-anisotropic materials. The advantage of this approach is the feasibility of generating solutions of the Maxwell equations for the complicated media, in terms of well-known solutions for simple media.

Specifically, the present study was motivated by a need to provide a theoretical framework for polarimetric glucometry methods, presently under investigation, in the hope of gaining better understanding of the systems and their limitations, as well as suggesting new configurations for acquiring better data. To that end, we chose to analyze gyromagnetic effects in lossless magneto-optical systems.

Some representative examples have been chosen, and the obtained results, for various situations involving plane and spherical waves, are discussed. It is shown that the specific configuration of the magnetic fields affect the solutions. Generally speaking, the magnetic fields create new multipoles in the resultant wave fields.

Another interesting feature of the present approach is the fact that we get the elementary Faraday rotation effect without resorting to a pair of two oppositely oriented circularly polarized waves, as usually done. Consequently we are able to discuss explicitly complicated situations involving non-planar waves and various external magnetic fields. The penalty is of course the restricted validity of the model to small non-isotropic effects.

1. INTRODUCTION

In general we write the Maxwell equations in the form

$$\begin{aligned} \partial_{\mathbf{r}} \times \mathbf{E} &= -\partial_t \mathbf{B} - \mathbf{j}_m \\ \partial_{\mathbf{r}} \times \mathbf{H} &= \partial_t \mathbf{D} + \mathbf{j}_e \\ \partial_{\mathbf{r}} \cdot \mathbf{D} &= \rho_e \\ \partial_{\mathbf{r}} \cdot \mathbf{B} &= \rho_m \end{aligned} \tag{1}$$

where the indices e, m refer to electric, magnetic sources, respectively. The magnetic sources are added for completeness, although they do not have an independent physical meaning. However they provide a mathematical tool for dealing with problems, e.g., the perturbation scheme used here.

In the time harmonic domain with the harmonic factor given by $e^{-i\omega t}$ the transformed set of (1) becomes

$$\begin{aligned} \partial_{\mathbf{r}} \times \mathbf{E} &= i\omega \mathbf{B} - \mathbf{j}_m \\ \partial_{\mathbf{r}} \times \mathbf{H} &= -i\omega \mathbf{D} + \mathbf{j}_e \\ \partial_{\mathbf{r}} \cdot \mathbf{D} &= \rho_e \\ \partial_{\mathbf{r}} \cdot \mathbf{B} &= \rho_m \end{aligned} \tag{2}$$

At this point we introduce the constitutive relations for simple isotropic media in the form

$$\begin{aligned}\mathbf{D} &= \varepsilon\mathbf{E}\\ \mathbf{B} &= \mu\mathbf{H}\end{aligned} \tag{3}$$

where the constitutive parameters ε, μ are scalars. The system of equations (2) is recast in the form

$$\begin{array}{l|l} \partial_{\mathbf{r}}\times\mathbf{E}_e = i\omega\mathbf{B}_e & \partial_{\mathbf{r}}\times\mathbf{E}_m = i\omega\mathbf{B}_m - \mathbf{j}_m\\ \partial_{\mathbf{r}}\times\mathbf{H}_e = -i\omega\mathbf{D}_e + \mathbf{j}_e & \partial_{\mathbf{r}}\times\mathbf{H}_m = -i\omega\mathbf{D}_m\\ \partial_{\mathbf{r}}\cdot\mathbf{D}_e = \rho_e & \partial_{\mathbf{r}}\cdot\mathbf{D}_m = 0\\ \partial_{\mathbf{r}}\cdot\mathbf{B}_e = 0 & \partial_{\mathbf{r}}\cdot\mathbf{B}_m = \rho_m \end{array} \tag{4}$$

where $\mathbf{E} = \mathbf{E}_e + \mathbf{E}_m$ etc., and the indices on the fields indicate whether they are e, or m, induced. Clearly the summation of the two sets of equations (4) yields back the original Maxwell equations (2). Substituting (3) into (4) yields

$$\begin{array}{l|l} \partial_{\mathbf{r}}\times\mathbf{E}_e = i\omega\mu\mathbf{H}_e & \partial_{\mathbf{r}}\times\mathbf{E}_m = i\omega\mu\mathbf{H}_m - \mathbf{j}_m\\ \partial_{\mathbf{r}}\times\mathbf{H}_e = -i\omega\varepsilon\mathbf{E}_e + \mathbf{j}_e & \partial_{\mathbf{r}}\times\mathbf{H}_m = -i\omega\varepsilon\mathbf{E}_m\\ \partial_{\mathbf{r}}\cdot\mathbf{E}_e = \rho_e/\varepsilon & \partial_{\mathbf{r}}\cdot\mathbf{E}_m = 0\\ \partial_{\mathbf{r}}\cdot\mathbf{H}_e = 0 & \partial_{\mathbf{r}}\cdot\mathbf{H}_m = \rho_m/\mu \end{array} \tag{5}$$

The two systems in (5) possess similitude properties

$$\mathbf{j}_e \Leftrightarrow -\mathbf{j}_m,\ \rho_e \Leftrightarrow -\rho_m,\ \mathbf{E}_e \Leftrightarrow \mathbf{H}_m,\ \mathbf{H}_e \Leftrightarrow \mathbf{E}_m,\ \varepsilon = -\mu \tag{6}$$

Accordingly, by substituting into the left set in (5) the corresponding fields indicated by (6), the right hand side of (5) is obtained, and *vice-versa*. These symmetry relations facilitate a more efficient manipulation of both sets.

Applying a divergence operator to the two vector equations of each set in (5) leads to

$$\partial_{\mathbf{r}}\cdot\mathbf{j}_{e,m} - i\omega\rho_{e,m} = 0 \tag{7}$$

written together for the e, m, indices and recognized as the relation usually referred to as the equation of continuity, or conservation of charge.

The general solution for the fields in terms of the source current is worked out carefully in many books, e.g., [1, 2]. Accordingly, we obtain the general formula

$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= \int\limits_{V(\mathbf{r}')} dV(\mathbf{r}')\left\{\tilde{\boldsymbol{\Gamma}}(\mathbf{r},\mathbf{r}')\cdot\mathbf{j}_e(\mathbf{r}')i\omega\mu - \partial_{\mathbf{r}}\times\tilde{\boldsymbol{\Gamma}}(\mathbf{r},\mathbf{r}')\cdot\mathbf{j}_m(\mathbf{r}')\right\}\\ \mathbf{H}(\mathbf{r}) &= \int\limits_{V(\mathbf{r}')} dV(\mathbf{r}')\left\{\tilde{\boldsymbol{\Gamma}}(\mathbf{r},\mathbf{r}')\cdot\mathbf{j}_m(\mathbf{r}')i\omega\varepsilon + \partial_{\mathbf{r}}\times\tilde{\boldsymbol{\Gamma}}(\mathbf{r},\mathbf{r}')\cdot\mathbf{j}_e(\mathbf{r}')\right\}\end{aligned} \tag{8}$$

where the two equations are similar subject to the transformation (6). In (8)

$$\begin{aligned}\tilde{\boldsymbol{\Gamma}}(\mathbf{r},\mathbf{r}') &= \left(\tilde{\mathbf{I}} + k^{-2}\partial_{\mathbf{r}}\partial_{\mathbf{r}}\right) G(\mathbf{r}-\mathbf{r}'), \quad k^2 = \omega^2\mu\varepsilon \\ G(\mathbf{r}-\mathbf{r}') &= \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}\end{aligned} \tag{9}$$

with k denoting the amplitude of the propagation vector $\mathbf{k}$; here G, $\tilde{\boldsymbol{\Gamma}}$ are the scalar, and its corresponding dyadic, respectively, free space Green's functions, satisfying the wave equations

$$\begin{aligned}\partial_{\mathbf{r}} \times \partial_{\mathbf{r}} \times \tilde{\boldsymbol{\Gamma}} - k^2\tilde{\boldsymbol{\Gamma}} &= \tilde{\mathbf{I}}\delta(\mathbf{r}-\mathbf{r}') \\ \partial_{\mathbf{r}}^2 G(\mathbf{r}-\mathbf{r}') + k^2 G(\mathbf{r}-\mathbf{r}') &= -\delta(\mathbf{r}-\mathbf{r}')\end{aligned} \tag{10}$$

where $\delta(\mathbf{r}-\mathbf{r}') = \delta(x-x')\delta(y-y')\delta(z-z')$ denotes the spatial Dirac's delta function which becomes singular at $\mathbf{r}-\mathbf{r}' = 0$. The theory summarized in (8)–(10) is amply used in the literature. The introduction of the dyadic Green's function (9) is attributed to [3]. See [2, 4] for theory and further references. Note that in (8), (9), the integration variable is $\mathbf{r}'$, while the derivatives $\partial_{\mathbf{r}}$ are in terms of the coordinates $\mathbf{r}$, hence the differential and integral operations are independent. As far as the differential operators are concerned, the vectors dependent on $\mathbf{r}'$ are to be taken as constant vectors. Consequently terms involving the curl operation can be written as $\left(\partial_{\mathbf{r}} \times \tilde{\boldsymbol{\Gamma}}\right) \cdot \mathbf{j}_{e,m}(\mathbf{r}') = (\partial_{\mathbf{r}} G) \times \mathbf{j}_{e,m}(\mathbf{r}')$, and similarly we have $(\partial_{\mathbf{r}}\partial_{\mathbf{r}})G \cdot \mathbf{j}_{e,m}(\mathbf{r}') = (\partial_{\mathbf{r}}\partial_{\mathbf{r}}G) \cdot \mathbf{j}_{e,m}(\mathbf{r}')$. These relations will be used to actually solve forms like (8) for the applications considered below. For more details see Appendix A.

2. THE PERTURBATION METHOD

A similar method to that below has been used to explore problems involving moving media [5]. Exponential approximations as used below have been introduced before [6]. In general, we consider bi-anisotropic media whose constitutive relations can be included in the form

$$\begin{aligned}\mathbf{D} &= \varepsilon\mathbf{E} + \tilde{\mathbf{a}} \cdot \mathbf{E} + \tilde{\mathbf{b}} \cdot \mathbf{H} \\ \mathbf{B} &= \mu\mathbf{H} + \tilde{\mathbf{c}} \cdot \mathbf{H} + \tilde{\mathbf{d}} \cdot \mathbf{E}\end{aligned} \tag{11}$$

where ε, μ are scalars, and $\tilde{\mathbf{a}}$, $\tilde{\mathbf{b}}$, $\tilde{\mathbf{c}}$, $\tilde{\mathbf{d}}$ are the pertinent dyadics. The only limitation on the dyadics, and this is the pivot for the present discussion, is that they can be considered as perturbing the simple media. I.e., when $\tilde{\mathbf{a}}$, $\tilde{\mathbf{b}}$, $\tilde{\mathbf{c}}$, $\tilde{\mathbf{d}}$ vanish, then we get back simple isotropic media.

It is therefore suggestive to implement a perturbation scheme. Thus we understand all the fields to have a zero order and a first order component, e.g., $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1$ etc. Substituting the fields into the constitutive relations (11), and separating zero order and first order terms, we obtain,

$$\begin{aligned}
\mathbf{D}_0 &= \varepsilon \mathbf{E}_0 \\
\mathbf{B}_0 &= \mu \mathbf{H}_0 \\
\mathbf{D}_1 &= \varepsilon \mathbf{E}_1 + \tilde{\mathbf{a}} \cdot \mathbf{E}_0 + \tilde{\mathbf{b}} \cdot \mathbf{H}_0 \\
\mathbf{B}_1 &= \mu \mathbf{H}_1 + \tilde{\mathbf{c}} \cdot \mathbf{H}_0 + \tilde{\mathbf{d}} \cdot \mathbf{E}_0
\end{aligned} \tag{12}$$

where terms of the order $\tilde{\mathbf{a}} \cdot \mathbf{E}_1$ etc. are neglected, provided the dyadics are sufficiently small. Substituting $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1$ etc. into (2) and separating zero order and first order terms, we obtain for the zero order fields (2) itself, with index zero attached to all field and source variables. The solution for the zero order fields is achieved by performing the integrations (8), with the dyadic Green's function (9) and with index zero attached to all field and source variables. The solutions for specific problems depend on the aforehand provided source fields, and in comparison to the solution for the full fledged bi-anisotropic system are simpler to derive.

The first order perturbation scheme prescribes

$$\begin{aligned}
\partial_{\mathbf{r}} \times \mathbf{E}_1 &= i\omega\mu \mathbf{H}_1 + i\omega \left(\tilde{\mathbf{c}} \cdot \mathbf{H}_0 + \tilde{\mathbf{d}} \cdot \mathbf{E}_0 \right) = i\omega\mu \mathbf{H}_1 - \mathbf{j}_{1,m} \\
\partial_{\mathbf{r}} \times \mathbf{H}_1 &= -i\omega\varepsilon \mathbf{E}_1 - i\omega \left(\tilde{\mathbf{a}} \cdot \mathbf{E}_0 + \tilde{\mathbf{b}} \cdot \mathbf{H}_0 \right) = -i\omega\varepsilon \mathbf{E}_1 + \mathbf{j}_{1,e} \\
\varepsilon \partial_{\mathbf{r}} \cdot \mathbf{E}_1 &= -\partial_{\mathbf{r}} \cdot \left(\tilde{\mathbf{a}} \cdot \mathbf{E}_0 + \tilde{\mathbf{b}} \cdot \mathbf{H}_0 \right) = \rho_{1,e} \\
\mu \partial_{\mathbf{r}} \cdot \mathbf{H}_1 &= -\partial_{\mathbf{r}} \cdot \left(\tilde{\mathbf{c}} \cdot \mathbf{H}_0 + \tilde{\mathbf{d}} \cdot \mathbf{E}_0 \right) = \rho_{1,m}
\end{aligned} \tag{13}$$

where on the right-hand side of equations (13) we have identified the first order current and charge sources, in terms of the supposedly already known fields obtained by solving for the zero order terms, using (8). Thus the new quantities $\mathbf{j}_{1,e}$, $\mathbf{j}_{1,m}$, $\rho_{1,e}$, $\rho_{1,m}$ are not additional unknowns, but rather known functions. As such, they allow us to consider them as the source (inhomogeneous) terms of the differential equations (13).

Similarly to (7) we now have

$$\partial_{\mathbf{r}} \cdot \mathbf{j}_{1;e,m} - i\omega \rho_{1;e,m} = 0 \tag{14}$$

indeed all the expressions for the zero order terms follow through, and therefore we are still dealing formally with an isotropic medium, and the solution for the first order perturbation fields is given by the analog of (8), i.e.,

$$\mathbf{E}_1(\mathbf{r}) = \int_{V(\mathbf{r}')} dV(\mathbf{r}') \left\{ \tilde{\boldsymbol{\Gamma}}(\mathbf{r},\mathbf{r}') \cdot \mathbf{j}_{1,e}(\mathbf{r}') i\omega\mu - \partial_{\mathbf{r}} \times \tilde{\boldsymbol{\Gamma}}(\mathbf{r},\mathbf{r}') \cdot \mathbf{j}_{1,m}(\mathbf{r}') \right\}$$
$$\mathbf{H}_1(\mathbf{r}) = \int_{V(\mathbf{r}')} dV(\mathbf{r}') \left\{ \tilde{\boldsymbol{\Gamma}}(\mathbf{r},\mathbf{r}') \cdot \mathbf{j}_{1,m}(\mathbf{r}') i\omega\varepsilon + \partial_{\mathbf{r}} \times \tilde{\boldsymbol{\Gamma}}(\mathbf{r},\mathbf{r}') \cdot \mathbf{j}_{1,e}(\mathbf{r}') \right\} \tag{15}$$

where the details of the differential operations in the integrals (15) are explained in the previous section.

The above results are complicated by a lot of detail, but nevertheless provide explicit formulas for the derivation of the first order anisotropic material effects from the zero order solution. In order to compact the notation, some shorthand conventions will be introduced. We introduce the solution in the form of integral dyadic operators acting on the zero-order solutions

$$\begin{aligned} \mathbf{E}_1 &= \left(\int \tilde{\boldsymbol{\mathcal{A}}}\right) \cdot \mathbf{E}_0 + \left(\int \tilde{\boldsymbol{\mathcal{B}}}\right) \cdot \mathbf{H}_0 \\ \mathbf{H}_1 &= \left(\int \tilde{\boldsymbol{\mathcal{C}}}\right) \cdot \mathbf{H}_0 + \left(\int \tilde{\boldsymbol{\mathcal{D}}}\right) \cdot \mathbf{E}_0 \end{aligned} \tag{16}$$

In (16) the fields on the left, $\mathbf{E}_1 = \mathbf{E}_1(\mathbf{r})$, $\mathbf{H}_1 = \mathbf{H}_1(\mathbf{r})$, are functions of the location vector $\mathbf{r}$, while on the right we have $\mathbf{E}_0 = \mathbf{E}_0(\mathbf{r}')$, $\mathbf{H}_0 = \mathbf{H}_0(\mathbf{r}')$, which are expressed in terms of the integration variables $\mathbf{r}'$, and thus operated upon within the integral by the dyadic operators defined by boldface script characters. The integral symbol stands for $\int = \int_{V(\mathbf{r}')} dV(\mathbf{r}')$ as in (8) and (15). The dyadic operators are given by

$$\begin{aligned} \tilde{\boldsymbol{\mathcal{A}}} &= \omega^2\mu\tilde{\boldsymbol{\Gamma}} \cdot \tilde{\mathbf{a}} + i\omega\partial_{\mathbf{r}'} \times \tilde{\boldsymbol{\Gamma}} \cdot \tilde{\mathbf{d}} \\ \tilde{\boldsymbol{\mathcal{B}}} &= \omega^2\mu\tilde{\boldsymbol{\Gamma}} \cdot \tilde{\mathbf{b}} + i\omega\partial_{\mathbf{r}'} \times \tilde{\boldsymbol{\Gamma}} \cdot \tilde{\mathbf{c}} \\ \tilde{\boldsymbol{\mathcal{C}}} &= \omega^2\varepsilon\tilde{\boldsymbol{\Gamma}} \cdot \tilde{\mathbf{c}} - i\omega\partial_{\mathbf{r}'} \times \tilde{\boldsymbol{\Gamma}} \cdot \tilde{\mathbf{b}} \\ \tilde{\boldsymbol{\mathcal{D}}} &= \omega^2\varepsilon\tilde{\boldsymbol{\Gamma}} \cdot \tilde{\mathbf{d}} - i\omega\partial_{\mathbf{r}'} \times \tilde{\boldsymbol{\Gamma}} \cdot \tilde{\mathbf{a}} \end{aligned} \tag{17}$$

For the special case where the zero order Maxwell equations are given in a sourceless domain, i.e.,

$$\begin{aligned} \partial_{\mathbf{r}} \times \mathbf{H}_0 &= -i\omega\varepsilon\mathbf{E}_0 \\ \partial_{\mathbf{r}} \times \mathbf{E}_0 &= i\omega\mu\mathbf{H}_0 \\ \partial_{\mathbf{r}} \cdot \mathbf{E}_0 &= 0 \\ \partial_{\mathbf{r}} \cdot \mathbf{H}_0 &= 0 \end{aligned} \tag{18}$$

it is feasible to use the first two equations (18) to express $\mathbf{H}_0$ in terms of $\mathbf{E}_0$ and *vice-versa*, hence, without going into the details, it is clear that (16), (17) can be rearranged in the form

$$\begin{aligned} \mathbf{E}_1 &= \left(\int \tilde{\boldsymbol{\mathcal{E}}}\right) \cdot \mathbf{E}_0 \\ \mathbf{H}_1 &= \left(\int \tilde{\boldsymbol{\mathcal{F}}}\right) \cdot \mathbf{H}_0 \end{aligned} \tag{19}$$

Again, without going into the details, the innate symmetry of Maxwell's equations (2) will produce some symmetries in (17), (19), which should be further investigated for specific cases.

To the first order we may write $e^{\alpha} = 1 + \alpha$. Consequently (19) can be exploited to represent the fields in yet another form

$$\begin{aligned} \mathbf{E} &= \left[\tilde{\mathbf{I}} + \left(\int \tilde{\boldsymbol{\mathcal{E}}}\right)\right] \cdot \mathbf{E}_0 = e^{\int \tilde{\boldsymbol{\mathcal{E}}}} \cdot \mathbf{E}_0 \\ \mathbf{H} &= \left[\tilde{\mathbf{I}} + \left(\int \tilde{\boldsymbol{\mathcal{F}}}\right)\right] \cdot \mathbf{H}_0 = e^{\int \tilde{\boldsymbol{\mathcal{F}}}} \cdot \mathbf{H}_0 \end{aligned} \tag{20}$$

which might be notationally more convenient.

We have accomplished our goal: The first-order solution of the field problem in the presence of an arbitrary anisotropy inclusion is given by the solution of the limiting case simple medium, and the first order correction is accomplished by operations on these solutions.

3. A SIMPLE PLANE WAVE EXAMPLE AND BOUNDARY CONSIDERATIONS

To better understand how the perturbation method works, we consider a simple case where an isotropic dielectric medium, possessing a different parameter, is embedded in the ambient medium. Accordingly we consider an infinite ambient space characterized by ε_1 in which a slab region of material possessing a dielectric constant $\varepsilon_1 + \varepsilon_2$ is embedded, with $\varepsilon_1 \gg \varepsilon_2$. The slab region extends from $z = a$ to $z = b$. The zero order field is a plane wave propagating in the z direction

$$\mathbf{E}_0 = \hat{\mathbf{x}} E_0 e^{i(kz - \omega t)} \tag{21}$$

According to (12), (13), the equivalent current source for the perturbation field is given by

$$\mathbf{j}_{1,e} = -i\omega \tilde{\mathbf{a}} \cdot \mathbf{E}_0 = -i\omega\varepsilon_2 \tilde{\mathbf{I}} \cdot \mathbf{E}_0 = -i\omega\varepsilon_2 \mathbf{E}_0 \tag{22}$$

combined with (21).

The specialization of (15) to the present case is a one dimensional integral (see Appendix A for more detail)

$$\mathbf{E}_1(z) = \omega^2 \mu \varepsilon_2 \hat{\mathbf{x}} E_0 \int_a^b dz' G(z'-z) e^{i(kz'-\omega t)}$$
$$G(z'-z) = \frac{i}{2k} \begin{cases} e^{ik(z'-z)}, & z' \geq z \\ e^{-ik(z'-z)}, & z' \leq z \end{cases} \tag{23}$$

see (A6) in Appendix A but note the primed and unprimed z-variable and its role in the integral. Thus for an observation point $z \geq b$ we obtain

$$\mathbf{E}_1(z) = \omega^2 \mu \varepsilon_2 \hat{\mathbf{x}} E_0 \frac{i}{2k} e^{i(kz-\omega t)} (b-a) = i\gamma \hat{\mathbf{x}} E_0 e^{i(kz-\omega t)} \tag{24}$$

which is once again a plane wave, like the zero order wave exciting it, except for a change in amplitude, depending on the extent of the active region $(b-a)$, and displaying a phase shift $i = e^{i\pi/2}$. An addition of a small phasor (24) perpendicularly in phasor space with respect to the original phasor (21) is tantamount to a phase shift of (21) by γ radians, i.e., the total field will now be given by

$$\mathbf{E}(z) = \mathbf{E}_0(z) + \mathbf{E}_1(z) = \mathbf{E}_0(z)(1+i\gamma) \approx \mathbf{E}_0(z) e^{i\gamma} \tag{25}$$

We are expecting a wave (21) passing through a region of length $(b-a)$ possessing a different propagation vector k' to acquire an extra phase shift

$$\gamma = (k'-k)(b-a), \; k' = \omega\sqrt{\mu(\varepsilon_1+\varepsilon_2)} \approx \omega\sqrt{\mu\varepsilon_1}\left(1+\frac{\varepsilon_2}{2\varepsilon_1}\right) \tag{26}$$

and it is easily verified that (25), (26) are consistent with (24).

For an observation point $z \leq a$ we obtain

$$\begin{aligned} \mathbf{E}_1(z) &= \omega^2 \mu \varepsilon_2 \hat{\mathbf{x}} E_0 \frac{i}{2k} e^{-i(kz+\omega t)} \int_a^b dz' e^{2ikz'} \\ &= \hat{\mathbf{x}} E_0 \frac{\varepsilon_2}{4\varepsilon_1} e^{-i(kz+\omega t)} \left(e^{2ikb} - e^{2ika}\right) \end{aligned} \tag{27}$$

displaying a backwards propagating plane wave, actually two reflected waves from the faces $z = a$, $z = b$ of the slab region, with opposite signs. Depending on the ratio a/b these two waves can reinforce or annihilate each other.

An important point to note is that the medium parameters themselves give rise to first order reflected waves, therefore subsequent transmission or reflection effects produced by the perturbed medium need not be considered. For example, in a situation where the slab region is backed up with a real reflector at $z = b$, the field (24) will be reflected, and must be taken into account. However, it is already a first order term, and cannot invoke new effects: On its passage back, the interaction of the first order field with the perturbed medium region is neglected, since this would lead to higher order perturbation terms which are consistently neglected in our discussion. On the other hand, the reflection of the zero order wave (21) and the interaction of the reflected wave with the perturbed medium, must be adequately taken into account.

One must also be aware of the discontinuity in the derivative of (23), and generally the singularity of the Green's function in (10), when performing integrations like (15), (23), when the observation point is within the perturbed medium.

This simple example outlines the method employed here and will be exploited for discussing various examples of plane and spherical waves in the presence of gyromagnetic media.

4. GYROMAGNETIC MEDIA

Gyromagnetic or magneto-optic media display the celebrated Faraday rotation effect. Accordingly the orientation of the field polarization† is changing. These media are characterized by a preferred direction, introduced by an external constant or slowly varying magnetic field, rendering the medium anisotropic, even though the ambient medium (in the absence of the externally imposed magnetic field) might be isotropic. A general overview [7] of magneto-optical effects in general optically active crystalline or amorphous materials indicates the various effects. In general we have the electromagnetic fields defined in special directions in space, but in addition the externally imposed magnetic field and the crystalline structure define additional preferred directions, rendering the medium to be anisotropic in a very complicated manner.

Our immediate interest is mainly in investigating such effects as perturbations, using the above theory. Note however that choosing the geometry of the perturbed region, as well as the direction of

† unfortunately the same word is used to indicate the direction of the fields in space, usually the **E** field, and also refers to the creation of induced dipoles, or the reorientation of existing permanent dipoles in the medium. To distinguish the two concepts we shall use the phrases "field polarization", and "dipole polarization", respectively.

the external field, affects the resulting first order field, as explained below. As mentioned above, we are restricting the present discussion to cases of negligible losses, which considerably simplifies the expressions involved.

For some representative (rather than exhaustive) references to the literature on electromagnetic properties of dielectric materials, see for example [8–11], who also provide references linking the subject to the vast existing literature. In essence, we are interested here in dipole polarization effects produced by time dependent electric fields, e.g., the electromagnetic waves propagating within the region of interest, in the presence of imposed static or *quasi* static slowly varying magnetic fields. There are a number of processes characterizing dipole polarization, see for example [8, 9]: One category is orientational polarization, evident in materials that in the absence of an applied field possess randomly oriented permanent dipole moments. The applied electric field exerts a torque that strives to align the dipoles along the electric field lines. The effect is similar to para-, and ferro-, magnetism, although the analogy should not be carried too far. Water is a good example for such polarizable materials; another effect is ionic or molecular dipole polarization, occurring in materials possessing positive and negative ions which tend to displace themselves when an electric field is applied. Salt (sodium chloride NaCl) is a good example for such media. Then there is electronic induced dipole polarization: This effect is evident in most materials, even when permanent dipoles are present, and it occurs when an applied electric field displaces the electron "cloud" center of an atom relative to the nucleus. Sometimes, as for example in ionized gases, the effect is strong enough to consider the creation of free electrons, forming an electron cloud in the presence of a background of sluggish heavy ions of positive net charge. By displacing part of the electron cloud, dipole polarization due to net positively and negatively displaced charges will be created, and the resulting Coulomb force will act to pull the charges back in order to reestablish charge neutrality. A good example for such dipole polarization are ionized gases, such as the ionospheric plasma medium. The electronic induced dipole polarization is somewhat similar to diamagnetic effects in magnetic media, i.e., the induced dipoles are oriented in the opposite direction relative to the field that creates them. In a naive fashion, we can think about two charged plates carrying opposite charges situated in the medium, and consider the effect of the created field: A permanent dipole will tend to be aligned with its negative charge pointing towards the initial positive charge, and thus the dipoles (by convention being depicted as an arrow pointing from their negative to the positive charge) point in the same direction as the field (by convention pointing from the positive to the negative charge). On the other hand, the induced dipoles are created by charge separation. Consider a field line connecting the two charged plates. The dipoles created start and end on these charges, and must therefore point in the opposite direction relative to the electric field. Once again, the analogy with diamagnetism should not be carried too far.

The response of material media to electric fields, i.e., dispersion, depends on the frequency. Loosely speaking, and without considering possible resonance mechanisms, at higher frequencies the inertia diminishes the response. Thus water, which for static or *quasi*-static electric fields displays a relative permittivity $\varepsilon_r \sim 81$, with a corresponding refractive index $n \sim 9$, at optical frequencies evinces a lower value, $n \sim 1.4$. The inertia effects are of course more pronounced for the heavier constituents of the medium. On the other hand electrons, due to their smaller mass, are easily moved by the applied electric fields. In other words, relaxation time of the electrons is much smaller compared to atomic and molecular constituents. At lower frequencies, when the orientational and ionic dipole polarization is dominant, the electronic dipole polarization is as a rule negligible, however as the frequency rises, the effect becomes more pronounced. Consequently the processes giving rise to Faraday rotation are mainly due to electronic dipole polarization as described above. This rather naive explanation motivates the choice taken here for the constitutive relations.

This naive picture is emerging from the mathematical modeling given in the literature, e.g., see [9]. Accordingly for a typical dielectric we start at low frequencies with the highest value of the dielectric constant, which becomes smaller as the three dipole polarization mechanisms manifest themselves, with typically one resonance frequency for each mechanism. For the electronic dipole polarization, which is an induced dipole polarization process, the dielectric parameter becomes negative, as our discussion above suggested.

In order to account for all processes, including the gyromagnetic effect, we use a formula close to those given in the literature, [7, 12],

$$\mathbf{D} = \varepsilon_0 \left(\varepsilon_m \tilde{\mathbf{I}} + \tilde{\mathbf{Y}} \right) \cdot \mathbf{E} \tag{28}$$

where $\tilde{\mathbf{I}}$ is the unit dyadic or matrix, ε_m lumps the relative permittivity due to heavier atoms or molecules, i.e., orientational, and ionic dipole polarization, and $\tilde{\mathbf{Y}}$ accounts for electronic dipole polarization which also gives rise to the gyromagnetic effects.

The geometry associated with $\tilde{\mathbf{Y}}$ is depicted in Fig. 1. The wave is

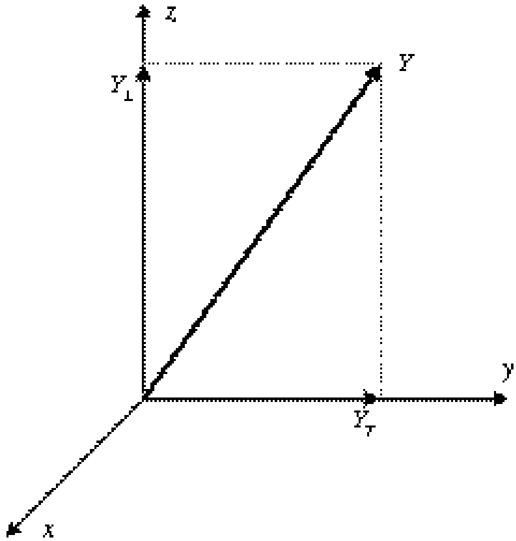

Figure 1. Geometry for the external magnetic field.

propagating in the z direction. The coordinate system is chosen such that the external magnetic field lies in the y-z plane.

Essentially using Kelso's [13] notation (but note — he is using $e^{i\omega t}$ for the time factor, cf. (2)) for the ionospheric plasma medium case, in (28) $\varepsilon_m = 1$, and $\tilde{\mathbf{Y}}$ is given by

$$\tilde{\mathbf{Y}} = -W \begin{bmatrix} 1 & iY_L & -iY_T \\ -iY_L & (1-Y_T^2) & -Y_TY_L \\ iY_T & -Y_TY_L & (1-Y_L^2) \end{bmatrix} \tag{29}$$

For brevity of notation we use a mixed notation of dyadic (boldface characters with a tilde) and matrix (denoted by square parentheses) expressions. The position of the matrix element, e.g., *1, 2* corresponds to a dyad, e.g., $\hat{\mathbf{x}}\hat{\mathbf{y}}$ characterized by two juxtaposed unit vectors (boldface, circumflex characters), etc. In (28) the conduction loss terms have already been neglected, resulting in a Hermitian matrix (29). The symbols in (29) are given as

$$\begin{gathered} W = \frac{X}{1-Y^2}, \quad X = \frac{\omega_N^2}{\omega^2}, \quad Y = \frac{\omega_H}{\omega}, \quad Y^2 = Y_L^2 + Y_T^2 \\ \omega_N^2 = \frac{Ne^2}{m\varepsilon_0}, \qquad \omega_H = \frac{|\mathbf{B}_e||e|\mu_0}{m} \end{gathered} \tag{30}$$

where in (30) ω_N denotes the (angular) plasma frequency, N, e, m, denote the electron number density, charge, and mass, respectively; ω_N is a measure of the electronic polarizability, and is therefore dependent on the effective number density of the active particles, the effective charge they muster for the particle polarization, and their inertia. The imposed magnetic field effect is related to the so called gyro frequency, sometimes called the cyclotron frequency, ω_H, which is dependent on the imposed magnetic field $\mathbf{B}_e$, and the effective charge and mass of the particles contributing to the effect. Treating Y as the length of a vector $\mathbf{Y} = \mathbf{Y}_L + \mathbf{Y}_T$ directed along the imposed magnetic field, the quantities $\mathbf{Y}_L$, $\mathbf{Y}_T$ are then understood as the longitudinal and transversal projections with respect to the direction of propagation of the plane wave propagating in the medium, respectively.

The ionospheric (cold magnetized plasma medium) model is based on the simple equation of motion of a single electron (e.g., see [13])

$$-m\omega^2 = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}_e) \tag{31}$$

i.e., the inertial force on the left, (31), is balanced by the Coulomb and the Lorentz forces on the right, (31), acting on the charge. In the full linear oscillator model, which is more appropriate for fluid and solid media, resonance phenomena are also taken into account (e.g., see [9] for the unmagnetized medium case, where the transition from negative to positive permittivity is also depicted). The resonance results from the interplay of the inertia force and the "spring action" returning force acting on the charges and causing free oscillations at the frequency ω_0 in the absence of other constraints. For a single resonance frequency the pertinent equation of motion now leads to

$$-m\left(\omega^2 - \omega_0^2\right) = -m\left(1 - \frac{\omega_0^2}{\omega^2}\right)\omega^2 = -m'\omega^2 = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}_e) \tag{32}$$

Comparing (32) and (31), it is evident that both can be written in the same form, provided a new term m' is introduced. The introduction of an equivalent mass term is a mere formal step which allows us to consider the formulas (29), (30) on the same footing, with the appropriate exchange of m' for m, thus accounting for the dispersive properties introduced by the resonance phenomenon. It is also clear that (29) is the limiting case of (30) when $\omega^2 \gg \omega_0^2$ can be assumed. As indicated in [9], the electronic dipole polarization resonance effect will be typically in the visible to ultraviolet region. Relevant information can be found in [14, 15], for example. It is important to note that whether the resonance effects are taken into account or not will not affect the subsequent analytical results.

Inasmuch as electronic dipole polarization is usually a small effect, W can be used as the perturbation parameter. When $Y = 0$ in (29), i.e., when the magnetic field vanishes, $\tilde{\mathbf{Y}}$ reduces to $\tilde{\mathbf{I}}W$ and (28) becomes

$$\mathbf{D} = \varepsilon_0(\varepsilon_m - W)\mathbf{E} \tag{33}$$

showing that the electronic dipole polarization has a negative sign, which is well known in the ionospheric case, where $\mathbf{D} = \varepsilon_0(1 - W)\mathbf{E}$, with the relative permittivity being lower than one.

Another approach would be to consider $\tilde{\mathbf{Y}}$ components as the perturbation parameter. In this case the first order perturbation would reduce (29) to

$$\tilde{\mathbf{Y}} = -X\tilde{\mathbf{I}} - X \begin{bmatrix} 0 & iY_L & -iY_T \\ -iY_L & 0 & 0 \\ iY_T & 0 & 0 \end{bmatrix} \tag{34}$$

where the term $-X\tilde{\mathbf{I}}$ can now be combined with $\varepsilon_m\tilde{\mathbf{I}}$ in (28). Thus we see that the full Faraday rotation phenomenon involves transversal as well as longitudinal components of the external magnetic field. Subtracting the effect due to (34) from the complete effect (29) leaves the effect due to second order terms only

$$\tilde{\mathbf{Y}} = -\frac{X}{1 - Y^2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & (1 - Y_T^2) & -Y_T Y_L \\ 0 & -Y_T Y_L & (1 - Y_L^2) \end{bmatrix} \tag{35}$$

Constitutive relations possessing second order effect are related to the Cotton-Mouton effect [7]. Subsequently we use (29) and consider $\tilde{\mathbf{Y}}$ in its entirety to be the perturbation parameter. This choice is less restrictive and allows for the discussion of more general cases.

If we now specialize (12) to the case (28), (29), we get

$$\tilde{\mathbf{a}} = \varepsilon_0\tilde{\mathbf{Y}}, \qquad \tilde{\mathbf{b}} = \tilde{\mathbf{c}} = \tilde{\mathbf{d}} = 0 \tag{36}$$

which for the present class of problem vastly simplifies the general formalism.

5. A NOTE ON SUPERPOSITION

Generally speaking, when we deal with linear differential equations, we can superpose sources, i.e., get separate solutions for various inhomogeneous terms, usually referred to as sources by the physicist-engineer, and sum up the individual solutions. The coefficients in the differential equations cannot be superposed, much like you cannot superpose elements in a circuit, although you can superpose (independent) voltage and current generators.

The formalism stated above is linear. It is therefore obvious that we may use superposition considerations, i.e., analyze the problem for various zero order fields individually, and sum them up, as well as the ensuing first order constituents to obtain a combined solution.

The question arises as to the superposition of perturbation parameters, which play here a double role: They are fields for the zero order case, but sources for the first order case. E.g., in the analysis for gyromagnetic media in the previous section, can we compute the results for various configurations of the external magnetic field, and then assume that the combined first order field is due to the combined external magnetic field? By inspection of (12) it becomes clear that if the zero fields are recast as a sum of fields, each component of the sum will be associated with a component of the effective first order current density, (13), resulting in a component first order field obtained from (15). If in (12) the perturbation constitutive dyadics are represented as sums, we can still define a sum of effective first order current densities in (13), and thus obtain the sum of the constituent first order fields in (15) — provided that if the zero order fields are split into sums as well, care is taken to include all the various cross product terms.

The situation is further complicated when the perturbation constitutive dyadics are nonlinearly dependent on a parameter: In (29), in view of the terms containing a product of Y or its components, the dependence on the external magnetic field is nonlinear. It is obvious that a superposition of different external magnetic fields will not include cross products, and is therefore invalid. On the other hand, restricting the discussion to first order effects in the external magnetic field, as expressed in (34), allows for separation or superposition.

6. APPLICATION TO PLANE WAVES IN GYROMAGNETIC MEDIA

The simple examples considered here belong to the class of problems where the initial fields are considered as the homogeneous solutions of (18), i.e., in a sourceless domain. We may therefore forgo the first step of computing the zero order fields according to (8) and proceed to compute the first order fields according to (15).

Consider the case of an incident plane wave (21), field polarized along the x axis.. Exploiting (29), (36), and the definition of first order effective currents (13), yields

$$\mathbf{j}_{1,e,x} = -i\omega\tilde{\mathbf{a}}\cdot\mathbf{E}_{0,x} = \omega\frac{X}{1-Y^2}\left(i\hat{\mathbf{x}}+\hat{\mathbf{y}}Y_L-\hat{\mathbf{z}}Y_T\right)E_{0,x}e^{i(kz-\omega t)} \quad (37)$$

where the index x keeps track of the direction of field polarization of the exciting wave. Similarly to what was done in (23), and using the general one dimensional expressions (A11)–(A13), the first order field $\mathbf{E}_1$ is derived. In the present context note that $\left(\tilde{\mathbf{I}}+k^{-2}d_z d_z\hat{\mathbf{z}}\hat{\mathbf{z}}\right)e^{ikz} = \left(\hat{\mathbf{x}}\hat{\mathbf{x}}+\hat{\mathbf{y}}\hat{\mathbf{y}}\right)e^{ikz}$. Consequently in integrals of the kind (23), (24), and (27), the z direction component of the effective current $\mathbf{j}_{1,e}$, (37), does not feature. Hence the only effect of the transverse magnetic field Y_T enters via the factor $1/(1-Y^2)$.

Similarly to (23), (24), we now get in the forward direction

$$\begin{aligned}\mathbf{E}_{1,x}(z) &= -\frac{\varepsilon_0\omega^2\mu}{2k}\frac{X\left(i\hat{\mathbf{x}}+\hat{\mathbf{y}}Y_L\right)}{1-Y^2}(b-a)E_{0,x}e^{i(kz-\omega t)}\\ &= \left(i\gamma_x\hat{\mathbf{x}}+\gamma_y\hat{\mathbf{y}}\right)E_{0,x}e^{i(kz-\omega t)} \end{aligned} \quad (38)$$

The first term in (38) is very similar to the simple example (24), (25), i.e., the combined field polarized in the x direction acquires an offset phase factor $e^{i\gamma_x}$ similar to the one in (25). This effect in our model depends on the magnetic field to the second order, and can be neglected in the case where $Y^2 \ll 1$. All that remains is the effect of the background dielectric parameter as calculated in (24), (25). The second term in (38) describes an additional wave component in the y direction, in phase with the zero order excitation field. When added up to the first order field, the resultant field polarization is still in the x-y plane, but is rotated relative to the x axis by an angle of γ_y radians. This is the Faraday rotation as we know it. Interestingly, in the present formalism no need arises to discuss two plane, circularly field polarized waves, in opposite directions of rotation, and possessing different propagation vectors, the way it is usually done in textbooks.

For a plane wave field polarized in the y direction (37) is now replaced by

$$\begin{aligned}\mathbf{j}_{1,e,y} &= -i\omega\tilde{\mathbf{a}}\cdot\mathbf{E}_{0,y}\\ &= \omega\frac{X}{1-Y^2}\left(-\hat{\mathbf{x}}Y_L+i\hat{\mathbf{y}}(1-Y_T^2)-i\hat{\mathbf{z}}Y_TY_L\right)E_{0,y}e^{i(kz-\omega t)} \end{aligned} \quad (39)$$

and the corresponding first order field now becomes

$$\begin{aligned}\mathbf{E}_{1,y}(z) &= \frac{\varepsilon_0\omega^2\mu}{2k}\frac{X\left(\hat{\mathbf{x}}Y_L-i\hat{\mathbf{y}}(1-Y_T^2)\right)}{1-Y^2}(b-a)E_{0,y}e^{i(kz-\omega t)}\\ &= \left(\delta_x\hat{\mathbf{x}}+i\delta_y\hat{\mathbf{y}}\right)E_{0,y}e^{i(kz-\omega t)} \end{aligned} \quad (40)$$

Comparison of (38), (40) reveals that $\gamma_y = -\delta_x$, corresponding to a Faraday rotation of the first order fields with respect to the corresponding zero order fields, in the same sense and through the same angle. If $Y_T^2 \ll 1$ then $\gamma_x = \delta_y$, indicating the same phase shift for the two first order waves, compared to their corresponding first order ones. However if Y_T^2 cannot be neglected, then the phase shift is not identical, hence for the first order waves we are dealing with two harmonic waves at the same frequency, whose fields are perpendicularly polarized in space, and which are out of phase — giving rise to elliptical polarization, with the amount of ellipticity depending on the phase offset between the waves and their amplitudes. Obviously this is a second order effect in powers of the components of Y, i.e., the elliptical polarization effect is superimposed on the zero order fields and the first order Faraday rotation effect.

7. A SIMPLE VECTOR SPHERICAL WAVE EXAMPLE

The efficacy of the present formalism is demonstrated for three dimensional vector problems. As opposed to plane wave solutions, the more general three dimensional ones facilitate the discussion of nonuniform fields, e.g., converging and diverging waves as encountered in optical systems. The present study focuses on simple sources described by spherical waves whose spatial radiation characteristics correspond to elementary short dipole and small loop antennas.

The present simple example is the analog of the one dimensional case discussed above: We assume an infinite ambient space characterized by ε_1 in which a spherical shell region of material possessing a dielectric constant $\varepsilon_1 + \varepsilon_2$ is embedded, with $\varepsilon_1 \gg \varepsilon_2$. The shell region extends from $r = a$ to $r = b$. Once again $\mathbf{j}_{1,e}$ is given by (22).

The simplest source wave that can be chosen here is a dipole field given by (B1), (B2), using the lowest possible indices $n = 1,\ m = 0$. Thus choosing for the source field

$$\begin{aligned}\mathbf{E}_0(\mathbf{r}) &= E_0 e^{-i\omega t} i\mathbf{M}_{1,0}(\mathbf{r})\\ &= E_0 e^{-i\omega t} i h_1(kr)\mathbf{C}_1^0(\hat{\mathbf{r}}) \cong E_0 e^{-i\omega t} h_0(kr)\mathbf{C}_1^0(\hat{\mathbf{r}}) \qquad (41)\\ \mathbf{C}_1^0(\hat{\mathbf{r}}) &= -\mathbf{r}\times\partial_{\mathbf{r}} Y_1^0(\hat{\mathbf{r}}) = -\hat{\boldsymbol{\varphi}}\partial_\theta P_1^0 = \hat{\boldsymbol{\varphi}}\sin\theta\end{aligned}$$

which is recognized as a magnetic dipole, i.e., the field radiated from a small current loop, behaving as an outgoing spherical wave. The field is polarized in the azimuthal direction, attaining its maximum at the equator and possessing a toroidal shaped radiation pattern, with a circular cross section according to $\hat{\boldsymbol{\varphi}}\sin\theta$. On the other hand, choosing

$$\begin{aligned}\mathbf{E}_0(\mathbf{r}) &= E_0 e^{-i\omega t}\mathbf{N}_{1,0}(\mathbf{r})\\ &= E_0 e^{-i\omega t}\left\{2h_1(kr)\mathbf{P}_1^0(\hat{\mathbf{r}}) + \partial_{kr}[krh_1(kr)]\mathbf{B}_1^0(\hat{\mathbf{r}})\right\}/kr\\ &= E_0 e^{-i\omega t}\left\{q_{1,P}(kr)\mathbf{P}_1^0(\hat{\mathbf{r}}) + q_{1,B}(kr)\mathbf{B}_1^0(\hat{\mathbf{r}})\right\}\\ &\cong E_0 e^{-i\omega t}h_0(kr)\mathbf{B}_1^0(\hat{\mathbf{r}})\\ \mathbf{B}_1^0(\hat{\mathbf{r}}) &= \hat{\mathbf{r}}\times\mathbf{C}_1^0(\hat{\mathbf{r}}) = -\hat{\boldsymbol{\theta}}\sin\theta,\ \mathbf{P}_1^0(\hat{\mathbf{r}}) = \hat{\mathbf{r}}\cos\theta\end{aligned} \tag{42}$$

we obtain an electric type dipole, i.e., a short dipole antenna oriented along the polar axis, once again possessing a toroidal shaped radiation pattern with a circular cross section. The field is polarized in the meridian plane according to $-\hat{\boldsymbol{\theta}}\sin\theta$.

We will demonstrate the solution of (15) for the simple zero order waves given in (41), (42). From (22), (41), (42) follow the expressions for the first order equivalent current source $\mathbf{j}_{1,e}$. Substituting (B5) into (15) and invoking the spatial and integral orthogonality relations (B4) prescribes for the present problem, it is obvious by inspection that the dyadic Green's function degenerates into a single mode

$$\tilde{\boldsymbol{\Gamma}}(\mathbf{r},\mathbf{r}') = \frac{ik}{4\pi}\frac{3}{2}\mathbf{M}_{1,0}(\mathbf{r})\mathbf{M}_{1,0}^{(1)}(\mathbf{r}') = \frac{3ik}{8\pi}h_1(kr)j_1(kr')\mathbf{C}_1^0(\hat{\mathbf{r}})\mathbf{C}_1^0(\hat{\mathbf{r}}') \tag{43}$$

Incorporating all the above in (15) we derive

$$\begin{aligned}\mathbf{E}_{1,>}(\mathbf{r}) &= \varepsilon_2\omega^2\mu\tfrac{3ik}{8\pi}E_0 e^{-i\omega t}h_1(kr)\mathbf{C}_1^0(\hat{\mathbf{r}})\tfrac{i}{k^3}f_>(b,a)g\\ f_>(b,a) &= \int_a^b d(kr')(kr')^2 j_1(kr')h_1(kr')\\ g &= \int d\Omega(\hat{\mathbf{r}}')\mathbf{C}_1^0(\hat{\mathbf{r}}')\cdot\mathbf{C}_1^0(\hat{\mathbf{r}}') = \frac{8\pi}{3}\end{aligned} \tag{44}$$

where g is the result of the orthogonality integral in (B4). The result (44) is valid outside the spherical shell defined by $a < r' < b$, for the forward direction $r > b > a$. After some manipulation of (44), where some factors were left in raw form to show how they are derived, we find

$$\begin{aligned}\mathbf{E}_{1,>}(\mathbf{r}) &= -\frac{\varepsilon_2\omega^2\mu}{k^2}E_0 e^{-i\omega t}h_1(kr)\mathbf{C}_1^0(\hat{\mathbf{r}})f_>(b,a)\\ &= -\frac{\varepsilon_2}{\varepsilon_1}E_0 e^{-i\omega t}h_1(kr)f_>(b,a)\hat{\boldsymbol{\varphi}}\sin\theta\\ &\cong \frac{\varepsilon_2}{\varepsilon_1}E_0 e^{-i\omega t}ih_0(kr)f_>(b,a)\hat{\boldsymbol{\varphi}}\sin\theta\end{aligned} \tag{45}$$

In (44), (45) $f_>(b,a)$ is a function of the limits of the integral, corresponding to $(b-a)$ appearing in (24). However note that it is an integration of a complex function between real limits, hence $f_>(b,a)$ is expected to be complex. We have tried to evaluate integrals of the kind $f_>(b,a)$ using the mathematical package "Mathematica" [16], and got explicit analytical representations in terms of known special functions, however the complicated forms do not warrant being quoted here. For large radii $ka,\ kb \gg 1$ the integrand of $f_>(b,a)$ becomes $-e^{ikr'}\sin(kr')$ hence for kr' large, and assuming a small loss term as discussed after (A3), we get

$$f_>(b,a) \cong -i(b-a)/2 \tag{46}$$

becoming the limiting case for the slab region discussed above for the plane waves one dimensional case. Note that in (45) $ih_0(kr)$ is a real quantity, hence for the limiting case (44), (45) can be approximated with a phase factor, as discussed for the plane wave case (24), (25).

The same steps can be retraced to integrate (15) for a zero order wave field given by (42) (or any combination of higher order multipoles as given by (B1), for that matter, although the details might be somewhat more complicated): Instead of (41) we now have

$$\begin{aligned}\tilde{\boldsymbol{\Gamma}}(\mathbf{r},\mathbf{r}') &= \frac{ik}{4\pi}\frac{3}{2}\mathbf{N}_{1,0}(\mathbf{r})\mathbf{N}^{(1)}_{1,0}(\mathbf{r}')\\ &= \frac{3ik}{8\pi}\Big(q_{1P}(kr)\mathbf{P}^0_1(\hat{\mathbf{r}})q^{(1)}_{1,P}(kr')\mathbf{P}^0_1(\hat{\mathbf{r}}')\\ &\qquad +q_{1,B}(kr)\mathbf{B}^0_1(\hat{\mathbf{r}})q^{(1)}_{1,B}(kr')\mathbf{B}^0_1(\hat{\mathbf{r}}')\Big)\end{aligned} \tag{47}$$

where the various functions $q_{1,P}(kr),\ q_{1,B}(kr)$ are defined in (42), and in the corresponding $q^{(1)}_{1,P}(kr'),\ q^{(1)}_{1,B}(kr')$ the nonsingular spherical Bessel functions $j_n(kr)$ replace the spherical Hankel functions $h_n(kr)$. In view of the similar orthogonality relations (B4) everything follows through, with the appropriate coefficients and the radial integrals corresponding to $f_>(b,a)$ above. If the circumstances allow the use of the approximate expression in (42), i.e., when $kr,\ kr' \gg 1$ can be assumed, then we get the approximate form (45) with $\mathbf{B}^0_1(\hat{\mathbf{r}}) = -\hat{\boldsymbol{\theta}}\sin\theta$ replacing $\mathbf{C}^0_1(\hat{\mathbf{r}}) = \hat{\boldsymbol{\varphi}}\sin\theta$.

The corresponding case for a source within the inner shell, i.e., $r < a < b$ follows the same lines, with the appropriate modifications: We start with the same zero order fields (41), (42), but in (43) and (47), primed and unprimed arguments must be interchanged. Consider the zero order field (41). Retracing the above argument, we derive

$$\mathbf{E}_{1,<}(\mathbf{r}) = -\frac{3}{8\pi}\frac{\varepsilon_2}{\varepsilon_1}E_0 e^{-i\omega t} j_1(kr)\mathbf{C}^0_1(\hat{\mathbf{r}})f_<(b,a)g$$

$$
\begin{aligned}
&= -\frac{\varepsilon_2}{\varepsilon_1} E_0 e^{-i\omega t} j_1(kr) f_<(b,a) \hat{\boldsymbol{\varphi}} \sin\theta \\
&\cong \frac{\varepsilon_2}{\varepsilon_1} E_0 e^{-i\omega t} \Im(h_0(kr)) f_<(b,a) \hat{\boldsymbol{\varphi}} \sin\theta \qquad (48)\\
f_<(b,a) &= \int_a^b d(kr')(kr' h_1(kr'))^2, \;\; g = \int d\Omega(\hat{\mathbf{r}}') \mathbf{C}_1^0(\hat{\mathbf{r}}') \mathbf{C}_1^0(\hat{\mathbf{r}}') = \frac{8\pi}{3}
\end{aligned}
$$

where $\Im(\;)$ denotes the imaginary part of the function in question, and the approximation applies as long as $kr,\; kr' \gg 1$ can be assumed.

8. APPLICATION TO VECTOR SPHERICAL WAVES IN GYROMAGNETIC MEDIA

Strictly speaking, the theory embodied in (29) (e.g., see [13]) applies to proper plane waves, and is based on an equation of motion involving inertia and the Lorentz force acting on charges. Furthermore, this theory applies to homogeneous media only, and assumes that the total (so called "material") derivative, and partial derivative are identical, or in other words, second order terms in material velocity, tantamount to nonlinear media effects, are negligible [17]. It follows that we have to use the integral plane wave representation (B11), set up a local coordinate system $\mathbf{p}(\xi,\eta,\zeta)$, corresponding to $\mathbf{r}(x,y,z)$ in Fig. 1, and find the components of the plane waves along these directions

$$
\begin{aligned}
[E_\xi(\mathbf{r}), E_\eta(\mathbf{r}), E_\zeta(\mathbf{r})] &= E_0 e^{-i\omega t} \frac{1}{2\pi} \int d\Omega(\hat{\mathbf{p}}) e^{ik\hat{\mathbf{p}}\cdot\mathbf{r}} \\
&\quad \cdot \left[-\frac{\hat{\mathbf{p}} \times \mathbf{Y}}{|\hat{\mathbf{p}} \times \mathbf{Y}|}, \frac{\left(\tilde{\mathbf{I}} - \hat{\mathbf{p}}\hat{\mathbf{p}}\right) \cdot \mathbf{Y}}{|\hat{\mathbf{p}} \times \mathbf{Y}|}, \hat{\mathbf{p}} \right] \cdot \mathbf{g}(\hat{\mathbf{p}}) \qquad (49)
\end{aligned}
$$

and apply (29) to these components under the integral operator. This procedure is too complicated, and can be abated if the curvature of the spherical waves is not too large, i.e., when kr in the associated spherical Bessel functions is not too small. For such cases the approximation

$$
[g_\xi(\hat{\mathbf{r}}), g_\eta(\hat{\mathbf{r}}), g_\zeta(\hat{\mathbf{r}})] = \left[\frac{\hat{\mathbf{r}} \times \mathbf{Y}}{|\hat{\mathbf{r}} \times \mathbf{Y}|}, \frac{\left(\tilde{\mathbf{I}} - \hat{\mathbf{r}}\hat{\mathbf{r}}\right) \cdot \mathbf{Y}}{|\hat{\mathbf{r}} \times \mathbf{Y}|}, \hat{\mathbf{r}} \right] \cdot \mathbf{g}(\hat{\mathbf{r}}) \qquad (50)
$$

is adequate, according to (B7), (B11), for obtaining the components of the scattering amplitude $\mathbf{g}(\hat{\mathbf{r}})$ along the directions $\mathbf{r}(\xi,\eta,\zeta)$ of the local Cartesian system of coordinates, respectively.

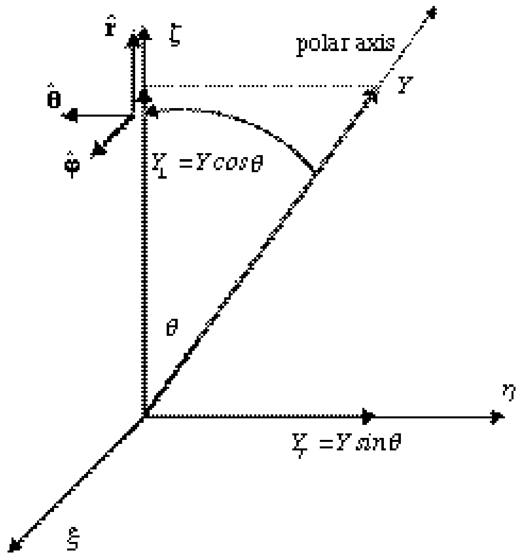

Figure 2. Geometry for spherical coordinates and external magnetic field along the $\theta = 0$ polar axis.

As a first example, consider the case of the external magnetic field directed along the polar axis $\theta = 0$, see Fig. 2, depicting the relation of the spherical coordinates to the local system $\mathbf{r}(\xi, \eta, \zeta)$

$$\begin{aligned}\mathbf{j}_{1,e,\xi} &= -i\omega\tilde{\mathbf{a}}\cdot\mathbf{E}_{0,\xi} = \omega W\left(i\hat{\boldsymbol{\xi}}\hat{\boldsymbol{\xi}} + \hat{\boldsymbol{\eta}}\hat{\boldsymbol{\xi}}Y_L - \hat{\boldsymbol{\zeta}}\hat{\boldsymbol{\xi}}Y_T\right)\cdot\mathbf{E}_{0,\xi}\\ &= \omega W\left(i\hat{\boldsymbol{\varphi}}\hat{\boldsymbol{\varphi}} - \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\varphi}}Y\cos\theta - \hat{\mathbf{r}}\hat{\boldsymbol{\varphi}}Y\sin\theta\right)\cdot E_0 e^{-i\omega t} i\mathbf{M}_{1,0}(\mathbf{r})\\ &= \omega W E_0 e^{-i\omega t} i h_1(kr)\left(i\hat{\boldsymbol{\varphi}}\hat{\boldsymbol{\varphi}} - \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\varphi}}Y\cos\theta - \hat{\mathbf{r}}\hat{\boldsymbol{\varphi}}Y\sin\theta\right)\cdot\mathbf{C}_1^0(\hat{\mathbf{r}})\\ &= \omega W E_0 e^{-i\omega t} i h_1(kr)\left(i\mathbf{C}_1^0(\hat{\mathbf{r}}) - \hat{\boldsymbol{\theta}}Y\frac{1}{2}\sin 2\theta - \hat{\mathbf{r}}Y\sin^2\theta\right) \quad (51)\end{aligned}$$

where W is given in (30). Comparing (51) to (37), with the first order fields given by (41), (21), respectively, it is seen that the first vector term in the parentheses (51) replicates the radiation pattern of the zero order wave, with an extra factor i, hence it amounts to adding a phase factor to the first order wave as in (25). In view of (41) which leads to (45), the first order field corresponding to this term is given by (45) with ε_2 replaced by $-iW$. See also (45), (46) and the associated discussion there. The last vector term in parentheses in (51) is longitudinal. And contains the factor $\sin^2\theta$. Using the Legendre functions $P_0 = 1$, $P_1 = \cos\theta$, $P_2 = (3/4)\cos 2\theta + 1/4$, we recast $\sin^2\theta = (2/3)(1-P_2)$, and derive the corresponding longitudinal vector spherical harmonics and vector spherical waves, (B3), and thus the contribution to the first order field can be computed from (15), using the dyadic Green's function (B5). It is noted that in the one dimensional case which led to (38), (40), the longitudinal field does not feature. In the present case, as seen from (B3), as kr increases the longitudinal wave component becomes negligible compared to the transversal constituent. Therefore the detailed calculation is avoided here.

The remaining term in (51) is of particular interest because it corresponds to the Faraday rotation phenomenon. Using the definitions in (B3), we recast

$$\hat{\boldsymbol{\theta}}\sin 2\theta = -\hat{\boldsymbol{\theta}}\frac{2}{3}\partial_\theta Y_2^0 = -\frac{2}{3}\mathbf{B}_2^0 \tag{52}$$

It is interesting to note that the wave corresponding to the vector spherical harmonic (52) is a higher multipole term, a quadrupole term in the present example, which unlike the zero order field possesses zeroes at $\theta = 0,\ \pi/4$. For this term the effective first order current (51) is given by

$$\mathbf{j}_{1,B} = \omega W E_0 e^{-i\omega t} i h_1(kr)\frac{1}{3}\mathbf{B}_2^0(\hat{\mathbf{r}}) \tag{53}$$

and instead of (47) we now have

$$\begin{aligned}\tilde{\boldsymbol{\Gamma}}(\mathbf{r},\mathbf{r}') &= \frac{ik}{4\pi}\frac{5}{6}\mathbf{N}_{2,0}(\mathbf{r})\mathbf{N}_{2,0}^{(1)}(\mathbf{r}')\\ &= \frac{5ik}{24\pi}\Big(q_{2,P}(kr)\mathbf{P}_2^0(\hat{\mathbf{r}})q_{2,P}^{(1)}(kr')\mathbf{P}_2^0(\hat{\mathbf{r}}')\\ &\qquad + q_{2,B}(kr)\mathbf{B}_2^0(\hat{\mathbf{r}})q_{2,B}^{(1)}(kr')\mathbf{B}_2^0(\hat{\mathbf{r}}')\Big)\end{aligned} \tag{54}$$

with the appropriate functions $q_{2,P}(kr)$, $q_{2,B}(kr)$ defined in (B3) and in the corresponding $q_{2,P}^{(1)}(kr')$, $q_{2,B}^{(1)}(kr')$ the nonsingular spherical Bessel functions $j_n(kr)$ replace the spherical Hankel functions $h_n(kr)$. As before, we will ignore the contribution of the longitudinal functions, assuming large arguments in the spherical functions. Keeping all the terms is not much more complicated, but the expressions are becoming cumbersome. Inserting all the above ingredients into (15) yields

$$\mathbf{E}_{1,B,>}(\mathbf{r}) = i\omega^2\mu W \frac{5ik}{24\pi} E_0 e^{-i\omega t} q_{2,B}(kr)\mathbf{B}_2^0(\hat{\mathbf{r}})\frac{i}{k^3} f_{B,>}(b,a) g_B$$

$$f_{B,>}(b,a) = \int_a^b d(kr')(kr')^2 q_{2,B}^{(1)}(kr') i h_1(kr') \tag{55}$$

$$g_B = \int d\Omega(\hat{\mathbf{r}}')\mathbf{B}_2^0(\hat{\mathbf{r}}') = \frac{24\pi}{5},\ q_{2,B}(kr) \cong -h_0(kr)$$

which can be somewhat simplified, but is left in the present form to facilitate an easy check of the various factors.

Thus we have demonstrated the feasibility of manipulating the various vector spherical terms, and the integration of (15) under the present circumstances. The fact that the first order field acquires new multipoles, i.e., displays different radiation patterns is an interesting and novel result of the present formalism.

The associated case of a first order field polarized along the η axis, Fig. 2, follows the same lines. For this case the zero order field will be taken as (42). The result is expected to follow the one dimensional case (39), (40), with the appropriate modifications as shown by comparing (37) and (51). Thus we have

$$\begin{aligned}
\mathbf{j}_{1,e,\eta} &= -i\omega\tilde{\mathbf{a}}\cdot\mathbf{E}_{0,\eta} \\
&= \omega W\left(-\hat{\boldsymbol{\xi}}\hat{\boldsymbol{\eta}}Y_L + i\hat{\boldsymbol{\eta}}\hat{\boldsymbol{\eta}}(1-Y_T^2) - i\hat{\boldsymbol{\zeta}}\hat{\boldsymbol{\eta}}Y_T T_L\right)\cdot\mathbf{E}_{0,\eta} \\
&= \omega W\left(\hat{\boldsymbol{\varphi}}\hat{\boldsymbol{\theta}}Y\cos\theta + i\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}(1-Y^2\sin^2\theta) + i\hat{\mathbf{r}}\hat{\boldsymbol{\theta}}Y\sin\theta\right) \\
&\qquad \cdot E_0 e^{-i\omega t}\mathbf{N}_{1,0}(\mathbf{r}) \\
&\cong -\omega W\left(\hat{\boldsymbol{\varphi}}\hat{\boldsymbol{\theta}}Y\cos\theta + i\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}\right)\cdot E_0 e^{-\omega t}\mathbf{B}_1^0(\hat{\mathbf{r}}) \\
&\cong \omega W\left(\hat{\boldsymbol{\varphi}}\hat{\boldsymbol{\theta}}Y\cos\theta + i\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}\right)\cdot E_0 e^{-\omega t} q_{1,B}(kr)\hat{\boldsymbol{\theta}}\sin\theta \\
&\cong \omega W\left(\hat{\boldsymbol{\varphi}}Y\frac{1}{2}\sin 2\theta + i\hat{\boldsymbol{\theta}}\sin\theta\right)E_0 e^{-i\omega t}h_0(kr)
\end{aligned} \tag{56}$$

where for brevity we decided to keep only the first order term in Y, ignore the longitudinal functions, and approximate the spherical functions. The similarity of (51) and (56) is then obvious, and the computation leading to (55) follows, and need not be displayed here in detail: The second term in parentheses in the last line, (56), replicates the zero order excitation wave vector spherical harmonic $\mathbf{B}_1^0(\hat{\mathbf{r}})$, and the initial zero order wave, multiplied by i; the first term, using the definition (B3), yields a vector spherical harmonic $\mathbf{C}_2^0(\hat{\mathbf{r}})$. This is the analog situation to (52)–(55).

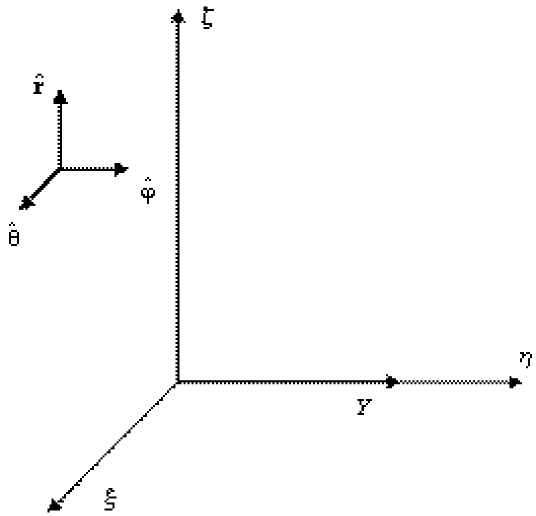

Figure 3. Geometry for spherical coordinates and circumferential external magnetic field.

There are other axisymmetric cases that can be easily coped with. Consider for example a circumferentially oriented field created by a long line current coinciding with the polar axis. For this case we have

$$Y_L = 0, \; Y_T = Y_0/r\sin\theta \tag{57}$$

See Fig. 3. For a field polarized perpendicularly to the magnetic field and the radial coordinate, we have

$$\begin{aligned} \mathbf{j}_{1,e,\xi} &= -i\omega\tilde{\mathbf{a}}\cdot\mathbf{E}_{0,\xi} = \omega W\left(i\hat{\boldsymbol{\xi}}\hat{\boldsymbol{\xi}} + \hat{\eta}\hat{\boldsymbol{\xi}}Y_L - \hat{\boldsymbol{\zeta}}\hat{\boldsymbol{\xi}}Y_T\right)\cdot\mathbf{E}_{0,\xi} \\ &= \mathbf{j}_{1,e,\theta} = \omega W\left(i\hat{\boldsymbol{\theta}} - \hat{\mathbf{r}}Y_T\right)\hat{\boldsymbol{\theta}}\cdot\mathbf{E}_{0,\theta} \end{aligned} \tag{58}$$

which prescribes that for the lowest dipole wave only (42) can be chosen. At large distances the longitudinal part in (58) becomes negligible, hence only a first order field polarized in the original direction of the zero order field is present. As was the case in (24), such a field can only create a phase shift, but no field polarization rotation. The physics of this situation is easily understood by considering the effect of the oscillating charges in the presence of the external magnetic field: The gyration takes place in the perpendicular plane, and the rotating charges will now produce a field along the direction of propagation, with a quarter period phase shift. On the other hand, for the case

$$\mathbf{j}_{1,e,\eta} = -i\omega\tilde{\mathbf{a}} \cdot \mathbf{E}_{0,\eta} = \omega W \left(-\hat{\boldsymbol{\xi}}\hat{\boldsymbol{\eta}} Y_L + i\hat{\boldsymbol{\eta}}\hat{\boldsymbol{\eta}}(1 - Y_T^2) - i\hat{\boldsymbol{\zeta}}\hat{\boldsymbol{\eta}} Y_T Y_L \right) \cdot \mathbf{E}_{0,\eta}$$
$$\mathbf{j}_{1,e,\varphi} = i\omega W \left(1 - Y_T^2\right) \hat{\boldsymbol{\varphi}}\hat{\boldsymbol{\varphi}} \cdot \mathbf{E}_{0,\varphi} = i\omega X \hat{\boldsymbol{\varphi}}\hat{\boldsymbol{\varphi}} \cdot \mathbf{E}_{0,\varphi} \tag{59}$$

only (41) can provide the lowest case dipole first order field. In this case the field is polarized parallel to the external magnetic field, and no gyration, hence no field polarization rotation is expected.

Other axisymmetric are easily conceived, e.g., the field of an external field magnetic dipole oriented along the polar axis $\mathbf{Y} = \hat{\boldsymbol{\theta}} Y_T + \hat{\mathbf{r}} Y_L = r^{-3} Y_0 (\hat{\boldsymbol{\theta}} \sin\theta + 2\hat{\mathbf{r}} \cos\theta)$. For a more general approach, see for example [19], discussing the general theory of electrostatic and magnetostatic multipoles derivable from scalar potential functions. However, calculations involving higher multipoles become increasingly cumbersome and will not be further pursued here. More generally, both the magnetic field and the zero order wave can be any desired functions, and in view of (B10) the effective first order current densities (13) can always be represented in terms of sums of vector spherical harmonics. Using the dyadic Green's function (B5), this facilitates the solution of the integrals (15). In general the solution of the integrals (B10) requires numerical techniques, therefore it is of some interest to guess simple canonical examples as done above.

Continuing this line of thought, and in view of (55) and the steps leading to it, it becomes clear that the structure of the magnetic field affects the multipole configuration of the first order field. For example, when the magnetic field is also dependent on the azimuthal angle φ, it is expected that the first order field will manifest φ dependence as well.

9. CONCLUDING REMARKS

The present study considers wave propagation in arbitrary bi-anisotropic media. The general formalism is based on a first order perturbation scheme which assumes that the zero order ambient medium is a simple isotropic material. Of course, the penalty for using this relatively simple formalism is its restricted validity for small bi-anisotropic effects. The present study was motivated by the problem of gyromagnetic effects in fluids where the proviso for small bi-anisotropic effects is met. Such problems arise, for example in ocular glucose polarimetry [18].

The material model chosen for the present study is the gyromagnetic effect in the presence of moving charges, as observed for magnetized cold plasma, e.g., the ionosphere, and fluid and amorphic solids, where in addition an ambient dielectric background is assumed. The present model is adequate for discussing various cases of interest. Inasmuch as the formalism is based on an integral involving the pertinent Green's function, it was felt that providing concrete examples could be important for the application oriented reader, and should be included. Examples shown include plane and spherical vector waves, and presented in a closely knit manner, although sometimes this required more discussion of details. General theoretical formulas, especially those involving the properties of spherical vector waves and harmonics are summarized in the appendices.

A general observation that emerges from the analysis is that the geometrical configuration of the magnetic field shapes the ensuing first order wave field — new multipole terms are generated that were not present in the initial zero order wave field. Another interesting observation is the fact that we derive the Faraday rotation effect with using the usual method of separating the incident plane wave into oppositely rotating circularly field polarized waves.

We made an effort to present the material in a way that could help the application oriented user. To that end, many details are included, and the paper is made self contained as much as a journal article can allow.

APPENDIX A.

The basic theory leading to the Green's function integral (8) is summarized and specialized to the one dimensional problem. The electromagnetic fields (5) can be expressed in terms of vector potentials. E.g., for the e-indexed fields (5),

$$\mathbf{H}_e = \mu^{-1}\partial_{\mathbf{r}} \times \mathbf{A}_e, \quad \mathbf{E}_e = i\omega\left(1 + k^{-2}\partial_{\mathbf{r}}\partial_{\mathbf{r}}\right)\mathbf{A}_e \tag{A1}$$

where the Lorentz condition $\partial_{\mathbf{r}} \cdot \mathbf{A}_e - i\omega\mu\varepsilon\varphi_e = 0$ has been incorporated to eliminate the scalar potential in the electric field definition $\mathbf{E}_e = -\partial_{\mathbf{r}}\varphi_e + i\omega\mathbf{A}_e = 0$. We now get from (5) and (A1) a wave equation on $\mathbf{A}_e$, to which we add a corresponding definition of the scalar Green's function in terms of its pertinent wave equation

$$\begin{gathered}\partial_{\mathbf{r}}^2\mathbf{A}_e(\mathbf{r}) + k^2\mathbf{A}_e(\mathbf{r}) = -\mu\mathbf{j}_e(\mathbf{r}) \\ \partial_{\mathbf{r}}^2 G(\mathbf{r}-\mathbf{r}') + k^2 G(\mathbf{r}-\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}')\end{gathered} \tag{A2}$$

where we have emphasized that the operations in (A2) are with respect to $\mathbf{r}$. to actually find the Green's function for various dimensionalities, media, and boundary conditions, is a separate subject which is outside the scope of the present study (e.g., see [2, 4]). Multiplying the first formula (A2) by G and the second by $\mathbf{A}_e$, subtracting and integrating over space, we get

$$\begin{aligned}
&\int \Big(G(\mathbf{r}'-\mathbf{r})\partial^2_{\mathbf{r}'}\mathbf{A}_e(\mathbf{r}') - \mathbf{A}_e(\mathbf{r}')\partial^2_{\mathbf{r}'}G(\mathbf{r}'-\mathbf{r})\Big)\, dV(\mathbf{r}') \\
= &\oint \left(G(\mathbf{r}'-\mathbf{r})\partial_{\mathbf{r}'}\mathbf{A}_e(\mathbf{r}') - \mathbf{A}_e(\mathbf{r}')\partial_{\mathbf{r}'}G(\mathbf{r}'-\mathbf{r})\right)\cdot d\mathbf{S}(\mathbf{r}') \\
= &\int \left(\mathbf{A}_e(\mathbf{r}')\delta(\mathbf{r}'-\mathbf{r}) - \mu\mathbf{j}_e(\mathbf{r}')G(\mathbf{r}'-\mathbf{r})\right) dV(\mathbf{r}') \\
= &\ \mathbf{A}_e(\mathbf{r}) - \mu\int \left(\mathbf{j}_e(\mathbf{r}')G(\mathbf{r}'-\mathbf{r})\right) dV(\mathbf{r}')
\end{aligned} \tag{A3}$$

where the arguments are included to clarify what operators act on what arguments. The Sommerfeld radiation condition (e.g., see [19]) ensures that the surface integral in (A3) vanishes. A simple way to get this vanishing of the waves at infinity is to assume [19] an imaginary component in k, however small. Consequently we have

$$\mathbf{A}_e(\mathbf{r}) = \mu\int \left(\mathbf{j}_e(\mathbf{r}')G(\mathbf{r}'-\mathbf{r})\right) dV(\mathbf{r}') \tag{A4}$$

and combining (A4) with (A1) the dyadic Green's function (9) is now introduced. Applying the same considerations to the m-indexed fields, one finally derives (8).

For the one-dimensional case (see for example [2]) the Green's function is particularly simple. Here (A2) reduces to the one dimensional wave equation, and the corresponding equation defining the Green's function is

$$\begin{aligned}
d^2_z\mathbf{A}_e(z) + k^2\mathbf{A}_e(z) &= -\mu\mathbf{j}_e(z) \\
d^2_z G(z-z') + k^2 G(z-z') &= -\delta(z-z')
\end{aligned} \tag{A5}$$

The pertinent Green's function is give by

$$G(z-z') = \frac{i}{2k}\begin{cases} e^{ik(z-z')}, & z \geq z' \\ e^{-ik(z-z')}, & z \leq z' \end{cases} \tag{A6}$$

To prove that (A6) is the solution of the second equation (A5) we integrate

$$\begin{aligned}
\int_{-\infty}^{\infty} \left(\left(d^2_{z'} + k^2\right) G(z'-z)\right) dz' &= \int_{-\infty}^{\infty} \left(\left(d^2_{z'} + k^2\right) G(z'-z)\right) d(z'-z) \\
&= -\int_{-\infty}^{\infty} \delta(z'-z)dz' = -1
\end{aligned} \tag{A7}$$

It is noted that everywhere except at $z = z'$ we are dealing with the homogeneous wave equation for a plane wave, i.e., the integrand $\left(d_{z'}^2 + k^2\right) G(z')$ vanishes. In the interval $-\alpha < z' - z < \alpha$ we use the fact that $G(z')$ is continuous at $z' - z = 0$, therefore according to the mean value theorem of integral calculus we get

$$\int_{-\alpha}^{\alpha} G(z') d(z' - z) \sim 2\alpha G(\alpha) \tag{A8}$$

and in the limit $\alpha \to 0$ (A8) vanishes. For the term in (A7) involving the second derivative, we use the fact that the integral of the first derivative of a function is the function itself, hence we have to evaluate

$$\int_{-\alpha}^{\alpha} d_{z'} \left(d_{z'} G(z' - z)\right) d(z' - z) = \left(d_{z'} G(z' - z)\right)\Big|_{-\alpha}^{\alpha} = -\frac{1}{2}\left(e^{ik\alpha} + e^{-ik\alpha}\right) \tag{A9}$$

which in the limit $\alpha \to 0$ becomes -1 as required.

It is now a straightforward task to adapt the one dimensional theory to the general form (8): The correspondence of (A5) to the three dimensional form (A2) suggests that in (A5) we multiply the equations and subtract, to get rid of the term $k^2 \mathbf{A}_e G$. We then integrate as in (A3). What amounts to a volume integration in (A3) becomes an integration on z'. The Green theorem converting the volume integral to a surface integral in (A3) follows through in a trivial manner, and the surface integral simply becomes the value of the integrand at $\pm z$. Using the rule for differentiating a product we obtain

$$\begin{aligned}
& \int \Big(G(z' - z) d_{z'}^2 \mathbf{A}_e(z') - \mathbf{A}_e(z') d_{z'}^2 G(z' - z)\Big) dz' \\
= & \int \left[d_{z'} \left(G(z' - z) d_{z'} \mathbf{A}_e(z')\right) - d_{z'} \left(\mathbf{A}_e(z') d_{z'} G(z' - z)\right)\right] dz' \\
= & \left(G(z' - z) d_{z'} \mathbf{A}_e(z') - \mathbf{A}_e(z') d_{z'} G(z' - z)\right)\Big|_{z_2}^{z_1} \\
= & \int \left(\mathbf{A}_e(z') \delta(z' - z) - \mu \mathbf{j}_e(z') G(z' - z)\right) dz' \\
= & \mathbf{A}_e(z) - \mu \int \left(\mathbf{j}_e(z') G(z' - z)\right) dz'
\end{aligned} \tag{A10}$$

As $z_1 \to \infty$, $z_2 \to -\infty$, the expressions vanish, due to the same arguments used in (A3), and we are left with

$$\mathbf{A}_e(z) = \mu \int \mathbf{j}_e(z') G(z' - z) dz' \tag{A11}$$

Adapting (A1) to the one dimensional case yields

$$\begin{aligned}\mathbf{H}_e &= \mu^{-1}\left(\hat{\mathbf{y}}d_z\left(\hat{\mathbf{x}}\cdot\mathbf{A}_e\right)-\hat{\mathbf{x}}d_z\left(\hat{\mathbf{y}}\mathbf{A}_e\right)\right)=\mu^{-1}d_z\left(\hat{\mathbf{y}}\hat{\mathbf{x}}-\hat{\mathbf{x}}\hat{\mathbf{y}}\right)\cdot\mathbf{A}_e \\ \mathbf{E}_e &= i\omega\left(\mathbf{A}_e+k^{-2}\hat{\mathbf{z}}d_zd_z\left(\hat{\mathbf{z}}\cdot\mathbf{A}_e\right)\right)=i\omega\left(\tilde{\mathbf{I}}+k^{-2}d_zd_z\hat{\mathbf{z}}\hat{\mathbf{z}}\right)\cdot\mathbf{A}_e\end{aligned} \tag{A12}$$

Finally for one dimension (8) follows through, with z, z' judiciously replacing $\mathbf{r}, \mathbf{r}'$. The dyadic Green's function corresponding to (9) will now be given by

$$\tilde{\boldsymbol{\Gamma}}(z,z')=\left(\tilde{\mathbf{I}}+k^{-2}d_zd_z\hat{\mathbf{z}}\hat{\mathbf{z}}\right)G(z-z') \tag{A13}$$

and the reason we are keeping vector expressions, although this is supposedly a one dimensional problem depending on the z coordinate only, stems from the fact that the current densities may have components in all directions, although the functional dependence is on the z coordinate only. The pertinent scalar Green's function pertinent to (15) is provided by (A6).

APPENDIX B.

The general Green's function integral (8) is specialized here for the three dimensional vector problem involving spherical vector waves and spherical vector harmonics.

Spherical vector waves are discussed in the literature [2, 19–23]. The main results used here are highlighted for reference and notation. The general solution of the wave equation for spherical vector electromagnetic fields in an isotropic medium is given as a superposition of the vector spherical waves modes

$$\mathbf{E}(\mathbf{r})=E_0e^{-i\omega t}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}i^n\left[c_{n,m}\mathbf{M}_{n,m}(\mathbf{r})-ib_{n,m}\mathbf{N}_{n,m}(\mathbf{r})\right] \tag{B1}$$

where the coefficients $c_{n,m}$, $b_{n,m}$ are the appropriate weights of the vector spherical waves $\mathbf{M}_{n,m}(\mathbf{r})$, $\mathbf{N}_{n,m}(\mathbf{r})$, respectively. The wave modes are related by

$$\partial_{\mathbf{r}}\times\mathbf{M}_{nm}(\mathbf{r})=k\mathbf{N}_{nm}(\mathbf{r}),\quad \partial_{\mathbf{r}}\times\mathbf{N}_{nm}(\mathbf{r})=k\mathbf{M}_{nm}(\mathbf{r}) \tag{B2}$$

and defined by

$$\begin{aligned}
\mathbf{M}_{nm}(\mathbf{r}) &= h_n(kr)\mathbf{C}_n^m(\hat{\mathbf{r}}) \\
\mathbf{C}_n^m(\hat{\mathbf{r}}) &= -\mathbf{r}\times\partial_{\mathbf{r}}Y_n^m(\hat{\mathbf{r}}) = \left(\hat{\boldsymbol{\theta}}\frac{\partial_\varphi}{\sin\theta} - \hat{\boldsymbol{\varphi}}\partial_\theta\right)Y_n^m(\hat{\mathbf{r}}) \\
\mathbf{N}_{nm}(\mathbf{r}) &= \left\{n(n+1)h_n(kr)\mathbf{P}_n^m(\hat{\mathbf{r}}) + \partial_{kr}\left[krh_n(kr)\right]\mathbf{B}_n^m(\hat{\mathbf{r}})\right\}/kr \\
\mathbf{P}_n^m(\hat{\mathbf{r}}) &= \hat{\mathbf{r}}Y_n^m(\hat{\mathbf{r}}) \\
\mathbf{B}_n^m(\hat{\mathbf{r}}) &= \hat{\mathbf{r}}\times\mathbf{C}_n^m(\hat{\mathbf{r}}) = r\partial_{\mathbf{r}}Y_n^m(\hat{\mathbf{r}}) = \left(\hat{\boldsymbol{\varphi}}\frac{\partial_\varphi}{\sin\theta} + \hat{\boldsymbol{\theta}}\partial_\theta\right)Y_n^m(\hat{\mathbf{r}}) \\
Y_n^m(\hat{\mathbf{r}}) &= P_n^m(\cos\theta)e^{im\varphi},\ Y_n^{-m}(\hat{\mathbf{r}}) = (-1)^m\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)e^{im\varphi}
\end{aligned} \tag{B3}$$

where Y_n^m, P_n^m, h_n, are the scalar spherical harmonics, associated Legendre functions, and the spherical Hankel functions of the first kind, respectively. The vector spherical harmonics possess orthogonality properties

$$\begin{aligned}
\mathbf{P}_n^m\cdot\mathbf{B}_n^m &= \mathbf{P}_n^m\cdot\mathbf{C}_n^m = \mathbf{B}_n^m\cdot\mathbf{C}_n^m = 0 \\
\int d\Omega\mathbf{C}_n^{-m}\cdot\mathbf{C}_\nu^\mu &= \int d\Omega\mathbf{B}_n^{-m}\cdot\mathbf{B}_\nu^\mu = n(n+1)\int d\Omega\mathbf{P}_n^{-m}\cdot\mathbf{P}_\nu^\mu \\
&= (-1)^m 4\pi\delta_{n\nu}\delta_{m\mu}\frac{n(n+1)}{2n+1} \\
\int d\Omega &= \int_0^{2\pi} d\varphi\int_0^{\pi} d\theta\sin\theta
\end{aligned} \tag{B4}$$

The dyadic Green's function (9) is given in terms of the vector spherical waves as

$$\begin{aligned}
&\tilde{\boldsymbol{\Gamma}}(\mathbf{r},\mathbf{r}') \\
&= \frac{ik}{4\pi}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left[\mathbf{M}_{n,m}(\mathbf{r})\mathbf{M}_{n,-m}^{(1)}(\mathbf{r}')+\mathbf{N}_{n,m}(\mathbf{r})\mathbf{N}_{n,-m}^{(1)}(\mathbf{r}')\right]\frac{2n+1}{n(n+1)}
\end{aligned} \tag{B5}$$

where in (B5) the superscript indicates that the spherical Hankel functions are replaced by the nonsingular spherical Bessel functions $j_n(kr)$. Equation (B5) applies to $\mathbf{r} > \mathbf{r}'$, the case $\mathbf{r} < \mathbf{r}'$ follows by exchanging the primed and unprimed argument in (B5). The proof of (B5) is amply given in the literature, essentially it follows from the initial definition (9) and application of the so called addition theorem for spherical Hankel functions [19].

It is noted, [19], that for large distances, i.e., $kr \gg 1$, the spherical Hankel functions behave as

$$i^n h_n(kr) \cong \frac{e^{ikr}}{ikr} = h_0(kr) \tag{B6}$$

Consequently

$$i^n \mathbf{M}_{nm}(\mathbf{r}) \cong h_0(kr)\mathbf{C}_n^m(\hat{\mathbf{r}}), \quad i^{n-1}\mathbf{N}_{nm}(\mathbf{r}) \cong h_0(kr)\mathbf{B}_n^m(\hat{\mathbf{r}}) \tag{B7}$$

and it is noted that for large distances the wave approaches a plane wave and the longitudinal wave functions $\mathbf{P}_n^m(\hat{\mathbf{r}})$ can be neglected. Hence in principle, arbitrary radiation patterns

$$\mathbf{g}(\hat{\mathbf{r}}) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left[c_{n,m}\mathbf{C}_n^m(\hat{\mathbf{r}}) + b_{n,m}\mathbf{B}_n^m(\hat{\mathbf{r}})\right], \quad \hat{\mathbf{r}} = \hat{\mathbf{r}}(\theta, \varphi) \tag{B8}$$

can be expressed in terms of the vector spherical harmonics functions. The coefficients $c_{n,m}$, $b_{n,m}$ can be retrieved from an arbitrary function $\mathbf{g}(\hat{\mathbf{r}})$, e.g., given as experimental data, by exploiting the spatial and functional orthogonality properties (B4), and from this the corresponding full wave solution (B1) may be reconstructed.

For completeness it is noted that (B8) can be augmented to include longitudinal functions as well

$$\mathbf{g}(\hat{\mathbf{r}}) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left[c_{n,m}\mathbf{C}_n^m(\hat{\mathbf{r}}) + b_{n,m}\mathbf{B}_n^m(\hat{\mathbf{r}}) + p_{n,m}\mathbf{P}_n^m(\hat{\mathbf{r}})\right] \tag{B9}$$

and from the orthogonality relations (B4)

$$\int d\Omega \left[\mathbf{C}_n^{-m}, \mathbf{B}_n^{-m}, \mathbf{P}_n^{-m}\right] \cdot \mathbf{g}(\hat{\mathbf{r}}) = (-1)^m 4\pi \frac{n(n+1)}{2n+1} \left[c_{nm}, b_{nm}, \frac{p_{nm}}{n(n+1)}\right] \tag{B10}$$

where for brevity (B10) includes the three formulas in an obvious manner.

Finally it is noted that a plane wave representation for arbitrary distances is available, e.g., see [19–23]

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &= E_0 e^{-i\omega t} \frac{1}{2\pi} \int d\Omega(\hat{\mathbf{p}}) e^{ik\hat{\mathbf{p}}\cdot\mathbf{r}} \mathbf{g}(\hat{\mathbf{p}}) \\ &\cong E_0 e^{-i\omega t} h_0(kr) \mathbf{g}(\hat{\mathbf{r}}) \\ \int d\Omega(\hat{\mathbf{p}}) &= \int_{-\pi}^{\pi} d\varphi_p \int_0^{(\pi/2)-i\infty} d\theta_p \sin\theta_p \end{aligned} \tag{B11}$$

where the integration is carried out on the indicated contours and involves waves propagating in complex directions. Noteworthy is the fact that for isotropic media, all the vector spherical harmonics appearing under the integral sign in (B11), are transverse with respect to the direction of propagation $\hat{\mathbf{p}}$. However the corresponding vector spherical waves (B1) involve the longitudinal vector spherical harmonics $\mathbf{P}_n^m(\hat{\mathbf{r}}) = \hat{\mathbf{r}} Y_n^m(\hat{\mathbf{r}})$. This is not surprising, considering the fact that the field (B1) describes a general wave field, which is not a plane wave, and only for large distances, according to (B7), it asymptotically becomes a *quasi* plane wave, and the longitudinal waves associated with $\mathbf{P}_n^m(\hat{\mathbf{r}})$ do not feature. For completeness is it noted that if longitudinal harmonics $\mathbf{P}_n^m(\hat{\mathbf{r}})$ are included in (B11), they give rise to longitudinal waves

$$\mathbf{L}_{nm}(\mathbf{r}) = [\partial_{kr} h_n(kr)]\, \mathbf{P}_n^m(\hat{\mathbf{r}}) + [h_n(kr)/kr]\mathbf{B}_n^m(\hat{\mathbf{r}}) \tag{B12}$$

By inspection of (B3) it is clear that $\mathbf{M}_{nm} \cdot \mathbf{N}_{nm}(\mathbf{r}) = \mathbf{M}_{nm} \cdot \mathbf{L}_{nm}(\mathbf{r}) = 0$ are spatially orthogonal, but $\mathbf{N}_{nm} \cdot \mathbf{L}_{nm}(\mathbf{r}) \neq 0$ are not.

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