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ELECTROMAGNETIC PULSE PROPAGATION IN DISPERSIVE MEDIA

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Abstract—We first consider the mathematical theory of boundary-initial value problems for Maxwell's equations with as illustration, an extensive discussion of 1D-problems, Then, with the objective to investigate electromagnetic pulse propagation in dispersive media, we analyse how to translate electromagnetic processes which take place in a bounded domain of space-time into a boundary-initial value problem of Maxwell's equations, still focusing on 1D-problems.

1. INTRODUCTION

To analyse electromagnetic wave propagation in any medium, most people stick to Sratton's philosophy [1] "A pulse or signal of any specified initial form, may be constructed by superposition of harmonic wave trains of infinite length and duration". A curious statement, germane to the idea that any function has a Fourier transform and that could be jeopardized by the behaviour of digital signals generated in the modern technology of Communication. We depart here from this philosophy.

When one considers the propagation of electromagnetic fields in a bounded region Ω of space, during a finite interval of time $[0, T)$, as it is the case for instance, for any computer modelization of electromagnetic processes, one has to deal with a boundary-initial value (B-I) problem, requiring initial data at $t = 0$ and another set of data on the boundary $\partial\Omega$. In order to make these problems well posed for Maxwell's equations, these data have to satisfy some constraints not always easy to formulate mathematically, that is why we investigate these constraints for 1D-problems where calculations are less intricate. Then, an important question that we also examine in this case, is how to translate an electromagnetic physical process into a correct B-I problem of Maxwell's equations. We assume that the material inside the region Ω is made of a Maxwell-Hopkinson (M-H for brief) dielectric which is the simplest dispersive medium that one can imagine. The objective of this paper is not to present a general theory of B-I problems in electromagnetism but only to show on simple examples the pitfalls to be avoided.

Then, the paper is organized as follows; Sec. 2 is devoted to the mathematical background of B-I problems for Maxwell's equations and to the constraints to be imposed on boundary and initial data in the frame of 1D-problems, Sec. 3 is concerned with the translation of electromagnetic propagation in $\Omega \times [0, T)$ into a B-I problem of Maxwell's equations still in the case of 1D-problems. Conclusive comments are given in Sec. 4.

2. B-I PROBLEMS FOR MAXWELL'S EQUATIONS

2.1. Mathematical Background

With the light velocity unity, Maxwell's equations in a nonconducting medium $\{\mathbf{x} = (x, y, z)\}$

$$\begin{gathered}\mathrm{curl}\mathbf{E}(\mathbf{x}, t) = -\partial_t\mathbf{B}(\mathbf{x}, t), \qquad \mathrm{curl}\mathbf{H}(\mathbf{x}, t) = \partial_t\mathbf{D}(\mathbf{x}, t), \\ \mathrm{div}\mathbf{E}(\mathbf{x}, t) = \mathrm{div}\mathbf{B}(\mathbf{x}, t) = 0\end{gathered} \tag{1}$$

become with $\mathbf{B} = \mu\mathbf{H}$, leaving aside the divergence equations

$$\mathrm{curl}\mathbf{E}(\mathbf{x},t) = -\mu\partial_t\mathbf{H}(\mathbf{x},t), \qquad \mathrm{curl}\mathbf{H}(\mathbf{x},t) = \partial_t\mathbf{D}(\mathbf{x},t) \tag{1a}$$

from which we get the following equation where Δ is the laplacian operator

$$\Delta\mathbf{E}(\mathbf{x},t) - \mu\partial_t^2\mathbf{D}(\mathbf{x},t) = 0 \tag{2}$$

Now, a M-H medium is characterized by the constitutive relation [2]

$$\mathbf{D}(\mathbf{x},t) = \varepsilon\mathbf{E}(\mathbf{x},t) + \int_0^t \Phi(t-\tau)\mathbf{E}(\mathbf{x},\tau)d\tau \tag{3}$$

where ε is a positive constant and $\Phi(t)$, $t > 0$, a monotonically decreasing function of t continuous for $0 < t < \infty$. Note that along this paper $\mathbf{E}$ is zero in the M-H medium for $t < 0$. The displacement field $\mathbf{D}$ satisfies an integrodifferential equation [3] which seems to have been largely unnoticed:

$$\varepsilon\mu\partial_t^2\mathbf{D}(\mathbf{x},t) = \Delta\mathbf{D}(\mathbf{x},t) + \int_0^\tau \Psi(t-\tau)\Delta\mathbf{D}(\mathbf{x},\tau)d\tau \tag{4}$$

in which $\Psi(t)$ is given by the iterative series whose convergence is discussed in [4]

$$\begin{aligned}\Psi(t) &= \sum_{n=1}^{\infty}(-1)^n\Psi_n(t), \qquad \Psi_1(t) = \varepsilon^{-1}\Phi(t),\\ \Psi_n(t) &= \int_0^t \Psi_1(t-\tau)\Psi_{n-1}(\tau), \quad n \geq 2\end{aligned} \tag{4a}$$

To look for the solutions of Eq. (2) with $\mathbf{D}$ given by (3), we use the Laplace transform [5]

$$f(s) = \int_0^\infty \exp(-st)F(t)dt, \qquad Re\cdot s > 0, \quad t \geq 0 \tag{5}$$

so that with evident notations and taking into account the property of the Laplace transform to change a convolution product into an ordinary product, the constitutive relation (3) becomes

$$\mathbf{d}(\mathbf{x},s) = \varepsilon\mathbf{e}(\mathbf{x},s) + \phi(s)\mathbf{e}(\mathbf{x},s) \tag{6}$$

Then, assuming the initial conditions

$$\mathbf{D}(\mathbf{x},0) = \varepsilon\mathbf{E}(\mathbf{x},0) = \mathbf{F}(\mathbf{x}), \qquad \partial_t\mathbf{D}(\mathbf{x},0) = \varepsilon\partial_t\mathbf{E}(\mathbf{x},0) = \mathbf{G}(\mathbf{x}) \tag{7}$$

and using the Laplace transform way of tackling derivatives, the integrodifferential equation (4) changes into

$$\Delta\mathbf{d}(\mathbf{x},s)+\psi(s)\mathbf{d}(\mathbf{x},s)-n_0^2[s^2\mathbf{d}(\mathbf{x},s)-s\mathbf{F}(\mathbf{x})-\mathbf{G}(\mathbf{x})]=0,\quad n_0^2=\varepsilon\mu \tag{8}$$

while Eq. (2) becomes

$$\Delta\mathbf{e}(\mathbf{x},s)-\mu[s^2\mathbf{d}(\mathbf{x},s)-s\mathbf{F}(\mathbf{x})-\mathbf{G}(\mathbf{x})]=0 \tag{9}$$

and substituting (6) into (9) gives

$$\Delta\mathbf{e}(\mathbf{x},s)-s^2n(s)\mathbf{e}(\mathbf{x},s)=\mathbf{V}(\mathbf{x},s) \tag{10}$$

$$n^2(s)=\mu[\varepsilon+\phi(s)],\qquad \mathbf{V}(\mathbf{x},s)=-\mu[s\mathbf{F}(\mathbf{x})+\mathbf{G}(\mathbf{x})] \tag{10a}$$

But for physical reasons and to satisfy causality, the partial differential equation (10) has to be hyperbolic and it has been proved [3] that this condition is fulfilled if $\phi(s)$ is the quotient of two polynomials $p(s)/q(s)$ with degree $q >$ degree p so that according to (10a) $n^2(s)$ is the quotient of two polynomials with the same degree and

$$\lim_{s\Rightarrow\infty} n^2(s)=n_0^2=\varepsilon\mu \tag{11}$$

For instance, in a M-H dielectric [2], $\phi(s)=\sum_1^N\alpha_n(s+a_n)^{-1}$ where α_n, a_n, are positive constants depending on the constitutive material of the dielectric and the Davis criterion is satisfied.

Now, as said in the introduction, one is interested in the solutions of Eq. (10) in some region Ω of space so that one has still to supply data on its boundary $\partial\Omega$ and the specification of these data can become quite involved [4] to pose physical problems in a convenient manner. The initial conditions impose some constraints on the solutions of (10) that also intervene to limit the set of possible boundary data required to define a well posed problem. To make calculations easier without losing the essentials, we discuss these constraints in the next two sections on the solutions of the 1D-partial differential equation

$$\partial_z^2\mathbf{e}(z,s)-s^2n^2(s)\mathbf{e}(z,s)=\mathbf{V}(z,s),\quad \mathbf{V}(z,s)=-\mu[s\mathbf{F}(z)+\mathbf{G}(z)] \tag{12}$$

As a simple example, the solution of the B-I problem for the 1D-scalar wave equation

$$\partial_z^2\Psi(z,t)-\partial_t^2\Psi(z,t)=0,\qquad z,t\geq 0 \tag{12a}$$

with the initial and boundary conditions

$$\Psi(0,t)=h(t)\quad t\geq 0,\quad \Psi(z,0)=f(z),\quad \partial_t\Psi(z,0)=g(z) \tag{12b}$$

where f, g, h, are given functions with continuous second derivatives and moreover

$$h(0) = f(0), \qquad h'(0) = g(0), \qquad h''(0) = f''(0) \tag{12c}$$

has continuous second derivatives even on the characteristic line $z - t = 0$.

Of course, once known $\mathbf{e}(\mathbf{x}, s)$, one has still to perform an inverse Laplace transform to obtain $\mathbf{E}(\mathbf{x}, t)$ either by using tables [6], by analytical calculations in the complex s-plane [7] or by numerical computations [8].

2.2. Constraints Imposed by Initial Conditions

With n written for $n(s)$, one proves in Appendix A that the general solution of (12) is

$$\begin{aligned}\mathbf{e}(z,s) &= \mathbf{a}(s)e^{-snz} + \mathbf{b}(s)e^{snz} - (1/2sn)\int_{-\infty}^{0} du\, e^{snu}\mathbf{V}(z+u,s)\\ &\quad - (1/2sn)\int_{0}^{\infty} du\, e^{-snu}\mathbf{V}(z+u,s)\end{aligned} \tag{13}$$

in which $\mathbf{a}(s)$ and $\mathbf{b}(s)$ are two arbitrary functions.

But, according to the Abel-Tauber theorem [5], $\lim_{s\Rightarrow\infty} s\, f(s) = \lim_{t\Rightarrow 0} F(t)$ when both limits exist, so the solutions (13) must satisfy the conditions obtained from (7)

$$\lim_{s\Rightarrow\infty} s\,\mathbf{e}(z,s) = \mathbf{E}(z,0) = \varepsilon^{-1}\mathbf{F}(z) \tag{14a}$$

$$\lim_{s\Rightarrow\infty} s\,[s\,\mathbf{e}(z,s) - \mathbf{V}(z,s)] = \partial_t\mathbf{E}(z,0) = \varepsilon^{-1}\mathbf{G}(z) \tag{14b}$$

which implies in particular

$$\lim_{s\Rightarrow\infty} s\, a(s)e^{-snz} = 0, \qquad \lim_{s\Rightarrow\infty} s\, b(s)e^{snz} = 0 \tag{15}$$

and assuming that $a(s)$ and $b(s)$ are not exponentially increasing or decreasing at infinity, we get

$$a(s) = 0 \quad z < 0, \qquad b(s) = 0 \quad z > 0 \tag{15a}$$

Taking into account (15), the relations (14a), (14b) become

$$\lim_{s\Rightarrow\infty}\left\{-(1/2n)\int_{-\infty}^{0} du\, e^{snu}\mathbf{V}(z+u,s) - (1/2n)\int_{0}^{\infty} du\, e^{-snu}\mathbf{V}(z+u,s)\right\} = \varepsilon^{-1}\mathbf{F}(z) \tag{16a}$$

$$\lim_{s\Rightarrow\infty}\left\{(\mu s/2n)\int_{-\infty}^{0} du\, e^{snu}\mathbf{G}(z+u) + (\mu s/2n)\int_{0}^{\infty} du\, e^{-snu}\mathbf{G}(z+u)\right\} = \varepsilon^{-1}\mathbf{G}(z) \tag{16b}$$

These relations are checked in Appendix B with the help of (11).

2.3. Constraints on Boundary Conditions

We consider three different situations according as the M-H dielectric fills in all the space $-\infty < z < \infty$ the half space $0 < z < \infty$ or the sheet $0 < z < z_0$.

2.3.1. Solutions in the Full Space

In this case, the electric field $\mathbf{E}$ must be zero at infinity which implies according to (15a) $a(s) = b(s) = 0$ so that Eq. (13) reduces to

$$\mathbf{e}(z,s) = -(1/2sn)\left[\int_{-\infty}^{0} du\, e^{snu}\mathbf{V}(z+u,s) + \int_{0}^{\infty} du\, e^{-snu}\mathbf{V}(z+u,s)\right] \tag{17}$$

with $\mathbf{V}(z,s)$ compelled to satisfy the condition $\lim_{z\Rightarrow\infty}\mathbf{V}(z,s) = 0$ which imposes a constraint on the initial conditions $\mathbf{F}(z)$ and $\mathbf{G}(z)$. As a simple example of pulse propagation in a M-H dielectric, we suppose the initial conditions

$$\mathbf{F}(z) = \mathbf{F}\exp(i\omega z), \quad \mathrm{Im}\omega > 0, \quad \mathbf{G}(z) = 0 \tag{17a}$$

where $\mathbf{F}$ is an arbitrary constant vector. Then, the solution of (17) is

$$\mathbf{e}(z,s) = \mathbf{F}s^2\left(s^2n^2 + \omega^2\right)^{-1}\exp(i\omega|z|) \tag{18}$$

and, in a Lorentz-like medium [1, 9] with the refractive index satisfying the Davis criterion

$$n^2(s) = s^{-2}\left(s^2 + s_0^2\right) \tag{18a}$$

this solution becomes

$$\mathbf{e}(z,s) = \mathbf{F}\,s\left(s^2 + s_0^2 + \omega^2\right)^{-1}\exp(i\omega|z|) \tag{19}$$

And the inverse Laplace transform of (19) has the simple expression [5] justifying the choice of (18a)

$$\mathbf{E}(z,t) = \mathbf{F}\cos\left[\left(s_0^2 + \omega^2\right)^{1/2} t\right]\exp(i\omega|z|) \tag{19a}$$

which illustrates in a simple way the B-I problems of Maxwell's equations in full space.

2.3.2. Solutions in the Half Space $0 < z < \infty$

The condition $\mathbf{E} = 0$ at infinity implies as in the last section $b(s) = 0$ and $\lim_{z \Rightarrow +\infty} \mathbf{V}(z, s) = 0$, but we now need boundary data on the face $z = 0$. And, as well known, they must make continuous the tangential components $E_j (j = 1, 2)$ of the electric field $(E_1 = E_x,\ E_2 = E_y)$ while the third component must guarantee the divergence equation $\mathrm{div}\mathbf{E} = 0$ that reduces here to $\partial_z E_z = 0$ and requires $\partial_z V_z(z, s) = 0$ giving two further constraints on the initial conditions $\partial_z F_z(z) = 0,\ \partial_z G_z(z) = 0$.

So, from now on, we only work with the tangential components E_j, V_j, of the vector fields $\mathbf{E}$, $\mathbf{V}$, and since $b_j(s) = 0$, the solution (13) reduces to

$$\begin{aligned} e_j(z, s) \ &= \ a_j(s) e^{-snz} - (1/2sn) \left[\int_{-\infty}^{0} du\, e^{snu} V_j(z + u, s) \right. \\ &\qquad \left. + \int_{0}^{\infty} du\, e^{-snu} V_j(z + u, s) \right], \quad j = 1, 2 \end{aligned} \tag{20}$$

where as just said $V_j(z, s)$ is compelled to satisfy $\lim_{z \Rightarrow +\infty} V_j(z, s) = 0$.

Now, let $E_j(0, t) = K_j(t)$ denote the boundary data on the face $z = 0$ then, with the Laplace transform $k_j(s)$ of $K_j(t)$ we get from (20)

$$k_j(s) = a_j(s) - (1/2sn) \left[\int_{-\infty}^{0} du\, e^{snu} V_j(u, s) + \int_{0}^{\infty} du\, e^{-snu} V_j(u, s) \right], \quad j = 1, 2 \tag{21}$$

These two equations supply $a_j(s)$ which achieves to determine the solution (20) of the B-I problem in the half space $0 < z < \infty$.

To illustrate (20), we suppose, with for simplicity $\varepsilon = \mu = n_0 = 1$, that the initial conditions take the form

$$F_j(z) = F_j \exp(-\lambda z), \qquad Re \cdot \lambda > 0, \qquad G_j(z) = 0 \tag{22}$$

Substituting (22) into (20) and (21) gives

$$e_j(z, s) = a_j(s) e^{-snz} + F_j s \exp(-\lambda z) \left(s^2 n^2 - \lambda^2 \right)^{-1} \tag{23}$$

$$k_j(s) = a_j(s) + F_j s \left(s^2 n^2 - \lambda^2 \right)^{-1} \tag{23a}$$

and we further assume $k_j(s)$ such that

$$a_j(s) = k_j(s) - F_j s \left(s^2 n^2 - \lambda^2 \right)^{-1} = a_j (s + i\varpi)^{-1}, \quad i = \sqrt{-1} \tag{24}$$

where a_j and ϖ are constant. Then, taking into account (24), the solution (23) becomes

$$e_j(z,s) = a_j e^{-snz}(s+i\varpi)^{-1} + F_j s \exp(-\lambda z)\left(s^2n^2-\lambda^2\right)^{-1} \quad (25)$$

Suppose now a medium with the refractive index (it is not claimed to represent a material medium)

$$n(s) = (s_0+s)/s, \qquad s_0 \text{ positive constant} \quad (25a)$$

then we get from (25)

$$\begin{aligned} e_j(z,s) &= a_j \exp[-(s+s_0)z](s+i\varpi)^{-1} \\ &\quad + F_j s \exp(-\lambda z)(s+s_0-\lambda)^{-1}(s+s_0+\lambda)^{-1} \quad (26) \end{aligned}$$

with the inverse Laplace transform [5] in which U is the unit step function

$$\begin{aligned} E_j(z,t) &= a_j \exp[-s_0 z + i\varpi(t-z)]U(t-z) \\ &\quad + F_j \exp[-(\lambda z + s_0 t)][\cos(\lambda t) - \sin(\lambda t)/s_0] \quad (26a) \end{aligned}$$

which must be considered as a simple illustration of a B-I problem in the half space $0 < z < \infty$.

We did not use the refractive index (18a) since the presence of a branch point in the exponential $\exp(-snz)$ makes calculations of the inverse Laplace transform much more difficult, leading to the kind of problem discussed a long time ago by Sommerfeld [10] and Brillouin [11] and correctly solved only recently [12].

Calculations in this section prove also that, contrarily to an opinion largely spread, one cannot dispense with divergence equations when one has to deal with electromagnetic B-I problems. This remark, made also recently [13, 14], is important for the numerical modelization of electromagnetic processes where the negligence of divergences can generate spurious solutions.

2.3.3. Solutions in the Sheet $0 < z < z_0$

From a mathematical point of view, this situation is no more difficult to tackle than the previous one, once known the boundary conditions on a face of the sheet, but for physical reasons to be discussed in the next section these conditions are difficult to obtain. So, mathematically with $K_j(t)$ given at $z = 0$ the equations (20) and (22) are still valid with no more constraint on $V_j(z,s)$ and, as soon as $a_j(s)$ is obtained from (22), the boundary conditions on the face $z = z_0$ follows from (20)

$$e_j(z_0, s) = a_j(s) \exp(-snz_0) - (1/2sn) \cdot \left[\int_{-\infty}^{0} du\, e^{snu} V_j(z_0+u, s) + \int_{0}^{\infty} du\, e^{-snu} V_j(z_0+u, s)\right] \quad (27)$$

Then, Descartes-Snell law and Fresnel formulae supply the electromagnetic field in the region $z > z_0$.

3. ELECTROMAGNETIC B-I PROBLEMS

3.1. Pulse Propagation in a Half-Space

The B-I problems of Maxwell's equations are discussed in Sec. 2 from a mathematical point of view, without taking into account the physical constraints imposed on initial and boundary data by the electromagnetic theory. It was only noticed that boundary data concern the tangential components of the electric field. Still working with 1D-problems, we now investigate on a simple example how to translate an electromagnetic process into a B-I problem.

We suppose that free space and M-H dielectric fill in respectively the half-space $-\infty < z < 0$ and $0 < z < \infty$. A linearly polarized, truncated, harmonic plane wave with amplitude A_i propagates in free space $z < 0,\ t < 0$

$$E_x(z, t) = A_i \exp[i\omega(t-z)][U(t-z) - U(t-t_0-z)], \quad E_y = E_z = 0 \quad (28)$$

and impinges on the M-H dielectric, t_0 is the duration of the pulse and U the unit step function. Leaving aside the reflected pulse, one is interested in the transmitted field which is the solution (20) for $t,\ z, > 0$ so that one needs the boundary-initial data $a_x(s)$ and $V_x(z, s)$ (from now on, the subscript x is suppressed since no confusion is possible).

For $z = 0$ and $t = 0$ that is just at the time when the pulse reaches the dielectric

$$E(z, t) = A \exp[i\omega(t - z)][U(t - z) - U(t - t_0 - z)] \quad (29)$$

in which according to Fresnel's formulae $A = 2(n_1 + n_0)^{-1},\ n_1$ and $n_0 = \sqrt{\varepsilon\mu}$ are respectively the refractive indices of free space and of the M-H dielectric at $t = 0$. So, we get from (29) at $t = 0_+$

$$E(z, 0_+) = A \exp(-i\omega z)[U(-z) - U(-t_0 - z)] \quad (30)$$

Now, from the definition $U(x) = \int_{-\infty}^{x} \delta(\xi) d\xi$ where δ is the Dirac distribution, we get $U(-x) = 1 - U(x)$ so that Eq. (30) becomes

$$E(z, 0_+) = A \exp(-i\omega z)[U(t_0 + z) - U(z)] \tag{30a}$$

This expression is zero for $z > 0$ since $U(t_0 + z) = U(z) = 1$ but for $z = 0 : U(t_0 + z) - U(z) = 1/2$ since the inverse Laplace transform $U(t) = L^{-1}(1/s)$ implies $U(0) = 1/2$ [15] while $U(t_0) = 1$. So, finally

$$F(z) = E(z, 0) = A/2 \tag{31}$$

From the time derivative of (29)

$$\partial_t E(z, t) = i\omega E(z, t) + A \exp[-i\omega(t - z)][\delta(t - z) - \delta(t - t_0 - z)] \tag{32}$$

we get for $t = 0_+$, using (31) and taking into account the relation $\delta(-z) = \delta(z)$

$$G(z) = \partial_t E(z, 0_+) = i\omega A/2 + A[\delta(z) - \exp(i\omega t_0)\delta(t_0 + z)] \tag{32a}$$

Then, substituting (31) and (32a) into the expression (12) of $V_j(z, s)$ gives the initial data

$$V(z, s) = -(\mu A/2)(s + i\omega) - \mu A[\delta(z) - \exp(i\omega t_0)\delta(t_0 + z)] \tag{33}$$

We now need the boundary condition $k(s)$ at $z = 0$ with according to (29)

$$K(t) = E(0_+, t) = A \exp(i\omega t)[U(t) - U(t - t_0)] \tag{34}$$

with the Laplace transform

$$k(s) = A(s - i\omega)^{-1} \{1 - \exp[-t_0(s - i\omega)]\} \tag{34a}$$

To obtain $a(s)$, one has just to substitute (33) and (34a) in Eq. (21) and we get in Appendix C

$$a(s) = k(s) - (\mu A/2s^2n^2)(s + i\omega) - (\mu A/2sn)\{1 - \exp[-t_0(s - i\omega)]\} \tag{35}$$

Finally, substituting (33) into (20) and using the relation (C6) of Appendix C give in the s-domain the equation of a truncated harmonic pulse propagating in a M-H dielectric:

$$\begin{aligned} e(z, s) \;\; = \;\; & a(s)e^{-snz} + (\mu A/2s^2n^2)(s + i\omega) \\ & +(\mu A/2sn)e^{-snz}\{1 - \exp[-t_0(s - i\omega)]\} \end{aligned} \tag{36}$$

which becomes with (35) and $k(s)$ given by (34a)

$$e(z, s) = k(s) \exp(-snz) + (\mu A/2s^2n^2)(s + i\omega)[1 - \exp(-snz)] \tag{36a}$$

Still using the refractive index (25a), we get from (36a)

$$e(z,s)=k(s)\exp[-(s+s_0)z]+[\mu A/2(s+s_0)^2](s+i\omega)\{1-\exp[-(s+s_0)z]\} \tag{37}$$

with the inverse Laplace transform $E^t(z,t)$ in which $E(z,t)$ is the field (29)

$$\begin{aligned} E^t(z,t) &= \exp(-s_0 z)E(z,t) + (\mu A/2)Q(t)U(t) \\ &\quad -(\mu A/2)\exp(-s_0 z)Q(t-z)U(t-z) \end{aligned} \tag{38}$$

$$Q(t) = \cos(s_0 t) + i\omega s^{-1}\sin(s_0 t) \tag{38a}$$

As noticed in Sec. 2.3.2, the inverse Laplace transform of (36a) would be more difficult to get with the refractive index (18a).

3.2. Pulse Propagation in a Sheet

Suppose now that the pulse (28) impinges on the sheet $0 < z < z_0$. In Sec. 2.3.3 the boundary condition on the face $z = 0$ was assumed to be known which made easy the solution of the B-I problem but the trouble is that this condition depends on the field reflected at $z = z_0$. So in fact one has to deal with a two-point B-I problem which is of a different nature.

To tackle two-point boundary value problems, very frequent for instance in nuclear industry to estimate the shielding against neutrons and gammas offered by sheets of different materials, Bellman and co-workers [16, 17] have developed many years ago the invariant imbedding technique which has been the object of many works (a bibliography till 1978 is given in [18, 19]). Recently this technique was applied to electromagnetism by Weston [20] and the swedish school of applied mathematics and a complete bibliography on the period 1990–2000 can be found in Gustafson's doctoral thesis [21]. An interested reader should consult these works mainly devoted to the conditions for the existence of a solution while in some papers a numerical implementation of the formalism is discussed when the incident field is a plane wave.

4. DISCUSSION

As said in the introduction, in order to centre on the essential difficulties of electromagnetic B-I problems in dispersive media, we worked with a very simple M-H dielectric but to deal with most of electromagnetic processes, one should have to consider media with more elaborate constitutive relations. As early as 1912, Volterra [22] extended the Maxwell-Hopkinson theory to situations where the dielectric is anisotropic, nonlinear and magnetized but still keeping an a-priori separation between electric and magnetic effects. Finally Toupin and Rivlin [23] removed this last restriction to get the constitutive relations in the form used to day [24]. Then, B-I problems of Maxwell's equations in media with these constitutive relations may be solved as in M-H dielectrics but at the expense of more calculations, specially in chiral media.

As also noticed in the introduction, it is easy to impose initial conditions while the specification of boundary data on the boundary $\partial\Omega$ of an arbitrary region Ω may be quite involved to pose a physical problem in a convenient and correct manner. One has in fact to deal with a ill-posed problem in the Hadamard sense [25] since the hereditary constitutive relations (a term coined by Volterra) such as (2) require to know all the past of the material to solve B-I problems. So, it is important to investigate how to choose the boundary conditions to make sure that the B-I problem has a unique solution. A mathematically inclined reader, familiar with Sobolev spaces will find some responses in two recent books [26, 27].

Because of the difficulty to formulate correctly the B-I problems of mathematical physics, most of the works devoted to these problems concern, as just said, the conditions for the existence and the uniqueness of a solution at the price of an heavy mathematical machinery [26]. It does not seem that practical problems were the object of thorough investigations, probably not only because of the above mentioned difficulties but also since many electromagnetic time-dependent processes discussed in infinite or bounded media do not need initial conditions: one has just to make sure that causality is respected which can be obtained by the use of retarded potentials. But this situation could change with the advent of digital signals for which time plays a dual role, on one hand that of an evolution parameter as for continuous pulses and on the other hand that of a duration counter. Consequences of causality intervene differently in both roles: think of the pulse (28), impinging on the M-H dielectric with the incidence H, the fact that all the parts of the signal do not reach the face $z = 0$ at the same time, generates already a distortion independently of what happens inside the dielectric.

The objective of this paper was to show on simple illustrative examples the traps to be avoided in the dealing with B-I problems but this should not deter people to work on these problems specially those thinking they could become important in the future.

REMARK

After completion of this work, an interesting paper [28] has appeared on the existence of a current density $\mathbf{j}$ such as the initial data $\mathbf{B}_0,\ \mathbf{D}_0$, at $t=0$ take in Ω the state $\mathbf{B},\ \mathbf{D}$, at $t=T$.

APPENDIX A.

One checks easily that the functions $\exp(\pm snz)$ are solutions of the homogeneous differential equations $\partial_z^2 e(z,s) - s^2n^2e(z,s) = 0$. We now prove that the solution of Eq. (12) is

$$\mathbf{e}(z,s) = -(1/2sn)\left[\int_{-\infty}^{0} du\, e^{snu}\mathbf{V}(z+u,s) + \int_0^{\infty} du\, e^{-snu}\mathbf{V}(z+u,s)\right] \tag{A1}$$

The z-derivative of (A1) is with $\mathbf{V}' = \partial_z\mathbf{V} = \partial_u\mathbf{V}$

$$\partial_z\mathbf{e}(z,s) = -(1/2sn)\left[\int_{-\infty}^{0} du\, e^{snu}\mathbf{V}'(z+u,s) + \int_0^{\infty} du\, e^{-snu}\mathbf{V}'(z+u,s)\right] \tag{A2}$$

and integrating by parts

$$\partial_z\mathbf{e}(z,s) = -\left\{(1/2sn)[e^{snu}\mathbf{V}(z+u,s)]_{-\infty}^{0} + (1/2sn)\left[e^{-snu}\mathbf{V}(z+u,s)\right]_0^{\infty}\right\}$$
$$+(1/2)\int_{-\infty}^{0} du\, e^{snu}\mathbf{V}(z+u,s) - (1/2)\int_0^{\infty} du\, e^{-snu}\mathbf{V}(z+u,s) \tag{A3}$$

but the quantity inside the curly bracket is zero so that

$$\partial_z^2\mathbf{e}(z,s) = (1/2)\int_{-\infty}^{0} du\, e^{snu}\mathbf{V}'(z+u,s) - (1/2)\int_0^{\infty} du\, e^{-snu}\mathbf{V}'(z+u,s) \tag{A4}$$

Still integrating by parts gives

$$\partial_z^2\mathbf{e}(z,s) = \left\{(1/2)[e^{snu}\mathbf{V}(z+u,s)]_{-\infty}^{0} - (1/2)\left[e^{-snu}\mathbf{V}(z+u,s)\right]_0^{\infty}\right\}$$
$$-(sn/2)\left[\int_{-\infty}^{0} du\, e^{snu}\mathbf{V}(z+u,s) + \int_0^{\infty} du\, e^{-snu}\mathbf{V}(z+u,s)\right] \tag{A5}$$

The term inside the curly bracket in (A5) is $\mathbf{V}(z,s)$ while the second term is $s^2n^2\mathbf{e}(z,s)$ and we get

$$\partial_z^2\mathbf{e}(z,s) = \mathbf{V}(z,s) + s^2n^2\mathbf{e}(z,s) \tag{A6}$$

which is the equation (12).

APPENDIX B.

To prove (16a), we first note that according to (10a) and (11) $\mathbf{V}(z,s) \approx -\mu s\mathbf{F}(z)$ while n reduces to n_0 for $s \Rightarrow \infty$. Then, the left hand side of (16a) may be written

$$\begin{aligned}\lim_{s\Rightarrow\infty} A(z,s) &= (\mu s/2n)\left[\int_{-\infty}^{0} du \exp(sn_0u)\mathbf{F}(z+u)\right.\\ &\quad \left.+\int_0^{\infty} du \exp(-sn_0u)\mathbf{F}(z+u)\right] \end{aligned} \tag{B1}$$

and integrating by parts gives

$$\begin{aligned}\lim_{s\Rightarrow\infty} A(z,s) &= \Big\{(\mu/2n_0^2)[\exp(sn_0u)\mathbf{F}(z+u)]_{-\infty}^{0}\\ &\quad -(\mu/2n_0^2)[e^{-snu}\mathbf{F}(z+u)]_0^{\infty}\Big\}\\ &\quad -\lim_{s\Rightarrow\infty}\left[(\mu/2n_0^2)\int_{-\infty}^{0} du \exp(sn_0u)\mathbf{F}'(z+u)\right.\\ &\quad \left.+(\mu/2n_0^2)\int_0^{\infty} du \exp(-sn_0u)\mathbf{F}'(z+u)\right] \end{aligned} \tag{B2}$$

The term inside the curly bracket is $\mu/2n_0^2\,\mathbf{F}(z) = \varepsilon^{-1}\mathbf{F}(z)$ while the quantity inside the square bracket is zero as easily seen by exchanging integration and limit.

One observes at once that (16b) is exactly (B1) with $\mathbf{F}(z)$ changed into $\mathbf{G}(z)$ so that (B2) implies (14b).

APPENDIX C.

We write (33): $\mathbf{V}(z,s) = \mathbf{V}_1(z,s) + \mathbf{V}_2(z,s)$ with

$$\mathbf{V}_1(z,s) = -\mu A(s+i\omega)/2, \quad \mathbf{V}_2(z,s) = -\mu A[\delta(z) - \exp(i\omega t_0)\delta(t_0+z)] \tag{C1}$$

Then a simple calculation gives

$$\int_{-\infty}^{0} du\, e^{snu}\mathbf{V}_1(u,s) + \int_0^{\infty} du\, e^{-snu}\mathbf{V}_1(u,s) = -\mu A(s+i\omega)/sn \tag{C2}$$

while using the relation $\int_{-\infty}^{0} f(x)\delta(x-x_0)dx = bf(x_0)$ with $b=1$ if $x_0 \in (-\infty, 0)$, $b = 1/2$ for $x_0 = 0$ and $b = 0$ when x_0 is in the interval $(0,\infty)$ we get

$$\int_{-\infty}^{0} du\, e^{snu}\mathbf{V}_2(u,s) + \int_0^{\infty} du\, e^{-snu}\mathbf{V}_2(u,s) = -\mu A\{1-\exp[-t_0(sn+i\omega)]\} \tag{C3}$$

and finally

$$\int_{-\infty}^{0} du\, e^{snu}\mathbf{V}(u,s) + \int_{0}^{\infty} du\, e^{-snu}\mathbf{V}(u,s)$$
$$= -\mu A(s+i\omega)/sn - \mu A\{1-\exp[-t_0(sn+i\omega)]\} \quad \text{(C4)}$$

Then, substituting (C4) into Eq. (21) gives

$$a(s) = k(s) - (\mu A/2s^2n^2)(s+i\omega) - (\mu A/2sn)\{1-\exp[-t_0(s-i\omega)]\} \quad \text{(C5)}$$

Changing u into $z+u$ in (C4) gives

$$\int_{-\infty}^{0} du\, e^{snu}\mathbf{V}(z+u,s) + \int_{0}^{\infty} du\, e^{-snu}\mathbf{V}(z+u,s)$$
$$= -\mu A(s+i\omega)/sn - \mu A e^{-snz}\{1-\exp[-t_0(sn+i\omega)]\} \quad \text{(C6)}$$

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