Progress In Electromagnetics Research B, Vol. 13, 41–58, 2009

MODIFIED INCOMPLETE CHOLESKY FACTORIZATION FOR SOLVING ELECTROMAGNETIC SCATTERING PROBLEMS

T.-Z. Huang, Y. Zhang, and L. Li

School of Applied Mathematics/Institute of Computational Science
University of Electronic Science and Technology of China
Chengdu 610054, Sichuan, China

W. Shao and S. Lai

School of Physical Electronics
University of Electronic Science and Technology of China
Chengdu 610054, Sichuan, China

Abstract—In this paper, we study a class of modified incomplete Cholesky factorization preconditioners LL^T with two control parameters including dropping rules. Before computing preconditioners, the modified incomplete Cholesky factorization algorithm allows to decide the sparsity of incomplete factorization preconditioners by two fill-in control parameters: (1) p, the number of the largest number p of nonzero entries in each row; (2) dropping tolerance. With RCM reordering scheme as a crucial operation for incomplete factorization preconditioners, our numerical results show that both the number of PCOCG and PCG iterations and the total computing time are reduced evidently for appropriate fill-in control parameters. Numerical tests on harmonic analysis for 2D and 3D scattering problems show the efficiency of our method.

1. INTRODUCTION

The coefficient matrix of the linear equations which stem from the finite-element analysis of high-frequency electromagnetic field simulations such as scattering [1–9] is generally symmetric and indefinite. At present, the incomplete Cholesky factorization [10–13] (IC) preconditioners applied with preconditioned Conjugate Gradient (PCG) method and preconditioned Conjugate Orthogonal Conjugate

Gradient (PCOCG) method are rather popular [14–16]. Here incomplete Cholesky factorization is studied in the case of finite element (FEM) matrices arising from the discretization of the following electromagnetic scattering problem:

$$\nabla \times \left(\frac{1}{\mu_r}\nabla \times E^{sc}\right) - k_0^2\varepsilon_r E^{sc} = -\nabla \times \left(\frac{1}{\mu_r}\nabla \times E^{inc}\right) + k_0^2 E^{inc}, \quad (1)$$

with some absorbing boundary conditions, where E^{sc} is the scattering field, E^{inc} is the incident field and μ_r and ε_r are relative permeability and permittivity, respectively.

The solution of Eq. (1) will result in a linear system

$$Ax = b, \quad (2)$$

where $A = (a_{ij})_{n\times n} \in C^{n\times n}$ is sparse complex symmetric (usually indefinite), $x, b \in C^n$.

In order to solve (2) effectively, incomplete LU factorization preconditioners are often associated with some preconditioned Krylov subspace methods such as BICGSTAB, QMR, TFQMR, CG, COCG [17–19]. To make full use of symmetry of the systems, incomplete Cholesky factorization is normally utilized with some preconditioned Krylov subspace methods such as PCG and PCOCG.

Incomplete Cholesky factorization was designed for solving symmetric positive definite systems. The performance of the incomplete Cholesky factorization often relies on drop tolerances [13, 17] to reduce fill-ins. The properties of the incomplete Cholesky factorization depend, in part, on the sparsity pattern S of the incomplete Cholesky factor $L = (l_{ij})_{n\times n}$, where L is a lower triangular matrix such that [10]

$$A = LL^T + R, \quad l_{ij} = 0 \text{ if } (i, j) \notin S.$$

The aim of the presented numerical tests is to analyze the performance of the studied incomplete Cholesky factorization algorithms. Our consideration is to focus on the performance of the proposed modified incomplete Cholesky factorization preconditioners with tuning of sparsity with PCG and PCOCG as accelerators. And we intend to find their impacts on these scattering problems discretized by FEM.

Many research papers about incomplete Cholesky factorization can be found, such as Lin and More [10], Fang and Leary [11], Margenov and Popov [12], the fixed fill factorization of Meijerink and Vorst [20], the ILUT factorization of Saad [13, 17]. For additional information on incomplete Cholesky factorizations, please refer to Saad [17]. Reordering methods are very important for incomplete factorization; see more in [21–26]. In [10], a new incomplete Cholesky factorization algorithm is proposed which is designed to limit the memory requirement by specifying the amount of additional memory. In contrast with drop tolerance strategies, the new approach in [10] is more stable in terms of number of iterations and memory requirements. In this paper, we intend to apply this approach as preconditioners for solving scattering problems and get more effective incomplete Cholesky factorization algorithm based on the work of Lin and More in [10]. Additionally, the reordering in the matrix plays an important role in the application of preconditioning technologies because the ordering of the matrix affects the fill in the matrix and thus the incomplete Cholesky factorization [21]. In this paper, both the AMD and RCM orderings [17, 21] are applied to reorder our linear system.

The rest of the paper is organized as follows: In section 2 we survey some relative preconditioning algorithms and Krylov subspace methods. The modified incomplete factorization algorithm is presented in section 3 with detailed description of its implementation. In Section 4, a set of numerical experiments are presented and short concluding remarks are given in Section 5.

2. PRECONDITIONERS AND ITERATIVE METHODS

Our implementation of the incomplete Cholesky factorization is based on the *jki* version of the Cholesky factorization shown below in Algorithm 1 [10]. Note that diagonal elements are updated as the factorization proceeds. Obviously, Algorithm 1 is based on the column-oriented Cholesky factorization for sparse matrices.

Algorithm 1: Column-oriented Cholesky Factorization [10, Algorithm 2.1]

for $j = 1 : n$
    $a_{jj} = \sqrt{a_{jj}}$
    for $k = 1 : j - 1$
        for $i = j + 1 : n$
            $a_{ij} = a_{ij} - a_{ik}a_{jk}$
        endfor
    endfor
    for $i = j + 1 : n$
        $a_{ij} = a_{ij}/a_{jj}$
        $a_{ii} = a_{ii} - a_{ij}^2$
    endfor
endfor

For a symmetric coefficient matrix A, we only need to store the lower or the upper triangular parts of A. In Algorithm 1, only the lower triangular part of A including the diagonal entries is needed. And the access to A is column by column. Therefore, in order to compute the IC-type preconditioner by Algorithm 1, we only need to store the lower triangular part L as the incomplete Cholesky factor. In order to show the detailed computation process of L, we transform Algorithm 1 into a comprehensible version with explicit computation of L:

Algorithm 2: Column-oriented Cholesky Factorization with Explicit Expression of L

```
for j = 1 : n
    l_jj = sqrt(a_jj)
    for k = 1 : j − 1
        for i = j + 1 : n
            l_ij = l_ij − l_ik l_jk
        endfor
    endfor
    for i = j + 1 : n
        l_ij = l_ij / l_jj
        a_ii = a_ii − l_ij^2
    endfor
endfor
```

In [10], the following Algorithm 3 has been discussed in details.

Algorithm 3. Column-oriented Cholesky Factorization [10, Algorithm 2.2]

```
for j = 1 : n
    a_jj = sqrt(a_jj)
    L_col_len = size(i > j : a_ij ≠ 0)
    for k = 1 : j − 1 and a_jk ≠ 0
        for i = j + 1 : n and a_ik ≠ 0
            a_ij = a_ij − a_ik a_jk
        endfor
    endfor
    for i = j + 1 : n and a_ij ≠ 0
        a_ij = a_ij / a_jj
        a_ii = a_ii − a_ij^2
    endfor
    Retain the largest L_col_len + p elements in a_{j+1:n,j}
endfor
```

Notice the symbol $a_{j+1:n,j}$ means these entries of the j-th column from row $j+1$ to row n of coefficient matrix A. For iterative solution of the symmetric linear system (2), we choose the precondtioned COCG and precondtioned CG methods (See more in [14–16]).

3. MODIFIED INCOMPLETE CHOLESKY FACTORIZATION ALGORITHM WITH ITS IMPLEMENTATION

In the light of ILUT algorithm in [17, p.287] and Algorithm 3 in [10, p.29], we present the following column-oriented MIC(p, τ) algorithm for obtaining the incomplete Cholesky factor L.

Algorithm 4. Modified Incomplete Cholesky factorization (MIC(p, τ))

```
for j = 1 : n
    l_jj = sqrt(a_jj)
    w = a_{j+1:n,j}
    for k = 1 : j − 1
        for i = j + 1 : n and when l_jk ≠ 0
            w_i = w_i − l_ik l_jk
        endfor
    endfor
    for i = j + 1 : n
        w_ij = w_ij / l_jj
    endfor
    τ_j = τ ||w||
    for i = j + 1 : n
        w_i = 0 when |w_i| < τ_j
    endfor
    An integer array I = (i_k)_{k=1,···,p} contains indices of the first
    largest p entries of |w_i|, i = j + 1 : n.
    for k = 1 : p
        l_{i_k j} = w_{i_k j}
    endfor
    for i = j + 1 : n
        a_ii = a_ii − l_ij^2
    endfor
endfor
```

In the case that the lower triangular matrix L keeps the same nonzero pattern as that of the lower triangular part of A, Algorithm 1 leads to IC(0) algorithm (i.e., incomplete Cholesky factorization preconditioners with the same nonzero pattern as that of coefficient matrix A).

In order to implement Algorithm 4, we store the upper triangular part of the coefficient matrix A in compressed sparse row (CSR) format. However, for convenience, Algorithm 4 needs to access lower triangular part of A in compressed sparse column (CSC) format. Observed from the data structures of CSR and CSC, the upper triangular part of the coefficient matrix A stored in CSR is exactly the lower triangular part of A stored in CSC. So, we don't need to perform the transform operation from the input matrix (the upper triangular part of the coefficient matrix A) into the CSC format of the lower triangular part of the coefficient matrix A. For simplicity, variable "L" here denotes the CSC format of L.

From Line 6 in Algorithm 4, in order to compute the linear combination vector w, we need to access the j-th row and k-th column of L. The access of $k-$th column of L is convenient because L is just stored in CSC. The difficulty in Line 6 is how to access the j-th row of L which is stored in CSC format. In order to get high efficiency of accessing rows of L, we introduce a temporary CSR variable "U" to store the CSR format of L.

In iterative methods, we need to solve the preconditioning system $LL^T x = y$. Normally, L and L^T are stored in CSR format, respectively. In fact, the transformation from the CSC format of L to the CSR foramt of L is unnecessary because variable "U" in CSR format is just the CSR format of L and variable "L" in CSC format is just the CSR format of L^T.

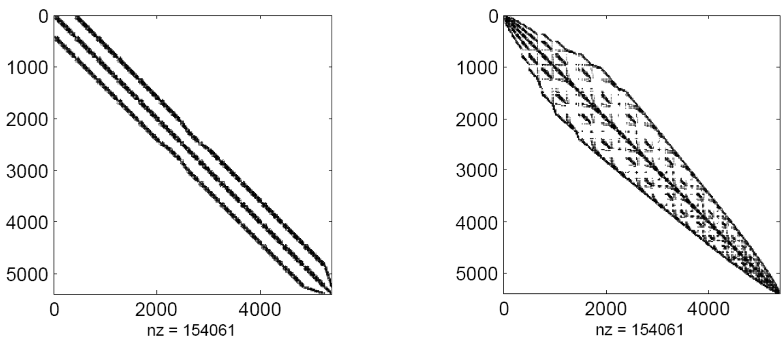
4. NUMERICAL TESTS

All numerical tests are performed on Linux operating system. All codes are programmed in C language and implemented on a PC, with 2 GB memory and a 2.66 GHz Intel(R) Core(TM)2 Duo CPU. In order to operate the complex type elements in computation, we declare "double complex" type variables which are supported directly by gcc compiler. The maximal iteration number is 1000. The iteration stops when $\|r^{(k)}\|/\|r^{(0)}\| < 10^{-8}$.

Since ordering is crucial to a good factorized preconditioner, the reverse Cuthill-McKee (RCM) reordering and Approximate Minimum Degree (AMD) reordering are applied before computing L. Denote the number of nonzero elements of a matrix as *nnz* (matrix name), the iteration number as *its*, the incomplete factorization CPU time as *P-t* and iteration CPU time as *I-t*, total computation time (preconditioning time plus iteration time) as *T-t*. All consuming time is measured in seconds. Denote sparse ratio of a preconditioner i.e., $\frac{nnz(L+L^T)-n}{nnz(A)}$ as *sp-r*.

Problem 1 (Harmonic Analysis for Plane Wave Scattering from a Metallic Plate): In this problem, we use edge-based FEM to calculate the RCS of a PEC plate ($1\lambda_o \times 1\lambda_o$) where λ_o stands for the free space wavelength of the incident plane wave. Applying PEC boundary condition, we need to solve a system of linear equations of size 5381 with a complex coefficient matrix containing 79721 nonzero elements in the upper triangular.

Problem 2 (Harmonic Analysis for Scattering of a Dielectric Sphere): In this problem, perfectly matched layers (PML) are used to truncate the finite element analysis domain in order to determine the radar cross section (RCS) from the scattering of a dielectric sphere. The relative permittivity of the dielectric sphere is $\varepsilon_r = 2.56$. To calculate the bistatic RCS of the dielectric sphere, the incident plane wave is taken as x-polarized with incident angles $\phi = 0°$ and $\theta = 0°$, which leads to a system of linear equations of size 130733 with a complex coefficient matrix containing 1105104 nonzero elements in the upper triangular.



(a) Original (b) RCM

Figure 1. Nonzero pattern of the coefficient matrix from Problem 1 with Original ordering and RCM reordering.

For Problems 1 and 2, without RCM reordering, PCOCG and PCG methods do not converge within 5000 iterations. In order to evaluate the performance of the proposed algorithm, the IC(0) preconditioner (i.e. $L + L^T$ has the same nonzero pattern with that of A) and the diagonal preconditioner are exploited (see Table 1 for details). Numerical results of the solution of Problem 1 and Problem 2 with PCOCG and PCG methods associated with RCM reordering are presented in Tables 2-7. For Problem 1, AMD reordering nearly failed in all cases with the same parameters of RCM reordering, except in two cases. For Problem 2, AMD reordering failed in all cases. So the results with AMD reordering are all ignored in this section.

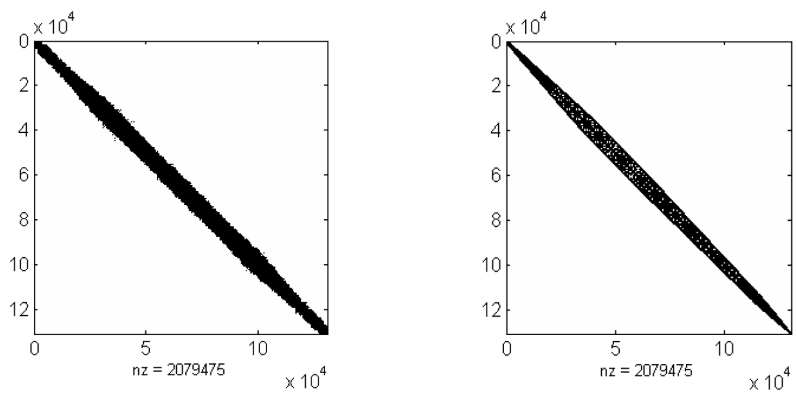
Table 1. PCOCG with IC(0) and diagonal preconditioners for Problem 1.

Preconditioner	IC(0)		Diagonal preconditioner	
ordering	NO	RCM	NO	RCM
Its	5000	2997	755	760

By comparing Table 1 with Table 2, it is obvious that our MIC (p, τ) preconditioner is much more efficient than IC(0) and diagonal preconditioners. Observed from Tables 2 and 3, the number of iterations and total computation time of the two kinds of iterative methods of PCOCG and PCG are almost the same in all cases with the same parameters in MIC(p, τ).

From Tables 2–4 and Fig. 1, it is noticed that the effect of dropping tolerance τ in such a wide range for Problem 1 is not prominent. Observed from Fig. 1, there is a jump of computation time with parameters $p = 30$ and $\tau = 10^{-3}$. From a general view, it is a specific case. However, from Tables 5-7 and Fig. 2 of the test of Problem 2, it is possible for us to draw the conclusion that the effect of dropping tolerance τ is obvious in the solution of larger scale problems. And the reasonable range for τ should arrange from 10^{-4} to 10^{-6}. In addition, the parameter τ has minor effect on memory requirement of our MIC preconditioner.

What affects remarkably is the parameter p which also decides both fill-ins and efficiency of MIC preconditioner. The larger p is,



(a) Original (b) RCM

Figure 2. Nonzero pattern of the coefficient matrix from Problem 2 with Original ordering and RCM reordering.

Table 2. PCOCG with RCM reordering and Algorithm 6 for Problem 1.

τ	p	$nnz(L)$	sp-r	its	P-t	I-t	T-t
10^{-3}	30	171563	2.19	143	0.39	0.77	1.16
	40	224771	2.88	54	0.57	0.35	0.92
	50	277754	3.57	47	0.77	0.36	1.13
	60	330495	4.26	39	0.97	0.33	1.30
	70	382817	4.93	31	1.16	0.29	1.45
	80	434926	5.61	28	1.38	0.29	1.67
	90	486670	6.28	25	1.57	0.30	1.87
	100	538009	6.95	23	1.78	0.27	2.05
10^{-4}	30	171563	2.19	323	0.38	1.74	2.12
	40	224774	2.88	57	0.58	0.37	0.95
	50	277756	3.57	42	0.76	0.32	1.08
	60	330467	4.26	38	0.96	0.32	1.28
	70	382843	4.94	30	1.18	0.27	1.45
	80	434934	5.61	27	1.38	0.27	1.65
	90	486670	6.28	25	1.58	0.27	1.85
	100	538017	6.95	22	1.77	0.26	2.03
10^{-5}	30	171563	2.19	304	0.38	1.63	2.01
	40	224774	2.88	56	0.57	0.36	0.93
	50	277756	3.57	41	0.78	0.30	1.08
	60	330467	4.26	38	0.98	0.32	1.30
	70	382821	4.93	29	1.16	0.27	1.43
	80	434934	5.61	27	1.37	0.27	1.64
	90	486658	6.28	24	1.59	0.26	1.85
	100	538017	6.95	22	1.78	0.27	2.05
10^{-6}	30	171563	2.19	309	0.39	1.65	2.04
	40	224774	2.88	56	0.57	0.36	0.93
	50	277756	3.57	44	0.77	0.33	1.10
	60	330467	4.26	38	0.98	0.31	1.29
	70	382821	4.93	30	1.17	0.28	1.45
	80	434934	5.61	26	1.36	0.26	1.62
	90	486658	6.28	24	1.56	0.27	1.83
	100	538017	6.95	22	1.77	0.26	2.03

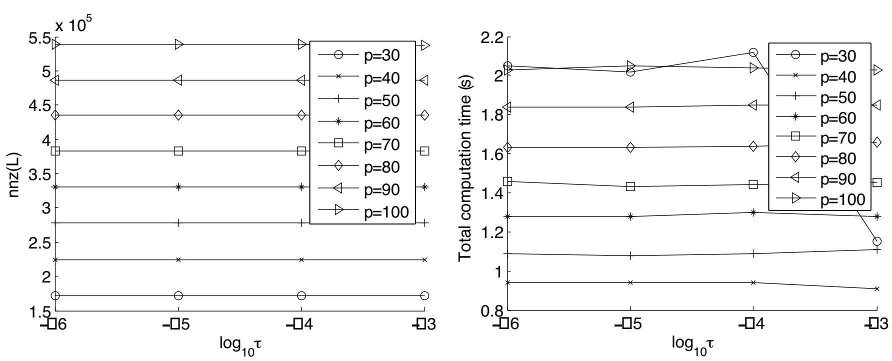
Table 3. PCG with RCM reordering and Algorithm 6 for Problem 1.

τ	p	$nnz(L)$	sp-r	its	P-t	I-t	T-t
10^{-3}	30	171563	2.19	143	0.39	0.76	1.15
	40	224771	2.88	54	0.57	0.34	0.91
	50	277754	3.57	47	0.77	0.34	1.11
	60	330495	4.26	39	0.96	0.32	1.28
	70	382817	4.93	31	1.17	0.28	1.45
	80	434926	5.61	28	1.38	0.28	1.66
	90	486670	6.28	25	1.58	0.27	1.85
	100	538009	6.95	23	1.77	0.26	2.03
10^{-4}	30	171563	2.19	323	0.39	1.73	2.12
	40	224774	2.88	57	0.57	0.37	0.94
	50	277756	3.57	42	0.78	0.31	1.09
	60	330467	4.26	38	0.98	0.32	1.30
	70	382843	4.94	30	1.17	0.27	1.44
	80	434934	5.61	27	1.37	0.27	1.64
	90	486670	6.28	25	1.58	0.27	1.85
	100	538017	6.95	22	1.78	0.26	2.04
10^{-5}	30	171563	2.19	304	0.39	1.63	2.02
	40	224774	2.88	56	0.58	0.36	0.94
	50	277756	3.57	41	0.77	0.31	1.08
	60	330467	4.26	38	0.97	0.31	1.28
	70	382821	4.93	29	1.16	0.27	1.43
	80	434934	5.61	27	1.36	0.27	1.63
	90	486658	6.28	24	1.57	0.27	1.84
	100	538017	6.95	22	1.78	0.27	2.05
10^{-6}	30	171563	2.19	309	0.39	1.66	2.05
	40	224774	2.88	56	0.57	0.37	0.94
	50	277756	3.57	44	0.77	0.32	1.09
	60	330467	4.26	38	0.97	0.31	1.28
	70	382821	4.93	30	1.18	0.28	1.46
	80	434934	5.61	26	1.37	0.26	1.63
	90	486658	6.28	24	1.58	0.26	1.84
	100	538017	6.95	22	1.77	0.26	2.03

Table 4. PCOCG with RCM reordering and Algorithm 6($\tau = 0$) for Problem 1.

p	$nnz(L)$	sp-r	its	P-t	I-t	T-t
30	171563	2.19	310	0.39	1.67	2.06
40	224774	2.88	57	0.57	0.38	0.95
50	277756	3.57	44	0.77	0.33	1.10
60	330467	4.26	38	0.97	0.32	1.29
70	382821	4.93	29	1.18	0.27	1.45
80	434934	5.61	26	1.37	0.28	1.65
90	486658	6.28	24	1.58	0.27	1.85
100	538017	6.95	22	1.78	0.27	2.05

the less the iteration number becomes while the more the fill-ins are required. However, the total computation time is not necessarily decreasing with the growth of p, which implies that it is crucial to select an appropriate parameter p. Generally, parameter p can be evaluated by setting the number of nonzero entries of incomplete Cholesky preconditioners, i.e., $p = \frac{nnz(L)}{n}$ where L is the incomplete Cholesky preconditioner and n is the dimension of coefficient matrix A. However, the number of nonzero entries of incomplete Cholesky preconditioners L is determined by the coefficient matrix A of linear system. For small-scale linear system such as Problem 1, proper set of the number of nonzero entries of L is about 2 times (or more) of that



(a) fill-in (b) Computation time with PCOCG

Figure 3. Comparisons using fill-in and total computation time for Problem 1.

Table 5. PCOCG with RCM reordering and Algorithm 6 for Problem 2.

τ	p	$nnz(L)$	sp-r	its	P-t	I-t	T-t
10^{-3}	180	23540520	22.58	1000	148.09	454.47	602.56
	190	24825650	23.81	1000	159.85	476.29	636.14
	200	26099574	25.04	1000	174.97	496.12	671.09
	210	27389064	26.28	1000	198.96	517.83	716.79
	220	28657799	27.5	1000	219.74	537.71	757.45
	230	29944507	28.74	1000	256.23	560.05	816.28
	240	31219668	29.96	1000	276.74	579.14	855.88
	250	32489468	31.18	1000	290.12	601.16	891.28
10^{-4}	180	23547572	22.58	1000	121.8	455.21	577.01
	190	24838383	23.83	1000	130.8	476.97	607.77
	200	26112689	25.05	1000	139.76	497.00	636.76
	210	27447965	26.34	278	140.69	143.99	284.68
	220	28678478	27.52	1000	158.86	539.23	698.09
	230	30015314	28.81	84	160.26	47.07	207.33
	240	31297382	30.04	63	171.29	36.74	208.03
	250	32578277	31.27	56	181.64	33.65	215.29
10^{-5}	180	23548029	22.59	1000	120.82	455.7	576.52
	190	24875794	23.86	783	122.89	372.6	495.49
	200	26162532	25.1	532	131.71	264.42	396.13
	210	27447791	26.34	210	141.15	108.78	249.93
	220	28731952	27.57	306	151.61	164.79	316.40
	230	30015228	28.81	80	160.69	44.81	205.50
	240	31297303	30.04	62	171.54	36.0	207.54
	250	32578207	31.27	56	182.7	33.72	216.42
10^{-6}	180	23574297	22.61	1000	118.37	456.53	574.90
	190	24826475	23.81	1000	131.18	477.2	608.38
	200	26121534	25.06	1000	140.29	497.7	637.99
	210	27447483	26.34	431	141.31	223.18	364.49
	220	28731705	27.57	508	152.5	273.6	426.10
	230	30014971	28.8	82	160.85	45.92	206.77
	240	31297074	30.04	62	171.61	35.99	207.60
	250	32577915	31.27	56	182.48	33.69	216.17

Table 6. PCG with RCM reordering and Algorithm 6 for Problem 2.

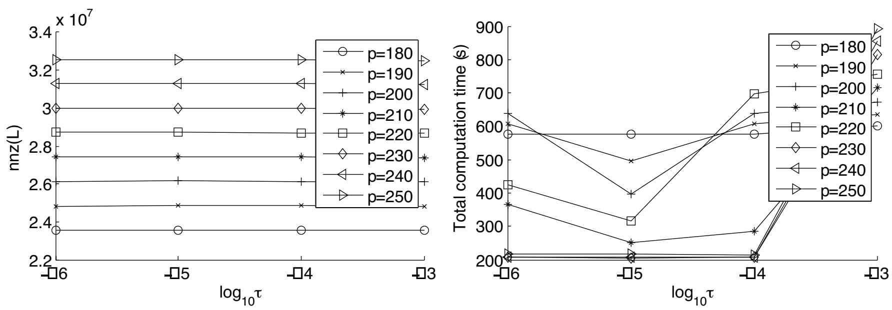
τ	p	$nnz(L)$	sp-r	its	P-t	I-t	T-t
10^{-3}	180	23540520	22.58	1000	145.29	454.33	599.62
	190	24825650	23.81	1000	161.02	479.51	640.53
	200	26099574	25.04	1000	175.92	498.69	674.61
	210	27389064	26.28	1000	198.87	519.67	718.54
	220	28657799	27.5	1000	219.67	537.06	756.73
	230	29944507	28.74	1000	255.94	559.45	815.39
	240	31219668	29.96	1000	276.37	579.78	856.15
	250	32489468	31.18	1000	300.56	600.82	901.38
10^{-4}	180	23547572	22.58	1000	121.97	455.65	577.62
	190	24838383	23.83	1000	130.31	476.83	607.14
	200	26112689	25.05	1000	139.8	497.33	637.13
	210	27447965	26.34	278	140.63	143.95	284.58
	220	28678478	27.52	1000	158.91	539.05	697.96
	230	30015314	28.81	84	160.3	47.27	207.57
	240	31297382	30.04	63	170.82	36.61	207.43
	250	32578277	31.27	56	181.45	33.67	215.12
10^{-5}	180	23548029	22.59	1000	120.75	456.19	576.94
	190	24875794	23.86	783	123.56	372.87	496.43
	200	26162532	25.1	532	131.93	264.7	396.63
	210	27447791	26.34	210	141.12	108.89	250.01
	220	28731952	27.57	306	151.26	164.62	315.88
	230	30015228	28.81	80	160.9	44.87	205.77
	240	31297303	30.04	62	171.42	36.0	207.42
	250	32578207	31.27	56	182.14	33.7	215.84
10^{-6}	180	23574297	22.61	1000	118.27	456.01	574.28
	190	24826475	23.81	1000	131.36	476.82	608.18
	200	26121534	25.06	1000	140.1	496.84	636.94
	210	27447483	26.34	431	141.34	223.45	364.79
	220	28731705	27.57	508	151.66	273.48	425.14
	230	30014971	28.8	82	160.88	45.91	206.79
	240	31297074	30.04	62	171.3	35.99	207.29
	250	32577915	31.27	56	182.26	33.66	215.92

Table 7. PCOCG with RCM reordering and Algorithm 6(τ=0) for Problem 2.

p	$nnz(L)$	sp-r	its	P-t	I-t	T-t
180	23588711	22.62	1000	114.830	457.230	572.06
190	24875808	23.86	1000	123.070	476.120	599.19
200	26161974	25.09	428	131.940	214.42	346.36
210	27447316	26.33	1000	144.280	518.570	662.85
220	28731527	27.57	212	151.670	114.240	265.91
230	30014648	28.80	81	160.750	45.350	206.10
240	31296689	30.03	63	172.080	36.700	208.78
250	32577809	31.26	56	184.360	33.680	218.04

of A. For middle-scale linear system such as Problem 2, the select of the number of nonzero entries of L could be 5 (or more) times of that of A.

In order to compare the performance of Algorithm 4 (MIC(p,τ) with that of Algorithm 3, numerical experiments with Algorithm 3 are also performed. Note that parameter p in Algorithms 3 and 4 has different meanings. Observed from Tables 8 and 9, Algorithm 3 needs more memory than Algorithm 4 under the requirement of the same total computation time. Take Problem 2 for example. The minimum computation time with Algorithm 3 is 211.92(s) and the fill-ins of L is 31098172. Nevertheless, using Algorithm 4, it consumes



(a) Fill-in (b) Computation time with PCOCG

Figure 4. Comparisons using fill-in and total computation time for Problem 2.

Table 8. PCOCG with RCM reordering and Algorithm 3 for Problem 1.

p	$nnz(L)$	sp-r	its	P-t	I-t	T-t
10	143492	1.83	1000	0.13	4.66	4.79
15	170548	2.18	985	0.19	5.12	5.31
20	197289	2.53	355	0.25	2.04	2.29
25	223975	2.87	213	0.33	1.34	1.67
30	250578	3.22	146	0.42	0.97	1.39
40	303620	3.91	55	0.61	0.42	1.03
50	356219	4.59	43	0.81	0.37	1.18
60	408503	5.27	38	1.00	0.36	1.36
70	460516	5.94	30	1.20	0.32	1.52
80	512103	6.61	26	1.40	0.30	1.70
90	563380	7.28	24	1.61	0.29	1.90
100	614174	7.94	23	1.82	0.31	2.13

Table 9. PCOCG with RCM reordering and Algorithm 3 for Problem 2.

p	$nnz(L)$	sp-r	its	P-t	I-t	T-t
180	24628860	23.62	1000	150.39	762.92	913.31
190	25913314	24.86	1000	156.25	767.82	924.07
200	27185244	26.08	1000	169.99	792.04	962.03
210	28532806	27.38	468	147.59	256.29	403.88
220	29752241	28.55	1000	184.54	799.48	984.02
230	31098172	29.85	78	166.37	45.55	211.92
240	32379376	31.08	61	177.49	36.88	214.37
250	33659524	32.31	56	189.05	35.10	224.15

205.5(s) with parameters $p = 230$ and $\tau = 10^{-5}$ and the fill-ins of L is 30015228. Additionally, as illustrated in Table 10 for Problem 1, Algorithm 4 is dramatically superior to Algorithm 3 in both aspects of total computation time and memory requirement.

Table 10. Comparison results with respect to the minimum of total computation time and the corresponding memory between Algorithms 3 and 6 for Problem 1.

min(T-t)			$nnz(L)$		
Alg. 3	Alg. 6	Reduction ratio(%)	Alg. 3	Alg. 6	Reduction ratio(%)
1.03	0.92	10.68	303620	224771	25.97

5. CONCLUSIONS

A column-oriented modified incomplete Cholesky factorization MIC (p, τ) with two controlling parameters for solution of systems of linear equations with sparse complex symmetric coefficient matrices resulted from finite-element analysis of the electromagnetic scattering problem (1) is presented in this paper. Proper choices of the controlling parameters in Algorithm 6 can evidently reduce the total computation time and memory requirements compared with Algorithm 3. It is worthwhile to emphasize that the involved parameter p, which prescribes the maximal fill-ins in each row of preconditioners, makes Algorithm 6 evidently superior to Algorithm 3 in the number of fill-ins, and helps to reduce total computation time of Algorithm 6. As shown in the numerical experiments, RCM ordering is obviously superior to AMD ordering. Moreover, RCM ordering is significant to our modified incomplete Cholesky factorization. Numerical experiments show that further developments of more proper incomplete factorization algorithms and reordering schemes for electromagnetic scattering problems are deserved to be taken into consideration in the future.

ACKNOWLEDGMENT

This research is supported by 973 Programs (2008CB317110), NSFC (10771030), the Scientific and Technological Key Project of the Chinese Education Ministry (107098), the Specialized Research Fund for the Doctoral Program of Higher Education (20070614001), Sichuan Province Project for Applied Basic Research (2008JY0052) and the Project for Academic Leader and Group of UESTC.

REFERENCES

1. Jin, J. M., *The Finite Element Method in Electromagnetics*, Wiley, New York, 1993.
2. Volakis, J. L., A. Chatterjee, and L. C. Kempel, *Finite Element Method for Electromagnetics: Antennas, Microwave Circuits and Scattering Applications*, IEEE Press, New York, 1998.
3. Ahmed, S. and Q. A. Naqvi, "Electromagnetic scattering of two or more incident plane waves by a perfect, electromagnetic conductor cylinder coated with a metamaterial," *Progress In Electromagnetics Research B*, Vol. 10, 75–90, 2008.
4. Fan, Z. H., D. Z. Ding, and R. S. Chen, "The efficient analysis of electromagnetic scattering from composite structures using hybrid CFIE-IEFIE," *Progress In Electromagnetics Research B*, Vol. 10, 131–143, 2008.
5. Botha, M. M. and D. B. Davidson, "Rigorous auxiliary variable-based implementation of a second-order ABC for the vector FEM," *IEEE Trans. Antennas Propagat.*, Vol. 54, 3499–3504, 2006.
6. Harrington, R. F., *Field Computation by Moment Method*, 2nd edition, IEEE Press, New York, 1993.
7. Choi, S. H., D. W. Seo, and N. H. Myung, "Scattering analysis of open-ended cavity with inner object," *J. of Electromagn. Waves and Appl.*, Vol. 21, No. 12, 1689–1702, 2007.
8. Ruppin, R., "Scattering of electromagnetic radiation by a perfect electromagnetic conductor sphere," *J. of Electromagn. Waves and Appl.*, Vol. 20, No. 12, 1569–1576, 2006.
9. Ho, M., "Scattering of electromagnetic waves from, vibrating perfect surfaces: Simulation using relativistic boundary conditions," *J. of Electromagn. Waves and Appl.*, Vol. 20, No. 4, 425–433, 2006.
10. Lin, C.-J. and J. J. More, "Incomplete cholesky factorizations with limited memory," *SIAM J. Sci. Comput.*, Vol. 21, 24–45, 1999.
11. Fang, H.-R. and P. O. Dianne, "Leary, modified Cholesky algorithms: A catalog with new approaches," *Mathematical Programming*, July 2007.
12. Margenov, S. and P. Popov, "MIC(0) DD preconditioning of FEM elasticity problem on non-structured meshes," *Proceedings of ALGORITMY 2000 Conference on Scientific Computing*, 245–253, 2000.
13. Saad, Y., "ILUT: A dual threshold incomplete LU factorization," *Numer. Linear Algebra Appl.*, Vol. 4, 387–402, 1994.

14. Freund, R. and N. Nachtigal, "A quasi-minimal residual method for non-Hermitian linear systems," *Numer. Math.*, Vol. 60, 315–339, 1991.
15. Van der Vorst, H. A. and J. B. M. Melissen, "A Petrov-Galerkin type method for solving $Ax = b$, where A is symmetric complex," *IEEE Trans. Mag.*, Vol. 26, No. 2, 706–708, 1990.
16. Barrett, R., M. Berry, T. F. Chan, J. Demmel, J. Donato, J. Dongarra, V. Eijkhout, R. Pozo, C. Romine, and H. van der Vorst, *Templates for the Solution of Linear Systems: Building Blocks for Iterative Methods*, 2nd edition, SIAM, Philadelphia, PA, 1994.
17. Saad, Y., *Iterative Methods for Sparse Linear Systems*, 2nd edition, SIAM, Philadelphia, PA, 2003.
18. Saad, Y., "Sparskit: A basic tool kit for sparse matrix computations," Report RIACS-90-20, Research Institute for Advanced Computer Science, NASA Ames Research Center, Moffett Field, CA, 1990.
19. Benzi, M., "Preconditioning techniques for large linear systems: A survey," *J. Comp. Physics*, Vol. 182, 418–477, 2002.
20. Meijerink, J. A. and H. A. van der Vorst, "An iterative solution method for linear equations systems of which the coefficient matrix is a symmetric M-matrix," *Math. Comp.*, Vol. 31, 148–162, 1977.
21. Lee, I., P. Raghavan, and E. G. Ng, "Effective preconditioning through ordering interleaved with incomplete factorization," *Siam J. Matrix Anal. Appl.*, Vol. 27, 1069–1088, 2006.
22. Ng, E. G. and P. Raghavan, "Performance of greedy ordering heuristics for sparse Cholesky factorization," *Siam J. Matrix Anal. Appl.*, Vol. 20, No. 2, 902–914, 1999.
23. Benzi, M., D. B. Szyld, and A. van Duin, "Orderings for incomplete factorization preconditioning of nonsymmetric problems," *SIAM J. Sci. Comput.*, Vol. 20, 1652–1670, 1999.
24. Benzi, M., W. Joubert, and G. Mateescu, "Numerical experiments with parallel orderings for ILU preconditioners," *Electronic Transactions on Numerical Analysis*, Vol. 8, 88–114, 1999.
25. Chan, T. C. and H. A. van der Vorst, "Approximate and incomplete factorizations," Preprint 871, Department of Mathematics, University of Utrecht, The Netherlands, 1994.
26. Zhang, Y., T.-Z. Huang, and X.-P. Liu, "Modified iterative methods for nonnegative matrices and M-matrices linear systems," *Computers and Mathematics with Applications*, Vol. 50, 1587–1602, 2005.