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THE CONDUCTANCE BANDWIDTH OF AN ELECTRICALLY SMALL ANTENNA IN ANTIRESONANT RANGES

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Abstract—Accurate approximations of the conductance and the conductance bandwidth of an electrically small antenna valid in resonant and antiresonant ranges were found. It was shown that the conductance bandwidth of an antenna tuned on maximal power of radiation is inversely proportional to the magnitude of the frequency derivative of the input impedance $|Z'(\omega_{cd})|$ of the antenna at frequency of maximal conductance. That is a generalization of the well known relationship, according to which, the conductance bandwidth of an antenna tuned on resonance in a resonant range is inversely proportional to the magnitude of the frequency derivative of the input reactance of the antenna $|X_0'(\omega_0)|$ at resonant frequency. Obtained approximate formulas display inverse proportionality of the conductance bandwidth to the approximate quality factor of the antenna in resonant and antiresonant ranges.

A differential definition of the fractional conductance bandwidth was formulated, which is convenient for the case of closely spaced resonances of an antenna.

As an example, numerical calculations for oblate spheroidal and spherical antennas in shells with negative permittivity in resonant and antiresonant ranges was used to confirm accuracy of the obtained approximations of the conductance and the conductance bandwidth of an electrically small antenna.

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1. INTRODUCTION

Evaluation of broadbanding potential of an antenna is one of the most important questions. Interconnection of bandwidth with the quality factor, electrical size, and the input impedance has been discussed in a number of papers [1–8]. Inverse proportionality of the conductance bandwidth of an antenna to the frequency derivative of the input reactance of electrically small antenna $|X_0'(\omega_0)|$ at resonant frequency ω_0 in resonant ranges is commonly accepted [1–3]. However, this relation is not valid in antiresonant ranges [1].

Frequency dependence of a flow of energy accepted by an antenna is defined by frequency dependence of the conductance of the antenna. The conductance bandwidth is defined by the input impedance of an antenna [1]. If feed line of an electrically small antenna is electrically small, accepted energy is solely defined by the input impedance of the antenna.

In case of an electrically long feed line, waves travelling through an interface of an antenna with its feed line are partially reflected at the interface [1, 6, 7]. Resonant properties of an antenna are modified by its feed line and matching network [6]. As a result, the fractional matched voltage-standing-wave-ratio (VSWR) bandwidth [1] is two times broader than the conductance bandwidth in resonant range [2, 3], while frequencies of their maximums do not coincide.

Broadening of the VSWR bandwidth is explained by an impact of the reflected wave on the input port of the antenna. An antenna is matched with its feed line only at resonant frequency. In vicinity of the resonant frequency, there is a frequency-dependent increase of voltage on the input port of the antenna conditioned by wave reflection. As a result, the bandwidth of the antenna widens as compared with the conductance bandwidth of the same antenna with an electrically small feed line.

A matching network combining resonant and antiresonant properties can also be used to widen the bandwidth of the antenna [6–8]. Because of electrically small sizes of discussed antennas, a practical matching network has to be electrically small and may be viewed as a part of the antenna [8]. Therefore, notion of the conductance bandwidth is fully defined and reasonable in the quasi-static model of an electrically small antenna.

Derivation of approximate formulas for the conductance bandwidth of an electrically small antenna functional in antiresonant ranges and verification of obtained results by comparison with exact conductance and the quality factor of the oblate spheroidal and spherical antennas in shells with negative permittivity are presented in the paper.

2. APPROXIMATION OF THE CONDUCTANCE AND THE CONDUCTANCE BANDWIDTH OF AN ANTENNA

2.1. Conductance and Inverse Conductance of an Antenna Tuned on Zero Reactance

Power accepted by an antenna tuned on zero reactance at a frequency ω_0 is [1]

$$P_A(\omega) = \frac{1}{2}|U_0(\omega)|^2 G_0(\omega), \tag{1}$$

where $|U_0(\omega)|$ — amplitude of voltage at the input port of the antenna, $G_0(\omega)$ —conductance of the antenna, $G_0(\omega) = Re[1/Z_0(\omega)]$, $Z_0(\omega)$ — impedance of the antenna. Resistance and reactance of an electrically small feed line and a voltage generator is considered small as compared with impedance of the electrically small antenna.

Conductance of an antenna tuned on zero reactance at a frequency ω_0 is

$$G_0(\omega) = \frac{R_0(\omega)}{[R_0(\omega)]^2 + [X_0(\omega)]^2}, \tag{2}$$

where $R_0(\omega)$, $X_0(\omega)$ — resistance and reactance of the antenna. There is a rapidly varying function $X_0^2(\omega)$ in vicinity of the resonant frequency ω_0 in the denominator of the conductance (2). In contrast, the inverse conductance is

$$g_0(\omega) = G_0^{-1}(\omega) = \frac{[R_0(\omega)]^2 + [X_0(\omega)]^2}{R_0(\omega)}. \tag{3}$$

Derivatives of the inverse conductance (3) with respect to frequency at $\omega = \omega_0$ are

$$\begin{aligned} g_0'(\omega_0) &= R_0'(\omega_0), \\ g_0''(\omega_0) &= \frac{R_0(\omega_0) R_0''(\omega_0) + 2[X_0'(\omega_0)]^2}{R_0(\omega_0)}, \end{aligned} \tag{4}$$

where, in contrast with [2, 3], derivatives of resistance is assumed not equal to zero to take into account energy of the antenna in antiresonant ranges.

According to (4), the inverse conductance, similar to the conductance, has no extremum at ω_0 in a general case ($R_0'(\omega_0) \neq 0$). Assuming that the maximum of $G_0(\omega)$ is at the frequency ω_{cd} ($G_0'(\omega_{cd}) = 0$), one can find

$$\omega_{cd} = \omega_0 - \triangle\omega, \tag{5}$$

where $\triangle\omega$ — difference between frequencies of resonance and maximal conductance. One can expand $g_0(\omega)$ in Taylor series in vicinity of the extremum

$$g_0(\omega) = g_0(\omega_{cd}) + \frac{1}{2} g_0''(\omega_{cd})(\omega - \omega_{cd})^2 + O[(\omega - \omega_{cd})^3], \qquad (6)$$

where $g_0(\omega_{cd}) = \frac{[R_0(\omega_{cd})]^2 + [X_0(\omega_{cd})]^2}{R_0(\omega_{cd})}$ in accordance with (3). According to (6), derivatives of the inverse conductance are

$$\begin{aligned} g_0'(\omega) &= g_0''(\omega_{cd})(\omega - \omega_{cd}) + O[(\omega - \omega_{cd})^2], \\ g_0''(\omega) &= g_0''(\omega_{cd}) + O[(\omega - \omega_{cd})]. \end{aligned} \qquad (7)$$

Using (4) and the second Equation (7) in the first Equation (7), one can find for frequency $\omega = \omega_0$

$$\triangle\omega \cong \frac{g_0'(\omega_0)}{g_0''(\omega_0)} = \frac{R_0'(\omega_0) R_0(\omega_0)}{R_0(\omega_0) R_0''(\omega_0) + 2[X_0'(\omega_0)]^2}, \qquad (8)$$

where terms of the first infinitesimal order in (7) were neglected. In contrast to (26) [1], the term $[R_0'(\omega_0)]^2$ is absent in the denominator of (8) that makes (8) more exact in antiresonant ranges, according to simulation results for LCR circuits. In resonant frequency ranges $[X_0'(\omega_0)]^2 \gg |R_0(\omega_0) R_0''(\omega_0)|$, whereas $\triangle\omega$ is [1]

$$\triangle\omega \cong \triangle\omega_X = \frac{R_0'(\omega_0) R_0(\omega_0)}{2[X_0'(\omega_0)]^2}, \qquad (9)$$

where $\triangle\omega_X$ — corresponds to the approximation valid in resonant range.

In antiresonant ranges $R_0(\omega_0)$ is a rapidly varying function. Using a piecewise approximation of $R_0(\omega_0)$ by a power dependency, it is easy to prove that $R_0'^2(\omega_0) \approx |R_0(\omega_0) R_0''(\omega_0)|$. Using the latter approximation in (8), one finds

$$\triangle\omega_Z = \frac{F(\omega_0) R_0(\omega_0) R_0'(\omega_0)}{2[X_0'(\omega_0)]^2 + 2[R_0'(\omega_0)]^2} = \frac{F(\omega_0) R_0(\omega_0) R_0'(\omega_0)}{2|Z_0'(\omega_0)|^2}, \qquad (10)$$

where $F(\omega_0)$ — a limited function of antiresonant conditions. Comparing (8) and (10), one finds

$$F(\omega_0) = \frac{2|Z_0'(\omega_0)|^2}{R_0(\omega_0) R_0''(\omega_0) + 2[X_0'(\omega_0)]^2}. \qquad (11)$$

2.1.1. The Approximated Inverse Conductance and Conductance

Using the second Equation (4) and (11) in the second relationship (7) at frequency $\omega = \omega_0$, one finds approximate value of $g_0''(\omega_{cd})$ expressed by means of resonant parameters

$$g_0''(\omega_{cd}) \cong \frac{2|Z_0'(\omega_0)|^2}{F(\omega_0)\, R_0(\omega_0)}. \tag{12}$$

Using (12) in (6), one finds an approximation for the inverse conductance

$$g_0(\omega) \cong g_0(\omega_{cd}) \left(1 + \left(\frac{|Z_0'(\omega_0)|\,(\omega - \omega_{cd})}{R_0(\omega_0)}\right)^2\right), \tag{13}$$

where $g_0(\omega_{cd}) \approx R_0(\omega_0)/F(\omega_0)$.

According to (13), the conductance bandwidth related to the approximate inverse conductance does not depend on $F(\omega_0)$, yet $F(\omega_0)$ is necessary to calculate $g_0(\omega_{cd})$ and $\triangle\omega_Z$. Using (10) in (13), one finds for $\omega = \omega_0$

$$F(\omega_0) \cong 1 + \frac{1}{4} q^2 F^2(\omega_0), \tag{14}$$

where $q = q(\omega_0) = \frac{R_0'(\omega_0)}{|Z_0'(\omega_0)|}$ — parameter of antiresonance conditions. Magnitude of q is close to zero in resonant ranges, and limited by the unit in antiresonant ranges ($|q| \leq 1$).

According to (14), the range of values of $F(\omega_0)$, presented as a function of $F(q(\omega_0))$, is

$$F(\omega_0) = F(q(\omega_0)) = \begin{cases} 1, & q = 0, \\ 2, & |q| = 1. \end{cases} \tag{15}$$

Using (14) and (15), in case of $|q| \cong 0$, one can find

$$F(q(\omega_0)) = 1 + \frac{1}{4} q^2 + O\left[q^2\right]. \tag{16}$$

Using the geometric series with the first two terms equal to terms in (16) as an approximation of $F(q(\omega_0))$, one finds

$$F(q(\omega_0)) \cong \sum_{n=1}^{\infty} \left(\frac{q^2}{2}\right)^{n-1} = \frac{2}{2 - q^2}, \tag{17}$$

which satisfies (15). However, in antiresonant ranges, $|q| > 0.7$, $F(q) > 1.14$, a more non-linear function than (17) better satisfies definition (11). The corrected $F(\omega_0)$ is

$$F(q(\omega_0)) \cong \frac{2}{2 - q^4}. \tag{18}$$

Using (13) in (3), one finds the approximate conductance of an antenna tuned on resonance at frequency ω_0 in resonant or antiresonant ranges

$$G_{0Z}(\omega) = G_{0Z}(\omega_{cd}) \left[1 + \left(\frac{|Z_0'(\omega_0)|(\omega - \omega_{cd})}{R_0(\omega_0)} \right)^2 \right]^{-1}, \qquad (19)$$

where $G_{0Z}(\omega_{cd}) \approx F(\omega_0)/R_0(\omega_0)$.

The approximate conductance of an antenna tuned on resonance in resonant ranges ($|q| \ll 1$) is

$$G_{0X}(\omega) = G_{0X}(\omega_{cd}) \left[1 + \left(\frac{|X_0'(\omega_0)|(\omega - \omega_{cd})}{R_0(\omega_0)} \right)^2 \right]^{-1}, \qquad (20)$$

where $G_{0X}(\omega_{cd}) \approx 1/R_0(\omega_0)$.

In vicinity of maximal conductance $\omega \cong \omega_{cd}$, one finds an exponential expression of the approximate conductance

$$G_{0EZ}(\omega) = G_{0Z}(\omega_{cd}) \exp \left[- \left(\frac{|Z_0'(\omega_0)|(\omega - \omega_{cd})}{R_0(\omega_0)} \right)^2 \right]. \qquad (21)$$

2.1.2. The Conductance Bandwidth

According to [1], the conductance bandwidth for an antenna tuned on resonance at a frequency ω_0 is defined as the difference between the two frequencies $\omega_\mp$ at which the power accepted by the antenna, excited by a constant value of voltage $U_0(\omega)$, is a given fraction of the power accepted at the frequency ω_0. In contrast to the definition in [1], the conductance bandwidth is defined here in regard to frequency of maximal conductance ω_{cd}, at which the power accepted by an antenna is maximal. If the accepted power at frequencies $\omega_\mp$ is the $(1-\alpha)$ part of the accepted power at ω_{cd}, one finds

$$G_{0Z}(\omega_\mp) = (1-\alpha) G_{0Z}(\omega_{cd}), \qquad (22)$$

where $0 \leq \alpha \leq 1$. Using (19) in (22), the half-width of frequency dependence of the conductance of an antenna tuned on resonance at a frequency ω_0 in resonant or antiresonant ranges is

$$\triangle\omega_{\mp Z} = \omega_+ - \omega_{cd} = \omega_{cd} - \omega_- = \frac{\beta^{0.5} R_0(\omega_0)}{|Z_0'(\omega_0)|}, \qquad (23)$$

where

$$\beta = \frac{\alpha}{1-\alpha} = \frac{G_{0Z}(\omega_{cd}) - G_{0Z}(\omega_\mp)}{G_{0Z}(\omega_\mp)}. \qquad (24)$$

Using (24) at frequency $\omega_{\mp} = \omega_0$, one finds

$$\beta_0 = F(\omega_0) - 1, \tag{25}$$

where β_0 — value of the parameter β corresponding to the conductance at a resonant frequency ω_0. According to (15), (24), and (25), in resonant ranges $F(\omega_0) \approx 1$, $\alpha_0 \approx 0$, while $\omega_0 \cong \omega_{cd}$. In antiresonant ranges, $F(\omega_0) \approx 2$, $\alpha_0 \approx 0.5$, while ω_0 corresponds to a half of the maximal conductance. Therefore, meanings of resonance in resonant and antiresonant ranges are essentially different, despite of zero reactance of the antenna in both cases.

According to (20), the half-width of the conductance bandwidth [1–3] in a resonant frequency range well away from a frequency of antiresonance ($|q| \ll 1$) is

$$\left|\triangle\omega_{\mp X}\right| = \frac{\beta^{0.5} R_0(\omega_0)}{X_0'(\omega_0)}. \tag{26}$$

The fractional conductance bandwidth of an antenna (FBW) in resonant or antiresonant ranges is

$$\mathrm{FBW}cdZ = \frac{2\left|\triangle\omega_{\mp Z}\right|}{\omega_{cd}} = \frac{2\beta^{0.5} R_0(\omega_0)}{\omega_{cd}\left|Z_0'(\omega_0)\right|}, \qquad \beta = \frac{\alpha}{1-\alpha}, \tag{27}$$

where frequency of maximal conductance ω_{cd} is the central frequency of the conductance bandwidth.

The fractional conductance bandwidth of an antenna tuned on resonance in a resonant range well away from an antiresonant frequency in accordance with [1–3] is

$$\mathrm{FBW}cdX = \frac{2\left|\triangle\omega_{\mp X}\right|}{\omega_0} = \frac{2\beta^{0.5} R_0(\omega_0)}{\omega_0\left|X_0'(\omega_0)\right|}. \tag{28}$$

In accordance with [1], the fractional conductance bandwidth for a resonant range (28) is not valid for an antenna tuned on resonance in an antiresonant frequency range.

The fractional conductance bandwidth based on the exponential approximation $G_{0EZ}(\omega)$ (21) is

$$\mathrm{FBW}cdE = \frac{2R_0'(\omega_0)}{\left|Z_0'(\omega_0)\right|}(-\ln(1-\alpha))^{0.5} \approx \frac{2\beta^{0.5} R_0(\omega_0)}{\left|Z_0'(\omega_0)\right|}, \tag{29}$$

where $\beta \ll 1$. The exponential approximation of the conductance (21) in case of $\beta \ll 1$ gives the same fractional conductance bandwidth as in the case of (19).

Frequency dependence of the inverse conductance is approximated by the polynomial of the second order (13). Therefore, it is possible to formulate a simple differential definition of the fractional conductance bandwidth. Using (6) for $\omega = \omega_{\mp}$, one can find

$$g_0(\omega_{\mp}) = g_0(\omega_{cd}) + \frac{1}{2} g_0''(\omega_{cd})(\omega_{\mp} - \omega_{cd})^2 + O[(\omega_{\mp} - \omega_{cd})^3]. \quad (30)$$

Using (22), one finds

$$g_0(\omega_{\mp}) = \frac{g_0(\omega_{cd})}{(1-\alpha)}. \quad (31)$$

According to (30) and (31), the fractional conductance bandwidth of an antenna in a differential form is

$$\text{FBW}cdD = \frac{2|\triangle\omega_{\mp}|}{\omega_{cd}} \cong \frac{2\beta^{0.5}}{\omega_{cd}} \left(\frac{2g_0(\omega_{cd})}{g_{0r}''(\omega_{cd})} \right)^{0.5}. \quad (32)$$

The definition (32) is directly applied for an antenna tuned on resonance in a resonant range ($\omega_0 \cong \omega_{cd}$). The differential definition of the fractional conductance bandwidth is equivalent to the definition of the conductance bandwidth (22), (31). The differential definition can be used in order to calculate the 3 dB conductance bandwidth ($\beta = 1$) in case of closely spaced resonances.

2.1.3. Conductance Bandwidth and the Quality Factor

The quality factor of an antenna tuned on zero reactance at frequency ω_0 is [1, 4, 5]

$$Q(\omega_0) = \frac{\omega|W(\omega_0)|}{P_A(\omega_0)}, \quad (33)$$

where $W(\omega_0)$ — time-averaged energy of non-propagating electromagnetic field. It is worth to notice that the quality factor of an untuned antenna is lower up to two times [4, 5] because of deficiency in energy of an electric or magnetic field. Calculations of the quality factor with (33) are based on rather complex numerical evaluations of $W(\omega_0)$ [2–5, 11]. Convenient form of the exact quality factor for numerical calculation is [1]

$$Q(\omega_0) = \frac{\omega_0}{2R_0(\omega_0)} \left| X_0'(\omega_0) - \frac{4}{|I_0|^2} [W_{\mathcal{L}}(\omega_0) + W_{\mathcal{R}}(\omega_0)] \right|, \quad (34)$$

where $W_{\mathcal{L}}(\omega_0)$ — time-averaged energy of losses of electromagnetic field, $W_{\mathcal{R}}(\omega_0)$ — energy of antenna radiation. The approximate quality factor of an antenna tuned at frequency ω_0 is [1]

$$Q_Z(\omega_0) \frac{\omega_0}{2R(\omega_0)} \sqrt{[R'(\omega_0)]^2 + \left[X'(\omega_0) + \left| \frac{X(\omega_0)}{\omega_0} \right| \right]^2}, \quad (35)$$

where primes stand for frequency derivatives.

The lower bound on the exact quality factor of an antenna tuned on resonance is [4]

$$Q_{lb} = \eta_r \left(\frac{1}{k^3 a^3} + \frac{1}{ka} \right), \tag{36}$$

where a — radius of a minimal imaginary sphere enclosing the antenna, k — the wave number of radiation, η_r — antenna radiation efficiency. The quality factor of an antenna normed on Q_{lb} shows efficiency of using of volume occupied by the antenna.

Comparing the approximate quality factor (35) with the approximate fractional conductance bandwidth (27), one finds

$$\mathrm{FBW}cdZ \cong \frac{\omega_0 \beta^{0.5}}{\omega_{cd} Q_Z(\omega_0)}. \tag{37}$$

Inverse proportionality of the bandwidth of an electrically small antenna to the approximate quality factor was used to define a measurable quality factor of an antenna [3, 6]. There is no essential difference between values of the approximate and the exact quality factors in resonant ranges. However, combining resonance and antiresonance of an antenna and matching network [6] or close resonances and antiresonance of an antenna [8], it is possible to obtain a wider bandwidth of antennas than one can expect, calculating the lower bound and the exact quality factor. Therefore, antiresonant ranges of antennas with small quality factors are especially interesting for simulation.

3. NUMERICAL RESULTS

3.1. Evaluation of the Approximation of the Conductance of an Electrically Small Antenna Tuned on Resonance

Electrically small antennas can be modeled by a simple LCR circuit. The model circuit with resonant and antiresonant properties comprises of a parallel LCR circuit in series with capacitance or inductance [1] in comparison with more complex models of antennas of bigger electrical size [12].

For example, electrically small dipole antennas in shells with negative permittivity [9, 10] can be used to evaluate exactness of obtained approximations of the conductance and the conductance bandwidth. Structure of the oblate spheroidal antenna in the shell is shown in Fig. 1. The antenna consists from two oblate semi-ellipses with the axis of rotation Z. Semi-ellipses are separated from each other by the narrow gap with sides coordinates — η', η', ($\eta' = 0.05$). The

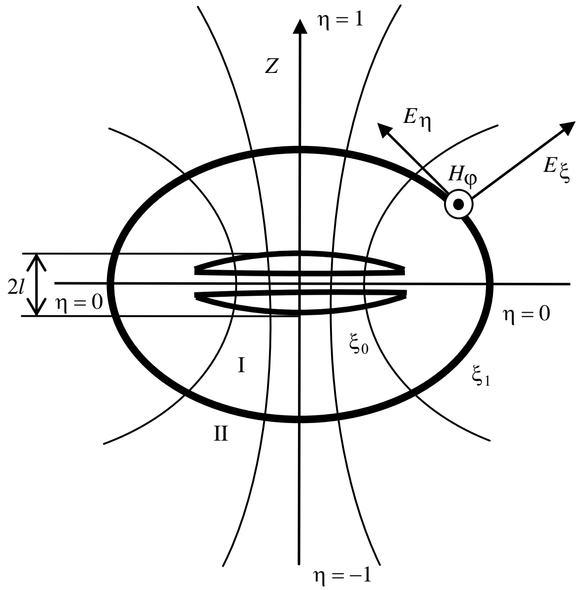

Figure 1. Oblate spheroidal coordinate system ξ, η, φ (cross-section by plane $\varphi = \text{const}$, $\text{const} + \pi$). Coordinate surfaces: Confocal oblate ellipsoids of rotation (in relation to the axis Z), one-sheet hyperboloids, and semi-planes accordingly; $\xi = \xi_0$ — surface of the antenna divided by a narrow feed gap, $2l$ — height of the antenna, $\xi = \xi_1$ — an outer boundary of the shell [9].

antenna is covered by a confocal layer of a substance (metamaterial, plasma) with Drude's permittivity. The height of the antenna is $2l$.

Numerical simulation of the antennas in confocal shells has been done by solving of a boundary-value problem in oblate spheroidal coordinate system. Results of numerical modeling for the oblate spheroidal and spherical antennas with maximal electrical size 0.19 (Electrical size of the oblate spheroidal antenna in direction of the height is $kl = 0.019$.) were obtained by solving of Maxwell's system of material and field equations for the lowest oblate spheroidal mode TM_{11} [9].

The parallel LCR circuit of those antennas providing antiresonant properties is comprised of inductive reactance of the shell with negative permittivity and capacitive reactance of free space around the shell [9]. Because of inductive reactance of the shells in series with the parallel LCR circuit, a resonant frequency ω_0 of those antennas is higher than an antiresonant frequency $\omega_0 > \omega_a$.

Resonant and antiresonant properties of the spherical and oblate spheroidal antennas are functions of negative permittivity of their shells [9]. Simulated dependences of the parameter of antiresonant conditions q for the spherical and oblate spheroidal antennas from magnitude of the negative real part of relative permittivity $|\varepsilon'|$ in Fig. 2 display resonant and antiresonant ranges of the antennas. Resonance is tuned in ranges of comparatively small magnitudes of negative permittivity where parameter q in Fig. 2 varies from 0 to $-1\,(X_0'(\omega_0) \geq 0)$. Parameter q varies from -1 to $+1$ in vicinity of the central antiresonant permittivity of each antenna where numerical value of q is equal to zero. Variation of q from -1 to $+1$ in Fig. 2 corresponds to antiresonant ranges $X_0'(\omega_0) < 0$. Ranges of permittivity corresponding to decrease of q from $+1$ relate to untuned antennas with inductive reactance.

Frequency dependencies of the exact conductance (2) of the oblate spheroidal antenna tuned on resonance ($X_0(\omega_0) = 0$, $X_0'(\omega_0) > 0$) in resonant ($q = -0.01$) and antiresonant ($q = -0.99$) ranges are shown in Fig. 3 and Fig. 4 together with corresponding graphs of the approximate conductance calculated by means of relations (19)–(21).

Figure 3 compares the conductance $G_0(\omega/\omega_0)$ (2) of the oblate spheroidal antenna in resonant range with approximations of the conductance $G_{0Z}(\omega/\omega_0)$ (19), $G_{0X}(\omega/\omega_0)$ (20), and $G_{0EZ}(\omega/\omega_0)$ (21) plotted as functions of $\frac{\omega}{\omega_0}$, ($\omega_0 = 6 \times 10^9$ rad/s). Relative permittivity of the shell $\varepsilon = -1 - 0.004i$ provides resonance frequency well away from the antiresonant range according to Fig. 2.

As it is shown in Fig. 3, curves of the conductance (2) and approximations of conductance (19), (20) coincide with accuracy better

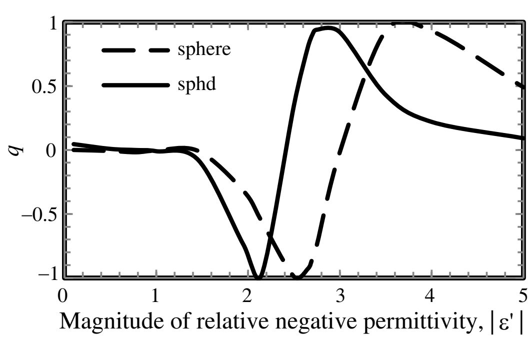

Figure 2. Parameter of antiresonant conditions $q = \frac{R_0'(\omega_0)}{|Z_0'(\omega_0)|}$ for the oblate spheroidal and spherical antennas in the shells as functions of magnitude of the real part of relative negative permittivity $|\varepsilon'|$.

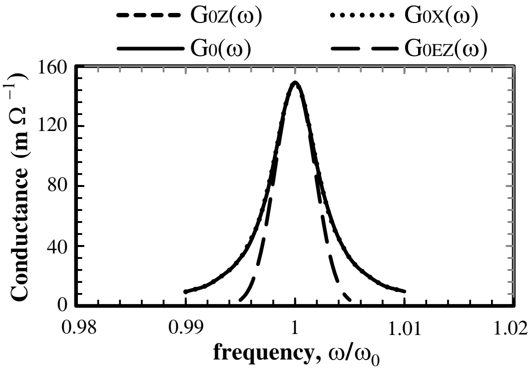

Figure 3. $G_0(\omega)$ — conductance of the oblate spheroidal antenna in the shell with relative permittivity $\varepsilon = -1-0.004i$ tuned on resonance at frequency ω_0 (2), $G_{0z}(\omega)$ — approximation of $G_0(\omega)$ with parameter $Z_0'(\omega_0)$ (19), $G_{0x}(\omega)$ — approximation of $G_0(\omega)$ with $X_0'(\omega_0)$ (20), $G_{0EZ}(\omega)$ — approximation of $G_0(\omega)$ with $Z_0'(\omega_0)$ (21).

than 0.5% ($\beta < 1$). Approximations based on $X_0'(\omega_0)$ and $Z_0'(\omega_0)$ are close throughout the entire frequency range. Found difference between frequencies ω_0 and ω_{cd} corresponds to value of $\triangle\omega$ (8) with accuracy 0.3%. Values of the approximate maximal conductance (19) and the exact maximal conductance (2) coincide with accuracy better than 0.2%. The fractional conductance bandwidths calculated with $Z_0'(\omega_0)$ (27) and $X_0'(\omega_0)$ (28) are equal to the exact fractional bandwidth with accuracy better than 0.3% ($\beta < 1$).

Figure 4 compares the conductance $G_0(\omega/\omega_0)$ (2) with approximations of the conductance $G_{0Z}(\omega/\omega_0)$ (19), $G_{0X}(\omega/\omega_0)$ (20), and $G_{0EZ}(\omega/\omega_0)$ (21) for the oblate spheroidal antenna tuned on resonance in antiresonant range ($q \cong -0.99$). Relative permittivity of the shell $\varepsilon = -2.05 - 0.004i$ provides the highest degree of convergence of resonance and antiresonance ($\omega_0 \approx \omega_a$, $X_0(\omega_0) \approx 0$, $X_0'(\omega_0) \approx 0$). The conductance has maximum at frequency $\omega_{cd} \approx 1.0139\omega_0$ in Fig. 4. As a result, the antenna is tuned on resonance at an edge of the conductance bandwidth.

As it is shown in Fig. 4, curves of the exact conductance (2) and approximations of the conductance (19) ($\beta < 1$), (21) ($\beta < 0.1$) based on $Z_0'(\omega_0)$ coincide with accuracy better than 5%. Approximation based on $X_0'(\omega_0)$ (20) gives a noticeably wider curve, which corresponds to a considerably overestimated bandwidth. Difference between ω_0 and ω_{cd} corresponds to value of $\triangle\omega$ (8) with accuracy better than 3%. The fractional conductance bandwidth based on $Z_0'(\omega_0)$ (27) ($\beta < 1$)

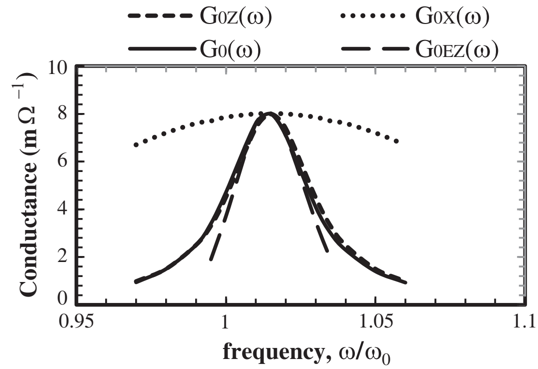

Figure 4. $G_0(\omega)$ — conductance of the of the oblate spheroidal antenna in the shell with relative permittivity $\varepsilon = -2.05 - 0.004i$ tuned on resonance at frequency ω_0 (2), $G_{0Z}(\omega)$ — approximation of $G_0(\omega)$ with parameter $Z_0'(\omega_0)$ (19), $G_{0X}(\omega)$ — approximation of $G_0(\omega)$ with $X_0'(\omega_0)$ (20), $G_{0EZ}(\omega)$ — exponential approximation of $G_0(\omega)$ with $Z_0'(\omega_0)$ (21).

and (29) ($\beta < 0.1$) coincides with the exact fractional bandwidth with accuracy better than 0.2%. Accuracy of calculation of the maximal conductance with approximation (19) is about 10%.

3.2. Approximation of FBW and the Quality Factor of Antennas Tuned on Maximal Power of Radiation

The conductance bandwidth can be calculated by means of the differential definition (32). That allows presenting of the conductance bandwidth in resonant and antiresonant ranges of the antennas in one graph. The same frequency ($\omega_{cd} = 6 \times 10^9$ rad/s) was employed for calculations of the conditions of maximal conductance being tuned by the size of the antenna. The size of the shell of the antenna was being defined by condition of maximal power of radiation for a given value of permittivity of the shell [9]. Conditions of maximal power of radiation are close to conditions of maximal power accepted by the antenna in case of low losses in the shell medium. Those conditions are also close to resonant conditions in resonant and antiresonant ranges $X_0'(\omega_0) \geq 0$ that provides almost zero reactance of antennas.

Condition of zero reactance in antiresonant range $X_0'(\omega_0) < 0$ corresponding to antiresonance was not employed because antiresonant conditions do not provide the conductance bandwidth. Convergence of resonance and antiresonance at frequency $\omega_0 = \omega_a$, $X_0(\omega_0) = 0$,

$X_0'(\omega_0) = 0$) means that resonant and antiresonant properties are always related. In fact, resonant frequency is connected with both antiresonant and maximal conductance frequencies (5). A frequency of maximal conductance ω_{cd} is always situated out of antiresonant range $X_0'(\omega_0) < 0$. Therefore, the quality factor of an antenna tuned on antiresonance does not correspond to the conductance bandwidth of the antenna in contrast with [13].

Analysis of results of [1, 6, 9, 10, 14–16] showed that antennas tuned on resonance in antiresonant ranges $X_0'(\omega_0) \geq 0$ provide comparatively low values of the quality factor corresponding to comparatively wide bandwidth. That can be explained by optimal use of the antennas volume.

Therefore, the fractional conductance bandwidth of the antennas tuned on maximal power of radiation was calculated by means of the differential definition (32) and relation (37) connected with the quality factor. Dependences of the fractional conductance bandwidths ($\beta = 1$) of the oblate spheroidal and spherical antennas are presented in Fig. 5 as functions of magnitude of the real part of relative permittivity $|\varepsilon'|$ of the shells. According with Fig. 2 and Fig. 5, values of the fractional conductance bandwidths of the antennas calculated by means of (32) and (37) are almost equal in resonant and antiresonant $X_0'(\omega_0) \geq 0$ ranges of permittivity. In antiresonant ranges $X_0'(\omega_0) < 0$, values of the fractional conductance bandwidth estimated by the use of approximate quality factor (37) in some degree exceed values calculated by means of the differential definition. However, inverse values of the approximate quality factor are better approximating the fractional conductance bandwidth than corresponding values of the exact quality factor. Therefore, the approximate quality factor is an especially useful parameter in antiresonant ranges in accordance with [6, 8].

Notion of resonance is not applicable in antiresonant ranges $X_0'(\omega_0) < 0$, whereas $\omega_{cd} \approx \omega_0$ in resonant ranges. Results of simulation of the conductance of the antennas tuned on maximal conductance showed that frequency of maximal conductance can be used instead of resonant frequency in relation (19). Therefore, the fractional conductance bandwidth of an electrically small antenna connected with power supply via an electrically small feed line in the form valid throughout resonant and antiresonant ranges is

$$\mathrm{FBW}cdZ\left(\omega_{cd}\right) \cong \frac{2\beta^{0.5} R\left(\omega_{cd}\right)}{\omega_{cd}\left|Z'\left(\omega_{cd}\right)\right|} \cong \frac{\beta^{0.5}}{Q_Z\left(\omega_{cd}\right)}, \qquad (\beta \ll 1). \qquad (38)$$

The differential definition stays valid in the range of comparatively high magnitudes of negative permittivity corresponding to the decrease of q from $+1$ in Fig. 2. At the same time, relations connected with the approximate quality factor (37) and (38) become invalid.

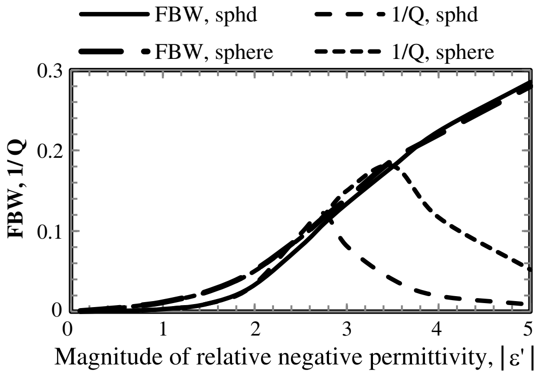

Figure 5. Fractional conductance bandwidth FBW_{cd} (FBW (32), $1/Q$ (37), (38)) ($\boldsymbol{\beta} = \mathbf{1}$) of the oblate spheroidal (sphd) and spherical (sphere) antennas in the shells as functions of magnitude of the negative real part of relative permittivity $|\varepsilon'|$.

4. CONCLUSION

The conductance and the fractional conductance bandwidth of an electrically small antenna tuned on resonance are efficiently approximated by (19), (27) in resonant and antiresonant ranges $X_0'(\omega_0) \geq 0$. The fractional conductance bandwidth of an antenna tuned on maximal conductance (38) is inversely proportional to the magnitude of the frequency derivative of the input impedance $|Z'(\omega_{cd})|$ at the frequency of maximal conductance throughout resonant and antiresonant ranges. That is a generalization of the well known relationship for the conductance bandwidth of an electrically small antenna, according to which the conductance bandwidth is inversely proportional to the magnitude of the frequency derivative of the input reactance of the antenna $|X_0'(\omega_0)|$ at resonant frequency in resonant ranges [1–3].

Obtained relations correspond to dependency of both the VSWR [1] and Bode-Fano [6] bandwidths from $1/|Z_0'(\omega_0)|$ in resonant and antiresonant $X_0'(\omega_0) \geq$ ranges. At the same time, the VSWR [1] and Bode-Fano [6] bandwidths are not fully compatible with notion of an electrically small antenna because of essential role of feed line with wave character of electromagnetic field. As a result, the conductance bandwidth of an electrically small antenna with frequency of maximal conductance ω_{cd} is two times narrower than the VSWR bandwidth with the maximum at resonant frequency ω_0.

It is worth to notice that calculation of reflection parameters cannot provide information about bandwidth of an antenna in case of quasi-static electromagnetic field of an antenna with electrically small feed line.

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