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AN OVERVIEW OF THE WATSON TRANSFORMATION PRESENTED THROUGH A SIMPLE EXAMPLE

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Abstract—One of the methods developed for accelerating the convergence speed of infinite series is the Watson transformation. It is a technique with an interesting theoretical background which is applied in a restricted number of cases due to its complexity. Most of the papers using this method do not extensively analyze every step of implementation. In this work we apply Watson transformation in a simple case and we focus on each aspect of the procedure.

1. INTRODUCTION

A basic principle in set theory is that a complete set of functions has infinite population. For this reason, the description of an unknown quantity is made usually through an arbitrarily weighted infinite sum. A common problem in mathematics and physics is the low convergence speed of these series and many techniques have been developed to overcome such a hindrance. Extrapolation methods convert the general term of the series into an equivalent one with greater convergence speed employing nonlinear operations. In [1] the Shanks transformation is used for accelerating the MoM solution for the current on a cylindrical antenna. The results are physically verified but it is remarked that the series are highly susceptible to noise and roudoff errors. In [2], the author presents the Euler transformation which is suitable for alternating series. It is pointed out that when the transformation is applied to a series with rounded coefficients, its behavior can differ substantially from that predicted theoretically. In [3] a multiport network is analyzed by utilizing fast multipole method. In order to avoid the relative convergence problem of other techniques based on mode matching, an alternative port treatment is used. Also in [4], the authors make use of the Kummer's method and Ewald's approach to accelerate the series defining the Green's function in a parallel-plate waveguide. It is noted that the second technique is better than the first one because it leads to smaller error not only for the quantity itself but for its derivative as well.

Another interesting method, mainly applicable to electrically large scattering problems is the so-called "Watson transformation" [5–7]. It is named after G. N. Watson who used a technique to compute the electric fields in the presence of the earth, back in the beginning of the previous century [8]. The basic characteristic of this approach is the conversion of the slowly converging "canonical" series to a rapidly converging "Watson" series which is defined as a sum of residues corresponding to an integrand's complex poles. In [9], the transformation is used for the treatment of an acoustic scattering problem where the surface waves, the phase velocities and the attenuations become easily computable. The Watson transformation is originating from mathematics and it is commonly applied in many variables for the manipulation of certain hypergeometric series [10]. A novel analysis of expanding the diffraction field in Watson series by using orthogonality relations is presented in [11]. The optimal coefficients are derived by approximate minimization of the square mean and by utilizing product expansions involving in the complex poles. Finally, in [12] the transformation is used to obtain an integral representation for the potential in an electrostatic problem. This integral is asymptotically evaluated in the far field and certain observations about its exponentially decaying behavior are made.

In the majority of the published works concerning Watson transformation, the technique is described properly except for the way of detecting the integrand's complex poles. It is certainly referred that the initial guesses for the integrand singularities are given from a simplified equation but the explicit process of this reduction is commonly omitted. For example in [13], the numerical procedure for determining the position of the poles is not indicated clearly and in [14] the explicit values of them are taken as precalculated without additional comments. It should be stated that this operation is not simple or straightforward as it contains a large amount of nontrivial approximations and assumptions. In the present work we apply the Watson transformation for the simple case of a perfectly conducting sphere (with electrically large radius) which scatters the field produced by an infinitesimal magnetic dipole located on its surface. The contour integrations on the complex plane are depicted in figures and the pole detection technique is extensively analyzed. All the intermediate steps are studied in detail and explanations are provided where it is necessary. The formulas are validated by comparison with the slow canonical solution which requires many dozens of terms for convergence. On the contrary, in case of the Watson series the first one or two terms are sufficient for the most investigated cases.

2. CANONICAL SOLUTION OF THE PROBLEM

Consider a spherical perfect electric conductor (PEC) of radius a placed inside vacuum area with wavenumber k_0 and intrinsic impedance ζ_0. An infinitesimal magnetic dipole of magnitude A measured in V/m is located slightly above the conducting surface (inside vacuum) and excites the spherical structure as shown in Fig. 1. Both the coordinate systems: cartesian (x, y, z) and spherical (r, θ, ϕ) can be used interchangeably and their origin coincides with the center of the sphere. The dipole is x-polarized, posed on the z axis, while the scatterer is electrically large ($k_0 a \gg 1$). A time dependence of the

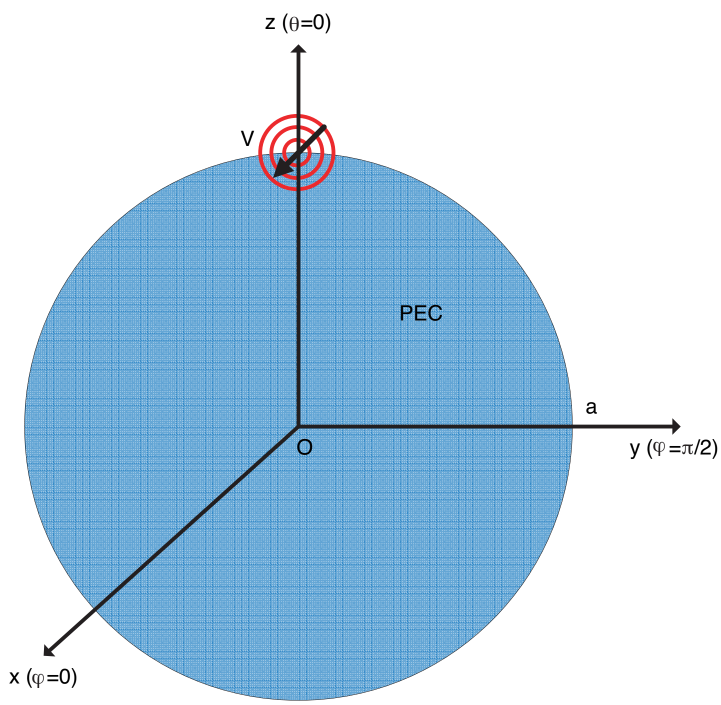

Figure 1. The physical configuration of the examined structure. A perfectly conducting (PEC), electrically large sphere of radius a is excited by a magnetic dipole located on its surface.

form $e^{-i\omega t}$ is adopted and suppressed throughout the analysis. The magnetic dipole excitation current $\mathbf{K}(\theta,\phi)$ at $r=a^+$ is expressed as follows:

$$\mathbf{K}(\theta,\phi)=\hat{x}\frac{A}{\sin\theta}\delta(\theta-0^+)\delta(\phi) \tag{1}$$

where $\delta(z)$ is the Dirac delta function. With use of the spherical eigenfunctions and their orthogonality properties specified in [15], the expansions for each component of the surface current (1) are obtained.

$$K_\theta(\theta,\phi) = -\frac{A}{4\pi}\sum_{n=1}^{+\infty}\frac{2n+1}{n(n+1)}\left(P_n^d(\theta)+\frac{P_n(\theta)}{\sin\theta}\right)\cos\phi \tag{2}$$

$$K_\phi(\theta,\phi) = \frac{A}{4\pi}\sum_{n=1}^{+\infty}\frac{2n+1}{n(n+1)}\left(P_n^d(\theta)+\frac{P_n(\theta)}{\sin\theta}\right)\sin\phi \tag{3}$$

The Legendre function of argument $\cos\theta$, degree n and unitary order is denoted by $P_n(\theta)$. Its derivative with respect to θ is symbolized as $P_n^d(\theta)$.

If one employs the vectorial solutions to the homogeneous Helmholtz equation in spherical coordinates and exploits the expansions (2), (3), the primary field in the absence of the scatterer is derived. Imposing the boundary condition for vanishing tangential electric field on the PEC sphere, leads to the canonical solution of the problem. The components of the total electric field are given below.

$$E_r(r,\theta,\phi)=-\frac{A}{4\pi}\frac{a}{r}\sum_{n=1}^{+\infty}(2n+1)\frac{h_n(k_0r)}{h_n^d(k_0a)}P_n(\theta)\sin\phi \tag{4}$$

$$E_\theta(r,\theta,\phi)=-\frac{A}{4\pi}\cdot$$
$$\sum_{n=1}^{+\infty}\frac{2n+1}{n(n+1)}\left[\frac{a}{r}\frac{h_n^d(k_0r)}{h_n^d(k_0a)}P_n^d(\theta)+\frac{h_n(k_0r)}{h_n(k_0a)}\frac{P_n(\theta)}{\sin\theta}\right]\sin\phi \tag{5}$$

$$E_\phi(r,\theta,\phi)=-\frac{A}{4\pi}\cdot$$
$$\sum_{n=1}^{+\infty}\frac{2n+1}{n(n+1)}\left[\frac{h_n(k_0r)}{h_n(k_0a)}P_n^d(\theta)+\frac{a}{r}\frac{h_n^d(k_0r)}{h_n^d(k_0a)}\frac{P_n(\theta)}{\sin\theta}\right]\cos\phi \tag{6}$$

The components of the total magnetic field are given below.

$$H_r(r,\theta,\phi)=-\frac{iA}{4\pi k_0r\zeta_0}\sum_{n=1}^{+\infty}(2n+1)\frac{h_n(k_0r)}{h_n(k_0a)}P_n(\theta)\cos\phi \tag{7}$$

$$H_\theta(r,\theta,\phi) = -\frac{iA}{4\pi k_0 r\zeta_0}\cdot$$

$$\sum_{n=1}^{+\infty}\frac{2n+1}{n(n+1)}\left[\frac{h_n^d(k_0r)}{h_n(k_0a)}P_n^d(\theta) - k_0^2ar\frac{h_n(k_0r)}{h_n^d(k_0a)}\frac{P_n(\theta)}{\sin\theta}\right]\cos\phi \tag{8}$$

$$H_\phi(r,\theta,\phi) = -\frac{iA}{4\pi k_0 r\zeta_0}\cdot$$

$$\sum_{n=1}^{+\infty}\frac{2n+1}{n(n+1)}\left[k_0^2ar\frac{h_n(k_0r)}{h_n^d(k_0a)}P_n^d(\theta) - \frac{h_n^d(k_0r)}{h_n(k_0a)}\frac{P_n(\theta)}{\sin\theta}\right]\sin\phi \tag{9}$$

where $h_n(z)$ is the spherical Hankel function of first type, order n and argument z. The Riccati-Hankel function is defined as $h_n^d(z) = [zh_n(z)]'$, where the prime denotes the derivative with respect to z. One can observe that for $r = a$ the functions $E_\theta(r,\theta,\phi), E_\phi(r,\theta,\phi)$ are not vanishing despite the inflicted requirement. That makes no wonder because the formulas (4)–(9) are valid for the area outside the source ($r > a$). On the contrary, the boundary condition is verified by the related expressions for $r = a$ (solution branch corresponding to the infinitesimal area between the PEC sphere and $r = a^+$ surface).

We are mainly interested in evaluating these field quantities within the near region but not too close to the scatterer because both primary (dipole singularity) and secondary (surface currents) sources exist on its boundary. We confine our analysis to observation points of the lower half space ($\pi/2 < \theta < \pi$) where the response of the electrically large sphere is the dominant component of the measured quantity. The infinite series (4)–(9) are computed by truncation of the first N terms with a maximum permissible error of 0.001%. In Fig. 2 we present N as function of the normalized radial distance r/a for various k_0a. One can observe that the greater the electrical radius gets, the more substantial is the necessary N in order to achieve convergence. The polar and the azimuthal angles (θ,ϕ) do not affect significantly the truncation limit (at least within the investigated region). In addition, the essential number of terms remains almost the same regardless of the field component that is calculated. Once the observation point gets distant from the spherical scatterer, less terms are required for a reliable evaluation, but they are still too many when $k_0a \gg 1$. Therefore, the direct summation of the canonical solution terms is not computationally efficient and another approach should be adopted instead.

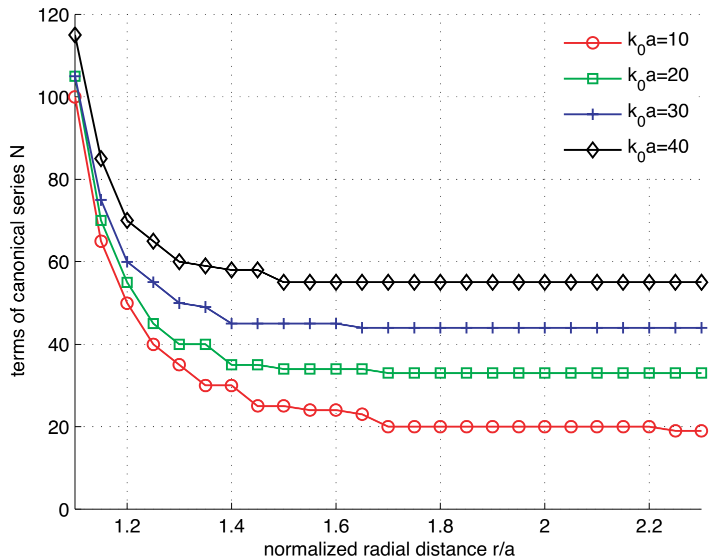

Figure 2. The required number of terms N which are necessary to achieve convergence as function of the normalized radial distance r/a for various electrical radii of the sphere k_0a. The canonical series is the used formula.

3. APPLICATION OF THE WATSON TRANSFORM

The Watson transformation is a technique of accelerating slowly convergent series by writing the infinite sum as an integral with complex integration path. It is an inspired method with theoretical completion and effective results even though it possesses a variety of difficulties in application. The purpose of this study is to implement this technique for a simple case and to explain in detail all the intermediate steps, a procedure not commonly appeared in bibliography. Each of the ten series defined in (4)–(9) (the transverse field components are written as sums of two series) owns the following form:

$$F = \sum_{n=1}^{+\infty} f(n) \tag{10}$$

where the geometrical parameters (r, θ, ϕ) are suppressed. In every case the function $f(n)$ contains a factor of a Legendre-type quantity ($P_n(\theta)$ or $P_n^d(\theta)$) and a Hankel-type component ($h_n(k_0a)$ or $h_n^d(k_0a)$) as denominator. The variable F can be written in terms of an unknown

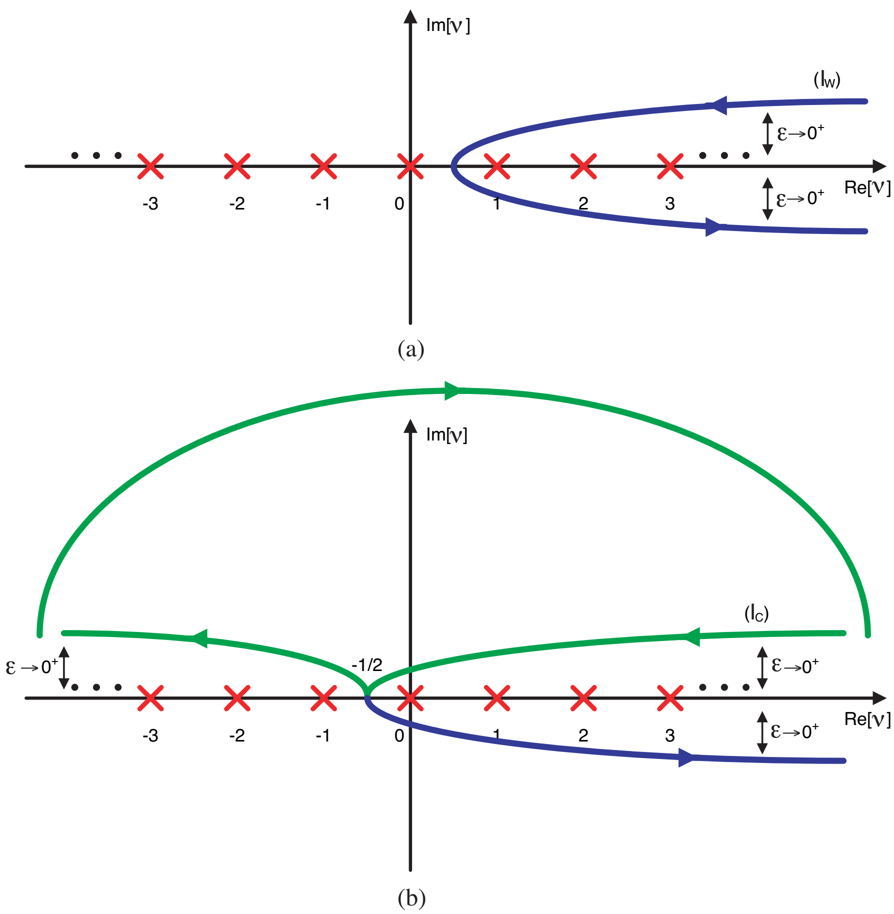

Figure 3. The integration paths: (a) the initial integration path (I_W) of the Watson integral tightly surrounds the positive real axis of the complex ν plane, (b) the modified integration path (I_C) of the Watson integral which occupies the upper half of the complex ν plane.

function $g(\nu)$ as below:

$$F = \int_{(I_W)} \gamma(\nu) d\nu = \int_{(I_W)} \frac{g(\nu)}{2\pi i \sin(\nu\pi)} d\nu \tag{11}$$

The integration path (I_W) encloses the positive semi axis of the complex ν plane as shown in Fig. 3(a). The curve is traced out in a positive direction and into the defined contour the integrand $\gamma(\nu)$ exhibits simple singularities only at the positive integer points

$\nu = n > 0$. Through the residue theorem, the following relation is derived:

$$F = \sum_{n=1}^{+\infty} (-1)^n g(n) \tag{12}$$

In order to apply the Watson method, a complex function $g(\nu)$ with $f(n) = g(n)(-1)^n$ for each integer n should be found. By taking into account the following relation concerning Legendre-type functions [16, eq. (8.2.3), p. 333]

$$P_n(\pi - \theta) = -(-1)^n P_n(\theta) \quad P_n^d(\pi - \theta) = (-1)^n P_n^d(\theta) \tag{13}$$

it is made obvious that each quantity $g(\nu)$ could be received by replacing $P_\nu(\theta)$ with $-P_\nu(\pi - \theta)$ and $P_\nu^d(\theta)$ with $P_\nu^d(\pi - \theta)$ in the formula of the corresponding $f(\nu)$.

A deformation of the integration path (I_W) is possible if the symmetry of the integrands with respect to $\nu = -1/2$ is exploited. The functions $(2\nu + 1), \nu(\nu + 1), \sin(\nu\pi)$ are odd, even and even (in the order given) with respect to $\nu = -1/2$. Consequently, with use of the following properties of the utilized special functions [16, eq. (8.2.1), p. 333]

$$h_{-\nu-1}(z) = = e^{i\nu\pi} h_\nu(z) \quad h_{-\nu-1}^d(z) = e^{i\nu\pi} h_\nu^d(z) \tag{14}$$

$$P_{-\nu-1}(z) = P_\nu(z) \quad P_{-\nu-1}^d(z) = P_\nu^d(z) \tag{15}$$

a symmetry relation for all the integrands $\gamma(\nu)$ is derived:

$$\gamma(-\nu - 1) = -\gamma(\nu) \tag{16}$$

It is well-known that the Legendre functions are vanishing by definition when the integer degree is smaller than the integer order, therefore $P_0(\theta) = P_0^d(\theta) = 0$. Accordingly, the integration path (I_W) can be expanded to include the singular point $\nu = 0$, as its residue equals zero. The new path passes through the symmetry point $\nu = -1/2$ and thus the integral along the lower branch equals the integral along the symmetric route traced out in the opposite direction as shown in Fig. 3(b). In this way, the integration path of the integral is now a straight line slightly above the real axis denoted by (I_C). In order to obtain a close contour integral the asymptotic behavior of the integrands $\gamma(\nu)$ for large complex $\nu \to \infty$ shall be extracted. In this attempt the approximate formulas below are useful [16, eq. (8.10.7), p. 336]:

$$P\nu(\theta), P_\nu^d(\theta) = O\left[\sin\left(\left(\nu + \frac{1}{2}\right)\theta + \frac{\pi}{4}\right)\right], \quad \nu \to \infty \tag{17}$$

$$\frac{h_\nu(k_0 r)}{h_\nu(k_0 a)}, \frac{h_\nu^d(k_0 r)}{h_\nu(k_0 a)}, \frac{h_\nu(k_0 r)}{h_\nu^d(k_0 a)}, \frac{h_\nu^d(k_0 r)}{h_\nu^d(k_0 a)} = O\left[\left(\frac{r}{a}\right)^\nu\right], \quad \nu \to \infty \tag{18}$$

By combining the aforementioned equations, the asymptotic expression for the magnitude of each integrand is received:

$$|\gamma(\nu)| = O\left[e^{-(\Re[\nu]\ln(r/a)+|\Im[\nu]|\theta)}\right], \quad \nu \to \infty \tag{19}$$

It exhibits exponentially decaying behavior along the upper semicircle of the complex ν plane ($\Re[\nu] > 0$) and this is the route via which the path (I_C) is closed as shown in Fig. 3(b).

The participating functions in the formulas of $\gamma(\nu)$ are entire for complex ν and thus the integrand singularities inside the contour (I_C) are exclusively owed to the roots of the denominators:

$$h_\nu(k_0 a) = 0 \quad h_\nu^d(k_0 a) = 0 \tag{20}$$

Suppose the exact values of ν verifying the transcendental equations above are denoted by ν_m and ν_m^d respectively for each positive integer m. By implementing the residue theorem again, an alternative series is obtained:

$$F = \int_{(I_C)} \frac{g(\nu)}{2\pi i \sin(\nu\pi)} d\nu = -\sum_{m=1}^{+\infty} \frac{G(\kappa_m)}{\sin(\kappa_m \pi)} \tag{21}$$

where $G(\nu)$ is the integrand $\gamma(\nu)$ if one replaces the denominators $h_\nu(k_0 a), h_\nu^d(k_0 a)$ by their derivatives with respect to the order ν $\frac{\partial h_\nu(k_0 a)}{\partial \nu}, \frac{\partial h_\nu^d(k_0 a)}{\partial \nu}$. The parameters κ_m equal ν_m or ν_m^d according to the denominator.

4. POLES ON THE COMPLEX ORDER PLANE

The purpose of this section is to solve approximately the following equations with respect to complex ν:

$$H_\nu(z) = 0 \qquad H'_\nu(z) = 0 \tag{22}$$

where $H_\nu(z)$ is the cylindrical Hankel function of the first kind and the prime denotes the derivative with respect to z. If one solves the Equations (22), it can easily find suitable initial guesses for the roots of (20). To manipulate the Hankel-type functions, it is necessary to introduce asymptotic expressions which are mainly available for three cases. When the order's magnitude $|\nu|$ is much greater than the absolute value of the argument $|z|$, the Hankel-type functions can be approximated by the following [16, eq. (9.3.1), p. 365]:

$$H_\nu(z) \sim \begin{cases} -i\left(\frac{2}{\pi\nu}\right)^{1/2}\left(\frac{ez}{2\nu}\right)^{-\nu}, & \Re[\nu]>0, |z| \ll |\nu| \\ -ie^{i\nu\pi}\left(-\frac{2}{\pi\nu}\right)^{1/2}\left(-\frac{ez}{2\nu}\right)^{\nu}, & \Re[\nu]<0, |z| \ll |\nu| \end{cases} \tag{23}$$

$$H'_\nu(z) \sim \begin{cases} \frac{\nu}{z}i\left(\frac{2}{\pi\nu}\right)^{1/2}\left(\frac{ez}{2\nu}\right)^{-\nu}, & \Re[\nu]>0, |z| \ll |\nu| \\ -i\frac{\nu}{z}e^{i\nu\pi}\left(-\frac{2}{\pi\nu}\right)^{1/2}\left(-\frac{ez}{2\nu}\right)^{\nu}, & \Re[\nu]<0, |z| \ll |\nu| \end{cases} \tag{24}$$

It is apparent that the aforementioned expressions do not vanish on the complex ν plane. When the magnitude of the argument $|z|$ is very larger than the order's magnitude $|\nu|$, then the Hankel-type functions possess the asymptotic expansions below:

$$H_\nu(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} e^{iz-i\nu\pi/2-i\pi/4}, \quad |\nu| \ll |z| \tag{25}$$

$$H'_\nu(z) \sim \frac{i}{z}\left(\frac{2}{\pi z}\right)^{1/2} e^{iz-i\nu\pi/2-i\pi/4}, \quad |\nu| \ll |z| \tag{26}$$

Again the quantities are nonzero for each finite complex ν. For this reason, we utilize a more sophisticated asymptotic formula covering cases that both the order's and argument's magnitudes are large and not very different. As far as the Hankel function $H_\nu(z)$ is concerned, we duplicate from [16, eq. (9.3.37), p. 368]:

$$\begin{aligned} H_\nu(z) \sim \frac{2}{\nu^{1/3}}e^{-i\pi/3}\left(\frac{4\nu^2\xi(\nu,z)}{\nu^2-z^2}\right)^{1/4}\Bigg[&Ai(e^{2\pi i/3}\nu^{2/3}\xi(\nu,z))\sum_{k=0}^{+\infty}\frac{a_k(\nu,z)}{\nu^{2k}} \\ &+ e^{2\pi i/3}Ai'(e^{2\pi i/3}\nu^{2/3}\xi(\nu,z))\sum_{k=0}^{+\infty}\frac{b_k(\nu,z)}{\nu^{2k+4/3}}\Bigg], \quad |\nu| \cong |z| \to +\infty \end{aligned} \tag{27}$$

where the function $\xi(\nu, z)$ is defined through the interlaced relation

$$\frac{2}{3}\left(-\xi(\nu,z)\right)^{3/2} = \left(\left(\frac{z}{\nu}\right)^2 - 1\right)^{1/2} - \arccos\left(\frac{\nu}{z}\right) \tag{28}$$

and the series coefficients are given by

$$a_k(\nu,z) = \sum_{s=0}^{2k}\mu_s\left(\xi(\nu,z)\right)^{-3s/2}u_{2k-s}\left(\frac{\nu}{(\nu^2-z^2)^{1/2}}\right)$$

$$b_k(\nu, z) = -\sum_{s=0}^{2k+1} \lambda_s \left(\xi(\nu, z)\right)^{-3s/2-1/2} u_{2k-s+1}\left(\frac{\nu}{(\nu^2 - z^2)^{1/2}}\right) \quad (29)$$

The parameters μ_s, λ_s are specific real numbers and the functions $u_k(t)$ are polynomials with degree $3k$. The Airy function and its derivative are denoted by $Ai(z), Ai'(z)$ respectively.

The asymptotic formula (27) is valid for $|\nu| \cong |z| \to +\infty$ but we are going to modify some of the participating quantities by supposing that ν is close to z as complex number, not simply as magnitude. This assumption is a significant one but we support that it cannot damage the effectiveness of the approximation due to the complexity and robustness of (27). With use of the following Taylor expansion:

$$(x^2 - 1)^{1/2} - \arccos\left(\frac{1}{x}\right) \sim \frac{2^{3/2}}{3}(x-1)^{3/2}, \quad x \cong 1 \quad (30)$$

the complicated Equation (28) can be simplified to give:

$$\xi(\nu, z) \sim 2^{1/3}\frac{\nu - z}{z}, \quad \nu \cong z \to \infty \quad (31)$$

With reference to the polynomials $u_k(t)$ the term with maximum degree ($3k$) will be dominant, due to their large argument ($\nu \to \infty$). The behavior of the series coefficients can be estimated by taking into account this fact

$$\begin{aligned} a_k(\nu, z) &= O\left[\left(\frac{\nu}{\nu - z}\right)^{3k} \frac{1}{2^{3k}} \sum_{s=0}^{2k} \frac{\mu_s}{2^{-s}}\right], \quad \nu \cong z \to \infty \\ b_k(\nu, z) &= O\left[\left(\frac{\nu}{\nu - z}\right)^{3k+2} \frac{1}{2^{3k}} \sum_{s=0}^{2k+1} \frac{\lambda_s}{2^{-s}}\right], \quad \nu \cong z \to \infty \quad (32) \end{aligned}$$

After carrying out straightforward algebra, one can produce the limiting relation connecting the two series in (27) as appeared below:

$$\sum_{k=0}^{+\infty} \frac{b_k(\nu, z)}{\nu^{2k+4/3}} = O\left[\frac{1}{\nu^{2/3}(\nu - z)}\right] \sum_{k=0}^{+\infty} \frac{a_k(\nu, z)}{\nu^{2k}}, \quad \nu \cong z \to \infty \quad (33)$$

Although ν and z are close each other on the complex plane, they never become equal (the difference between them can be also substantial given the fact that both numbers tend to infinite). As $|\nu|$ takes very large values, it is sensible that the term proportional to the Airy function will be of much larger magnitude than the term proportional to the Airy prime function in (27). Considering that the two functions have values of the same order, the roots of the first equation of (22) are close to the ones of the following

$$Ai\left(e^{2\pi i/3}\nu^{2/3}\xi(\nu,z)\right)=0 \tag{34}$$

As far as the Hankel derivative is concerned, a similar to (27) asymptotic expansion is available [16, eq. (9.3.45), p. 369] and given by:

$$\begin{aligned} H'_\nu(z) \sim \frac{4}{z\nu^{2/3}}e^{2i\pi/3}\left(\frac{4\nu^2\xi(\nu,z)}{\nu^2-z^2}\right)^{-1/4}&\Bigg[Ai(e^{2\pi i/3}\nu^{2/3}\xi(\nu,z))\sum_{k=0}^{+\infty}\frac{c_k(\nu,z)}{\nu^{2k+2/3}} \\ &+ e^{2\pi i/3}Ai'(e^{2\pi i/3}\nu^{2/3}\xi(\nu,z))\sum_{k=0}^{+\infty}\frac{d_k(\nu,z)}{\nu^{2k}}\Bigg], \quad |\nu|\cong|z|\to+\infty \end{aligned} \tag{35}$$

where

$$\begin{aligned} c_k(\nu,z) &= -\sum_{s=0}^{2k}\mu_s\left(\xi(\nu,z)\right)^{-3s/2+1/2}v_{2k-s+1}\left(\frac{\nu}{(\nu^2-z^2)^{1/2}}\right) \\ d_k(\nu,z) &= \sum_{s=0}^{2k+1}\lambda_s\left(\xi(\nu,z)\right)^{-3s/2}v_{2k-s}\left(\frac{\nu}{(\nu^2-z^2)^{1/2}}\right) \end{aligned} \tag{36}$$

The functions $v_k(t)$ are polynomials with degree $(3k)$. In a similar way the roots of the second equation of (22) are found to be close to the ones of the following:

$$Ai'\left(e^{2\pi i/3}\nu^{2/3}\xi(\nu,z)\right)=0 \tag{37}$$

This information is sufficient for estimating the solution of the equation $h_\nu^d(z)=0$. That is because the spherical Hankel function and its derivative possess similar magnitudes and therefore the derivative is the dominant quantity because it is multiplied by the large complex number z.

5. NUMERICAL IMPLEMENTATION

The well-known roots of the Airy function and its derivative are negative, denoted by τ_m and τ'_m respectively where m is positive integer increasing with the distance of the root from the origin. Consequently, by equalizing these parameters with the arguments of (34) and (37) we can find initial guesses $\tilde{\nu}_m$ and $\tilde{\nu}_m^d$ close to the exact roots ν_m and ν_m^d of (20). If one takes into account the assumption $\nu \cong z$ and the result of (31), the approximations for the positions of the singularities belonging on the upper half of the complex plane are readily derived:

$$\tilde{\nu}_m = -\frac{1}{2} + k_0 a + \left(\frac{k_0 a}{2}\right)^{1/3} e^{-2\pi i/3}\tau_m \tag{38}$$

$$\tilde{\nu}_m^d = -\frac{1}{2} + k_0 a + \left(\frac{k_0 a}{2}\right)^{1/3} e^{-2\pi i/3}\tau'_m \tag{39}$$

With this input, a numerical solver will determine the actual roots ν_m, ν_m^d provided the fact that it can compute Hankel and Legendre functions with complex order and degree respectively. The numerical examples are implemented in the computing environment $MATHEMATICA$ with use of its build-in routine $FindRoot$. In Fig. 4 we depict a contour plot of the quantity $1/|h_\nu(10)|$ on the

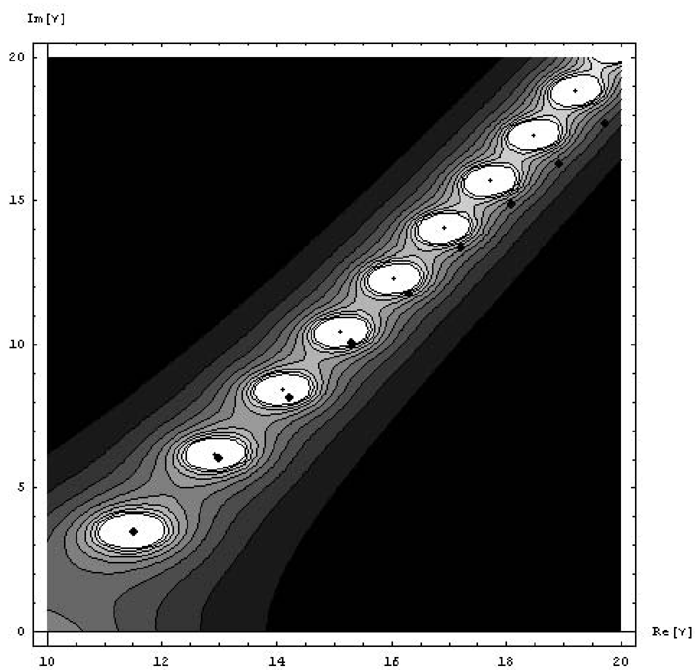

Figure 4. The contour plot of the quantity $1/|h_\nu(10)|$ on the complex ν plane. The white areas correspond to large values and the dark regions to small ones. The small crosses correspond to the exact positions of the poles, while the large dots represent the initial guesses.

complex ν plane. The large values are represented by white color (root regions) and the small ones by black color (shadowed area). The small crosses symbolize the exact positions ν_m of the denominator singularities, while the large dots represent the approximations $\tilde{\nu}_m$ of (38). The numbering m is increasing towards the upper right direction, starting with $m = 1$ (lower leftmost root). It is observed that for small m, when the corresponding residue terms have a large contribution to the Watson sum (21), the estimations are closer to the exact roots. Mind that the roots lie exclusively on the first quadrant, whereas the dominant first singularity is close to the positive argument $z = k_0 a = 10$ a fact that verifies our assumption ($\nu \cong z$).

After the detection of each integrand's poles, the evaluation of the series (21) is possible as the differentiations of the denominators with respect to the complex order are carried out numerically (the corresponding functions are entire with respect to ν). In Fig. 5 we present the required number of terms M to achieve convergence for (21) with a maximum tolerance of 0.001%. We examine the same cases as in Fig. 2 and it is surprising that just the first term is adequate for computing the field quantities for several radial distances and

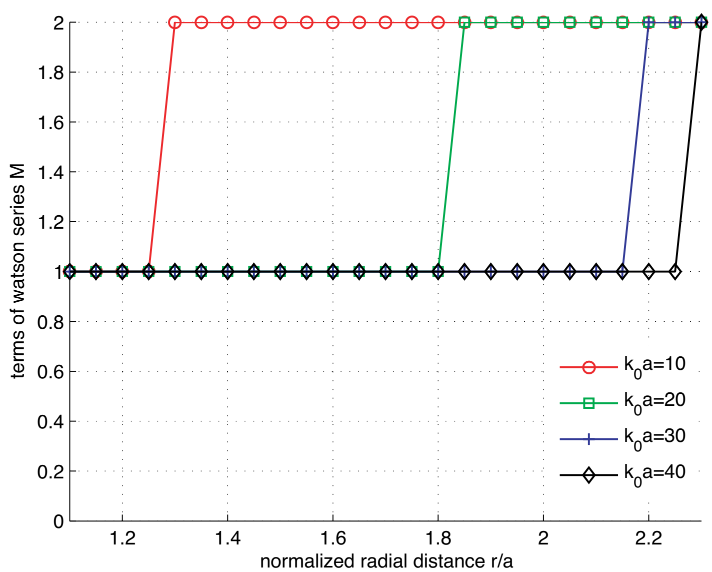

Figure 5. The required number of terms M which are necessary to achieve convergence as function of the normalized radial distance r/a for various electrical radii of the sphere $k_0 a$. The Watson series is the used formula.

scatterer sizes. This is an impressive result concluding in a closed-form expression with observation points into the investigated region. One notices that the required number of terms is increasing with increasing r/a and with decreasing electrical radius of the sphere. This behavior is opposite compared to the one of the canonical series.

6. CONCLUSIONS

The Watson transformation is implemented for the simple case of the scattering of a spherical wave by a spherical scatterer. The procedure of determining the complex poles of the integrand is fully described. Numerical tests verify the achieved acceleration in convergence of the series defining the total field.

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