Progress In Electromagnetics Research B, Vol. 73, 1–16, 2017

Axially-Symmetric TM-Waves Diffraction by Sphere-Conical Cavity

Dozyslav B. Kuryliak, Zinoviy T. Nazarchuk, and Oksana B. Trishchuk*

Abstract—The problem of axially-symmetric TM-wave diffraction from the perfectly conducting sphere-conical cavity is analysed. The cavity is formed by a semi-infinite truncated cone; one of the sectors of this cone is covered by the spherical diaphragm. The problem is formulated in terms of scalar potential for spherical coordinate system as a mixed boundary problem for Helmholtz equation. The unknown scalar potential of the diffracted field is sought as expansion in series of eigenfunctions for each region, formed by the sphere-conical cavity. Using the mode matching technique and orthogonality properties of the eigenfunctions, the solution to the problem is reduced to an infinite set of linear algebraic equations (ISLAE). The main part of asymptotic of ISLAE matrix elements determined for large indexes identifies the convolution type operator. The corresponding inverse operator is represented in an explicit form. The convolution type operator and corresponding inverse operator are applied to reduce the problem to the ISLAE of the second kind. This procedure determines the new analytical regularization method for the solution of wave diffraction problems for the sphere-conical cavity. The unknown expansion coefficients, which are determined from the ISLAE by the reduction, belong to the space of sequences that allow obtaining the solution which satisfies all the necessary conditions with the given accuracy. The particular cases, such as transition from sphere-conical cavity to the open hemispherical resonator, as well as the low frequency approximation, are analysed. The numerically obtained results are applied to the analysis of TM-waves radiation through the circular hole in the cavity.

1. INTRODUCTION

Cavities connected with the open space through the slot/hole in their walls are widely applied to communication and measuring systems [1, 2], particles accelerators [3], nano-technologies and materials diagnostics [4, 5] to achieve the necessary radiation characteristics in the wide frequency band; the millimeter and sub-millimeter wavelengths are also included into this range. The theoretical methods for analysis of the radiating properties of the slot/hole were first elaborated in the mid-1940s [6–10]. These methods, however, considered only the small holes and offered approximate formulations of the boundary problems. Further development of theoretical methods to study the radiation from/through the hole is based on mathematically correct formulation of the boundary value problems. The direct numerical method [11], integral equations [12] and hybrid methods [13, 14] were developed for the analysis of the slot/hole radiation problems. Unfortunately these powerful tools require long computation time for determination of the diffraction characteristics and verification of the results, particularly for resonance regimes, quasi-optical diapasons, as well as for the near field calculations. For simple geometries, the mode matching technique is usually applied to electrodynamics analysis. But in most cases, the application of this technique is formal and also requires validation. Therefore, it is important to expand the number of models allowed for mathematically rigorous electrodynamics analysis. Correct accounting

Received 9 December 2016, Accepted 13 January 2017, Scheduled 2 February 2017
* Corresponding author: Oksana B. Trishchuk (trishchuk@ipm.lviv.ua).
The authors are with the Department of Physical Bases of Materials Diagnostics, Karpenko Physico-Mechanical Institute, National Academy of Sciences of Ukraine, 5 Naukova St., Lviv 79060, Ukraine.

of field singularities near the edges and tips, as well as deduction of the rules for reduction of the ISLAE, are included in this analysis. The Wiener-Hopf method and Fourier transform have been used in [15–17] for rigorous analysis of the radiation from the flat and cylindrical cavities. The rigorous analysis of radiation from the spherical volumes with the circular hole has been studied by using the analytical regularization technique in [18, 19]. Spherical dielectric resonators with metal semi-infinite and finite conical inserts have been considered in [20–23]. Quantum-scale sphere-conical resonators have been analysed in [24]. Closed sphere-conical resonators have been examined in [25, 26]. A new approximate technique for the analysis of electromagnetic waves radiation through the slots in resonators is developed in [27]. Experimental research of resonators containing conical structures has been offered in [28].

This paper discusses a new open resonant structure which allows for the rigorous electrodynamics analysis. The cavity is formed by a semi-infinite truncated circular cone, in which one of the sectors is covered by spherical diaphragm. In other words, the sphere-conical resonator with a circular hole is considered. For various geometrical parameters, such a structure allows simulation of a number of radiated elements: probes [29], hemispherical cavities [30], tapered and metal-coated optical fiber tips [31], etc. We apply the field expansion in the series of eigenfunctions and mode matching techniques to reduce the problem to the ISLAE. The key idea, which we develop here, consists in separating the singular operator from the initial ISLAE and derivation of the operator which allows for inversion of this singular operator analytically. This allows for reduction of the problem to the second kind of the ISLAE and to justify this reduction procedure rigorously. The proposed approach is called the method of analytical regularization or semi-inversion method. This approach was used earlier for the studies of the diffraction of acoustic and electromagnetic waves from conical, bi-conical, and wedge structures [21–23, 32–40].

2. STATEMENT OF THE PROBLEM

Let us consider a perfectly conducting semi-infinite truncated circular cone with the infinitely thin wall. One of the two conical sectors is closed by perfectly conducting spherical diaphragm, and it forms an open sphere-conical resonator (Fig. 1(a)). In the spherical coordinates (r, θ, φ), this structure can be expressed as

$$\mathbf{Q} = \left\{ (r, \theta, \varphi) | r \in \left\{ \begin{array}{ll} (c_1, \infty), & \theta = \gamma + 0 \\ (c_1, c), & \theta = \gamma - 0 \end{array} \right\}, \theta = \gamma \right\} \bigcup_{\varphi \in [0, 2\pi)} \{ (r, \theta, \varphi) | r = c, \theta \in [0, \gamma] \}, \qquad (1)$$

where γ is the spherical aperture angle; c_1 is the radial coordinates of the rib of the hole; c is the radius of the spherical diaphragm.

Let $\mathbf{Q}$ be excited by TM-wave, produced by radial electric dipole which is located outside of the structure, on the axis of symmetry at $r = r_0 < c_1$ and E_r, E_θ, $H_\varphi \neq 0$. Then the total field radiated from resonance cavity will also have the property of axial symmetry. Time factor $e^{-i\omega t}$ is suppressed throughout this paper. The problem is to find the distribution of the field components established in the presence of cone $\mathbf{Q}$.

The problem is formulated in terms of Debye scalar potential $U = U(r, \theta)$ that satisfies the Helmholtz equation. The components of the field are expressed as follows

$$E_r = -\frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial U}{\partial \theta} \right), \quad E_\theta = \frac{1}{r} \frac{\partial^2}{\partial r \partial \theta} (rU), \quad H_\varphi = ikZ^{-1} \frac{\partial U}{\partial \theta}. \qquad (2)$$

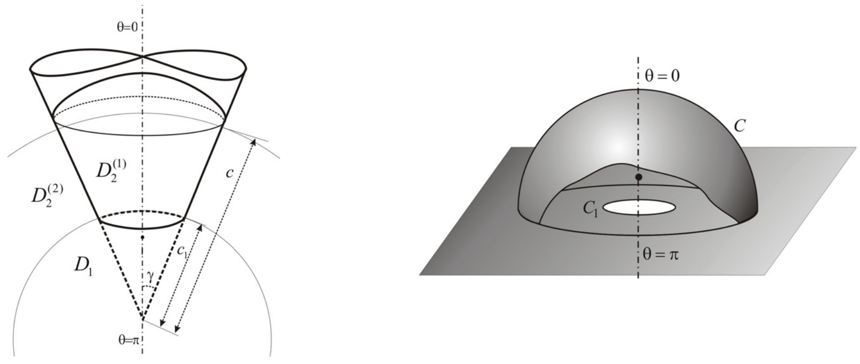
Here k is the wave number ($k = k' + ik'' = \omega\sqrt{\varepsilon\mu}$, k', $k'' > 0$, $i = \sqrt{-1}$; ε and μ are the permittivity and permeability of the continuum), and $Z = \sqrt{\mu/\varepsilon}$ is the wave resistance.

In view of Eq. (2), the diffraction problem is reduced to the boundary value problem proceeding from the Helmholtz equation

$$\Delta U + k^2 U = 0 \qquad (3)$$

with boundary conditions

$$\frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial U^t}{\partial \theta} \right) = 0, \quad \text{if} \quad \left\{ \begin{array}{ll} r \in (c_1, \infty) & \text{for} \quad \theta = \gamma + 0 \\ r \in (c_1, & \text{for} \quad \theta = \gamma - 0 \end{array} \right., \qquad (4)$$



(a) (b)

Figure 1. Geometrical schemes, (a) open sphere-conical resonator, (b) hemispherical resonator with a hole.

$$\frac{1}{r}\frac{\partial^2(rU^t)}{\partial r\partial\theta} = 0, \quad \text{if} \quad r = c,\ \theta \in\ [0,\gamma]. \tag{5}$$

Here $U^t(r,\theta) = U(r,\theta) + U^i(r,\theta)$ is the potential of the total field; $U = U(r,\theta)$, $U^{(i)}(r,\theta)$ are diffracted field and incident field illuminated by the radial dipole respectively;

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right). \tag{6}$$

We search for the solution of the mixed boundary value problem in Eqs. (3)–(5) in the class of functions that satisfy the Silver-Muller radiation condition in form

$$\lim_{r\to\infty} r\left[\vec{i}_r \times \vec{H} + Z^{-1}\vec{E}\right] = 0, \tag{7}$$

as well as energy bounded condition as

$$\int\limits_V \left(\varepsilon|\vec{E}|^2 + \mu|\vec{H}|^2\right) dv < \infty. \tag{8}$$

Here V is any finite volume of integration.

Next, we represent the potential of the dipole field radiation in spherical coordinates as

$$U^i(r,\theta) = \frac{1}{\sqrt{sr sr_0}}\sum_{n=1}^{\infty} A_n^{(0)} P_{z_n-1/2}(\cos\theta)\left\{\begin{array}{c} I_{z_n}(sr_0)K_{z_n}(sr), r \geq r_0 \\ K_{z_n}(sr_0)I_{z_n}(sr), r \leq r_0 \end{array}\right\}, \tag{9}$$

where $A_n^{(0)} = 2z_n p_0 Z/r_0$, $p_0 = I_r^{(e)}h$ is the dipole moment, $I_r^{(e)}$ the electric current, h the dipole length; $s = -ik$; $I_{z_n}(\cdot)$ and $K_{z_n}(\cdot)$ are the modified Bessel and Macdonald functions respectively, $P_{z_n-1/2}(\cdot)$ is Legendre function with subscripts $z_n = n + 1/2$, $n = 1, 2, 3, \ldots$.

3. SOLUTION OF THE PROBLEM

To solve the boundary value problem in Eqs. (3)–(5), we split the solution space into the partial domains

$$\begin{array}{c} D_1 : \{r \in (0, c_1);\ \theta \in\ [0,\pi];\ \varphi \in [0, 2\pi)\}, \\ D_2^{(1)} : \{r \in (c_1, c);\ \theta \in\ [0,\gamma);\ \varphi \in\ [0, 2\pi)\},\ \ D_2^{(2)} : \{r \in (c_1, \infty);\ \theta \in (\gamma, \pi];\ \varphi \in\ [0, 2\pi)\}. \end{array} \tag{10}$$

Assuming that the source of the incident field is located in D_1 domain, we expand the desired potential in eigenfunctions of the Helmholtz equation as applied to partial domains in Eq. (10) as follows

$$U(r,\theta)=\begin{cases} U^i(r,\theta)+\dfrac{1}{\sqrt{\rho}}\displaystyle\sum_{n=1}^{\infty}\bar{x}_n^{(1)}P_{z_n-1/2}(\cos\theta)\frac{I_{z_n}(\rho)}{I_{z_n}(\rho_1)}, \quad \rho,\theta\in D_1; \\ \dfrac{1}{\sqrt{\rho}}\displaystyle\sum_{p=1}^{\infty}P_{\nu_p-1/2}(\cos\theta)\left[y_p^{(2,1;1)}\frac{K_{\nu_p}(\rho)}{K_{\nu_p}(\rho_1)}+y_p^{(2,1;2)}\frac{I_{\nu_p}(\rho)}{I_{\nu_p}(\rho_1)}\right], \quad \rho,\theta\in D_2^{(1)}; \\ \dfrac{1}{\sqrt{\rho}}\displaystyle\sum_{k=1}^{\infty}y_k^{(2,2)}P_{\mu_k-1/2}(-\cos\theta)\frac{K_{\mu_k}(\rho)}{K_{\mu_k}(\rho_1)}, \quad \rho,\theta\in D_2^{(2)}. \end{cases} \tag{11}$$

Here $\bar{x}_n^{(1)}$, $y_p^{(2,1;1)}$, $y_p^{(2,1;2)}$, $y_k^{(2,2)}$ are unknown expansion coefficients; $\rho = sr$, $\rho_1 = sc_1$, $\{\nu_p\}_{p=1}^{\infty}$ and $\{\mu_k\}_{k=1}^{\infty}$ are the growing sequences of the positive roots of the transcendental equations

$$P_{\nu-1/2}(\cos\gamma)=0, \tag{12a}$$

$$P_{\mu-1/2}(-\cos\gamma)=0. \tag{12b}$$

Expression (11) corresponds to the total and diffracted field potential in domains D_2 and D_1, respectively.

The unknown coefficients will be found in the class of sequences which provide the absolute and uniform convergence of series in Eq. (11), as well as their first-order derivatives with respect to r and θ variables. The second-order derivatives of series in Eq. (11) involved E_r and E_θ components, which are normal to the edges. These components admit the integrable singularity: E_r, $E_\theta = O(\widehat{\rho}^{\,-1/2})$ for $\widehat{\rho}\to 0$, where $\widehat{\rho}$ is the distance to the edge of the cone in the local coordinates.

The representation in Eq. (11) satisfies the radiation condition at infinity and the boundary conditions on the conical surface.

Using the boundary condition in Eq. (5) and representation (11), we arrive at the equation as

$$\begin{aligned} &\frac{1}{2\sqrt{\rho_c}}\sum_{p=1}^{\infty}P^1_{\nu_p-1/2}(\cos\theta)\left[y_p^{(2,1;1)}\frac{K_{\nu_p}(\rho_c)}{K_{\nu_p}(\rho_1)}+y_p^{(2,1;2)}\frac{I_{\nu_p}(\rho_c)}{I_{\nu_p}(\rho_1)}\right]+ \\ &+\sqrt{\rho_c}\sum_{p=1}^{\infty}P^1_{\nu_p-1/2}(\cos\theta)\left[y_p^{(2,1;1)}\frac{K'_{\nu_p}(\rho_c)}{K_{\nu_p}(\rho_1)}+y_p^{(2,1;2)}\frac{I'_{\nu_p}(\rho_c)}{I_{\nu_p}(\rho_1)}\right]=0, \quad \gamma\le\theta\le\pi. \end{aligned} \tag{13}$$

Here $\rho_c = sc$; the prime indicates the derivation of the modified Bessel and Macdonald functions with respect to the argument, $P^1_{z_n-1/2}(\cdot)$ is associated Legendre function of the first order defined as $P^1_{\nu-1/2}(\pm\cos\theta) = \pm d/d\theta[P_{\nu-1/2}(\pm\cos\theta)]$ [41].

From Equation (13) we get the correlation between the coefficients as

$$y_p^{(2,1;2)} = y_p^{(2,1;1)}\Upsilon_{\nu_p}(\rho_1,\rho_c), \tag{14}$$

where $p = 1, 2, 3, \ldots$;

$$\Upsilon_{\nu_p}(\rho_1,\rho_c) = -\frac{I_{\nu_p}(\rho_1)K_{\nu_p}(\rho_c)}{K_{\nu_p}(\rho_1)I_{\nu_p}(\rho_c)}\left[\frac{1+2\rho_c K'_{\nu_p}(\rho_c)/K_{\nu_p}(\rho_c)}{1+2\rho_c I'_{\nu_p}(\rho_c)/I_{\nu_p}(\rho_c)}\right]. \tag{15}$$

The unknown coefficients in Eq. (11) can be found using the continuity conditions of total field tangential components on the spherical surface $r = c_1$ containing the circular rib of the hole. These conditions lead to the series equations for finding the unknowns. In order to take into account the singularity of $E^t_\theta(r,\theta)$ at the conical edge, we present these equations by way of

$$\lim_{N\to\infty}\sum_{n=1}^{N}P^1_{z_n-1/2}(\cos\theta)\left[\bar{x}^{(1)}_n+A_nK_{z_n}(\rho_1)I_{z_n}(\rho_0)\right]=$$
$$=\begin{cases}\lim\limits_{P\to\infty}\sum\limits_{p=1}^{P}y^{(2,1;1)}_pP^1_{\nu_p-1/2}(\cos\theta)\left[1+\Upsilon_{\nu_p}(\rho_1,\rho_c)\right], & \theta\in\ [0,\gamma),\\ -\lim\limits_{K\to\infty}\sum\limits_{k=1}^{K}y^{(2,2)}_kP^1_{\mu_k-1/2}(-\cos\theta), & \theta\in(\gamma,\pi],\end{cases}\tag{16}$$

$$\lim_{N\to\infty}\sum_{n=1}^{N}P^1_{z_n-1/2}(\cos\theta)\left[\bar{x}^{(1)}_n\frac{I'_{z_n}(\rho_1)}{I_{z_n}(\rho_1)}+A_nK'_{z_n}(\rho_1)I_{z_n}(\rho_0)\right]=$$
$$=\begin{cases}\lim\limits_{P\to\infty}\sum\limits_{p=1}^{P}y^{(2,1;1)}_pP^1_{\nu_p-1/2}(\cos\theta)\left[\frac{K'_{\nu_p}(\rho_1)}{K_{\nu_p}(\rho_1)}+\frac{I'_{\nu_p}(\rho_1)}{I_{\nu_p}(\rho_1)}\Upsilon_{\nu_p}(\rho_1,\rho_c)\right], & \theta\in\ [0,\gamma),\\ -\lim\limits_{K\to\infty}\sum\limits_{k=1}^{K}y^{(2,2)}_kP^1_{\mu_k-1/2}(-\cos\theta)\frac{K'_{\mu_k}(\rho_1)}{K_{\mu_k}(\rho_1)}, & \theta\in(\gamma,\pi],\end{cases}\tag{17}$$

where $\rho_0=sr_0$, $A_n=A^{(0)}_n/\sqrt{\rho_0}$.

In order to reduce series Equations (16), (17) to the infinite system of linear algebraic equations (ISLAE), we use the property of orthogonality of Legendre functions, which leads to

$$P^1_{z_n-1/2}(\cos\theta)=q(z_n,\gamma)\lim_{P(K)\to\infty}\sum_{p=1}^{P(K)}\frac{\eta_p\alpha^{\pm}(\eta_p,\gamma)}{\eta_p^2-z_n^2}P^1_{\eta_p-1/2}(\pm\cos\theta).\tag{18}$$

Here upper and lower signs correspond to the angle regions $\theta\in[0,\gamma)$ with $\eta_p=\nu_p$ and $\theta\in(\gamma,\pi]$ with $\eta_p=\mu_p$ respectively;

$$q(z_n,\gamma)=(z_n^2-1/4)P_{z_n-1/2}(\cos\gamma),\tag{19a}$$
$$\alpha^{\pm}(\eta_p,\gamma)=\mp2\left\{(\eta_p^2-1/4)\partial P_{\eta_p-1/2}(\pm\cos\gamma)/\partial\eta\right\}^{-1}.\tag{19b}$$

Then, we prove the following theorem on the convergence of the series in Eq. (18):

Theorem. For any γ, which belongs to $0\le\gamma\le\pi$, the upper and lower series on the right-hand part of Eq. (18) are uniformly convergent to the function $P^1_{z_n-1/2}(-\cos\theta)/q(z_n,\gamma)$ for any $\theta\in[0,\gamma]$ and $\theta\in\ [\gamma,\pi]$ respectively.

Proof. Let us consider the integral

$$J^{\pm}_n(\theta)=\frac{1}{2\pi i}\int\limits_{C_R}\frac{tP^1_{t-1/2}(\pm\cos\theta)dt}{(t^2-z_n^2)(t^2-1/4)P_{t-1/2}(\pm\cos\gamma)}.\tag{20}$$

Here C_R is the circular integration path in complex plane t; the points $t=0$ and $t=R$ are the center and radius of this circle, respectively; C_R outline encompasses the simple poles of the integrand at $t=\pm z_n$ and $t=\pm\nu_k$ ($k=1,\,2,\,3,\,\ldots$). For $|t|\to\infty$ the integrand as a function of t tends to zero not slower than t^{-2}, therefore $J^{\pm}_n(\theta)\to0$ if $R\to\infty$. Then, applying the residues theorem, we arrive at the statement of the theorem.

Let us substitute series in Equation (18) into Equations (16) and (17). Next, limiting the finite number of unknowns and excluding $y^{(2,1;1)}_p$, $y^{(2,2)}_k$, we come to the finite system of linear algebraic equations as follows

$$\sum_{n=1}^{N}x^{(1)}_n\left\{\frac{\rho_1W[K_{\nu_p},I_{z_n}]_{\rho_1}}{(\nu_p^2-z_n^2)K_{\nu_p}(\rho_1)I_{z_n}(\rho_1)}+\frac{\rho_1W[I_{\nu_p},I_{z_n}]_{\rho_1}\Upsilon_{\nu_p}(\rho_1,\rho_c)}{(\nu_p^2-z_n^2)I_{\nu_p}(\rho_1)I_{z_n}(\rho_1)}\right\}=f^{(1)}_{\nu_p},\quad p=1,\ldots,P,\tag{21a}$$
$$\sum_{n=1}^{N}x^{(1)}_n\frac{\rho_1W[K_{\mu_k},I_{z_n}]_{\rho_1}}{(\mu_k^2-z_n^2)I_{z_n}(\rho_1)K_{\mu_k}(\rho_1)}=f^{(1)}_{\mu_k},\quad k=1,\ldots,K.\tag{21b}$$

Here $x_n^{(1)} = \bar{x}_n^{(1)} q(z_n, \gamma)$; $W[f_\nu, \varphi_\mu]_x = f_\nu(x)\varphi'_\mu(x) - f'_\nu(x)\varphi_\mu(x)$; $N = P + K$,

$$f_{\nu_p}^{(1)} = \sum_{n=1}^{N} A_n q(z_n, \gamma) K_{z_n}(\rho_1) I_{z_n}(\rho_0) \times \\ \times \left\{ \frac{\rho_1 W[K_{z_n}, K_{\nu_p}]_{\rho_1}}{(\nu_p^2 - z_n^2) K_{z_n}(\rho_1) K_{\nu_p}(\rho_1)} + \frac{\rho_1 W[K_{z_n}, I_{\nu_p}]_{\rho_2} \Upsilon_{\nu_p}(\rho_1, \rho_c)}{(\nu_p^2 - z_n^2) K_{z_n}(\rho_1) I_{\nu_p}(\rho_1)} \right\}, \tag{22a}$$

$$f_{\mu_k}^{(1)} = \sum_{n=1}^{N} A_n q(z_n, \gamma) K_{z_n}(\rho_1) I_{z_n}(\rho_0) \frac{\rho_1 W[K_{z_n}, K_{\mu_k}]_{\rho_1}}{(\mu_k^2 - z_n^2) K_{z_n}(\rho_1) K_{\mu_k}(\rho_1)}. \tag{22b}$$

We introduce this limitation to provide the correct transition from Equation (21) to the ISLAE, which solution satisfies the Meixner condition at the conical edge. For this purpose, we introduce a growing sequence of roots $\{\nu_k\}_{k=1}^{\infty}$, $\{\mu_p\}_{p=1}^{\infty}$ of transcendental Equation (12) as

$$\{\xi_p\}_{p=1}^{\infty} = \{\nu_p\}_{p=1}^{\infty} \cup \{\mu_k\}_{k=1}^{\infty}. \tag{23}$$

Next, in Equation (21) we pass to the limit $P, K, N \to \infty$ $(N = P + K)$ and arrange the ISLAE according to the sequence (23) as

$$(A_{11} + B_{11}) X^{(1)} = F_1. \tag{24}$$

Here $X_1 : \{x_n^{(1)}\}_{n=1}^{\infty}$ is the unknown vector, $A_{11} : \{a_{pn}^{(1,1)}\}_{p,n=1}^{\infty}$, $B_{11} : \{b_{pn}^{(1,1)}\}_{p,n=1}^{\infty}$ are infinite matrixes with the elements:

$$a_{pn}^{(1,1)} = \frac{\rho_1 W[K_{\xi_p}, I_{z_n}]_{\rho_1}}{(\xi_p^2 - z_n^2) K_{\xi_p}(\rho_1) I_{z_n}(\rho_1)}; \tag{25}$$

$$b_{pn}^{(1,1)} = \begin{cases} \frac{\rho_1 W[I_{\xi_p}, I_{z_n}]_{\rho_1} \Upsilon_{\xi_p}(\rho_1, \rho_c)}{(\xi_p^2 - z_n^2) I_{\xi_p}(\rho_1) I_{z_n}(\rho_1)}, & \text{if } \xi_p \in \{\nu_p\}_{p=1}^{\infty}, \\ 0, \text{ if } \xi_p \in \{\mu_k\}_{k=1}^{\infty}, \end{cases} \tag{26}$$

$F_1 : \{f_p^{(1)}\}_{p=1}^{\infty}$ is the known vector; $\{f_p^{(1)}\}_{p=1}^{\infty} \equiv \{f_{\xi_p}^{(1)}\}_{p=1}^{\infty} = \{f_{\nu_p}^{(1)}\}_{p=1}^{\infty} \cup \{f_{\mu_k}^{(1)}\}_{k=1}^{\infty}$, $l = 1, 2$ is the sequence formed by the elements $f_{\nu_p}^{(1)}$, $f_{\mu_k}^{(1)}$ (see Equation (22)) that are placed according to the condition in Eq. (23).

Next, we apply the analytical regularisation procedure for reducing of the ISLAE (24) to the ISLAE of the second kind.

4. REGULARISATION OF THE ISLAE

Taking into account the asymptotic properties of the modified Bessel and Macdonald functions for large indices, we find that

$$a_{pn}^{(1,1)} = \frac{1}{\xi_p - z_n} + \begin{cases} O\left(\frac{1}{\xi_p z_n (\xi_p - z_n)}\right), & \xi_p, \ z_n \gg |sc_1|; \\ O\left((sc_1/2)^2\right), & |sc_1| \to 0. \end{cases} \tag{27}$$

Asymptotic behavior of the matrix elements $b_{pn}^{(1,1)}$ from Eq. (26) in the case is $\xi_p, z_n \gg |sc_1|$ or $|sc_1| \to 0$, $|sc| \to 0$, $c > c_1$ looks as

$$b_{pn}^{(1,1)} = O\left(\frac{(c_1/c)^{2\xi_p}}{\xi_p + z_n}\right). \tag{28}$$

Let us introduce the operator formed with the main parts of the asymptotic expression (27) and the corresponding inverse operator as [21, 35, 36]

$$A : \left\{ a_{pn} = \frac{1}{\xi_p - z_n} \right\}_{p,n=1}^{\infty}, \tag{29}$$

$$A^{-1} : \left\{ \tau_{kp} = \left\langle [M_-^{-1}(\xi_p)]' \, M'_-(z_k)(z_k - \xi_p) \right\rangle^{-1} \right\}_{k,p=1}^{\infty}. \tag{30}$$

Here, the product of operators in Eqs. (29), (30) represents the identity matrix I: $A^{-1}A = I$; $M'_-(\eta_k) = d/d\eta\,[M_-(\eta)]_{\eta=\eta_k}$; $M_-(\nu)$ is determined from the factorization of the even meromorphic function $M(\nu)$, which is regular in the strip $\Pi : \{|Re\nu| < 1/2\}$ with simple zeroes and poles at $\nu = \pm z_k$, $\nu = \pm\xi_j$ that are located on the real axis out of the Π;

$$M(\nu) = M_+(\nu) \cdot M_-(\nu) = \frac{\cos\pi\nu}{(\nu^2 - 1/4)P_{\nu-1/2}(\cos\gamma)P_{\nu-1/2}(-\cos\gamma)}, \tag{31}$$

where $M_+(\nu)$, $M_-(\nu)$ are split functions, regular in the right ($Re\nu > -1/2$) and in the left ($Re\nu < 1/2$) half-planes respectively; $M(\nu) = O(\nu^{-1})$ and $M_+(\nu) = M_-(-\nu) = O(\nu^{-1/2})$, if $|\nu| \to \infty$ in the regularity region;

$$M_-\,(\nu) = \left\{B_0\,(1/2-\nu)\,\Gamma\,(1/2-\nu)\,e^{-\nu\chi}\prod_{p=1}^{\infty}(1-\nu/\xi_p)\,e^{\nu/\xi_p}\right\}^{-1}. \tag{32}$$

Here

$$\begin{aligned}
B_0 &= -i\pi^{-1/2}\left[P_{-1/2}(\cos\gamma)P_{-1/2}(-\cos\gamma)\right]^{1/2},\\
\chi &= \frac{\gamma}{\pi}\ln\frac{\gamma}{\pi} + \frac{\pi-\gamma}{\pi}\ln\frac{\pi-\gamma}{\pi} - \psi\,(3/4) - S(\gamma) - S(\pi-\gamma),\\
S(\gamma) &= \sum_{n=1}^{\infty}\left[\frac{\gamma}{\pi\,(n-1/4)} - \frac{1}{\nu_n}\right],\quad S(\pi-\gamma) = \sum_{n=1}^{\infty}\left[\frac{\pi-\gamma}{\pi\,(n-1/4)} - \frac{1}{\mu_n}\right],
\end{aligned}$$

$\Gamma(\cdot)$ and $\psi(\tau)$ are gamma function and its logarithmic derivative, respectively. The formulas for effective calculation of the matrix elements in Eq. (30) are presented in **Appendix A**.

Next, we formulate original diffraction problem via the ISLAE of the second kind as follows

$$X^{(1)} = A^{-1}[(A - A_{11}) - B_{11}]X^{(1)} + A^{-1}F_1. \tag{33}$$

The ISLAE (33) is valid for $\gamma \neq \pi/2$.

The technique described above is elaborated in [33–35, 39, 40] and called the analytical regularization procedure. The ISLAE (33) admits the solution in the class of sequences $b(\sigma) : \{||X|| = \sup_n |x_n|,\ \lim_{n\to\infty}|x_n n^\sigma| \to 0\}$ for $0 \le \sigma < 1/2$. This fulfils all the necessary conditions for the existence of a unique solution of the ISLAE (33), including the Meixner condition on the edges.

The proof of these statements is based on the use of asymptotic estimates for matrix elements in Eqs. (25), (26), and (30), which are given in the expressions (27), (28) and by the formula

$$\tau_{kp} = O\left(\frac{\xi_p^{-1/2}z_k^{1/2}}{z_k - \xi_p}\right),\ \text{if}\ k, p \to \infty.$$

We represent the other unknown coefficients through the solution of Eq. (33) by way of

$$\begin{aligned}
y_p^{(2,1,1)} &= \frac{\nu_p\alpha^+(\nu_p,\gamma)}{1+\Upsilon_{\nu_p}(\rho_1,\rho_c)}\sum_{n=1}^{\infty}\frac{1}{\nu_p^2 - z_n^2}\left[x_n^{(1)} + A_n q(z_n,\gamma)I_{z_n}(\rho_0)K_{z_n}(\rho_1)\right],\\
y_p^{(2,1;2)} &= y_p^{(2,1;1)}\Upsilon_{\nu_p}(\rho_1,\rho_c),\\
y_k^{(2,2)} &= -\mu_k\alpha^-(\mu_k,\gamma)\sum_{n=1}^{\infty}\frac{1}{\mu_k^2 - z_n^2}\left[x_n^{(1)} + A_n q(z_n,\gamma)I_{z_n}(\rho_0)K_{z_n}(\rho_1)\right],\\
\bar{x}_n^{(1)} &= x_n^{(1)}/q(z_n,\gamma).
\end{aligned} \tag{34}$$

Taking into account the correlations in Eqs. (2), (11) and (34), we get the definitive expressions for field representation anywhere in spherical and conical regions.

5. TRANSITION TO THE HEMISPHERICAL CAVITY ($\gamma = \pi/2$)

For this particular case, the wave diffraction problem is reduced to the ISLAE (33) using the same procedure as in the general case. Here instead of using Equation (12), we use the transcendental equation $P_{\nu-1/2}(0) = 0$.

Taking into account the representation of the Legendre function for $\gamma = \pi/2$ as

$$P_{\nu-1/2}(0) = \frac{\sqrt{\pi}}{\Gamma(\nu/2+3/4)\,\Gamma(-\nu/2+3/4)},$$

we derive from Eq. (31) that

$$M(\nu) = M_+(\nu)\cdot M_-(\nu) = \frac{\Gamma^2(\nu/2+3/4)\,\Gamma^2(-\nu/2+3/4)}{\Gamma(\nu+1/2)\,\Gamma(-\nu+1/2)\,(\nu^2-1/4)}. \tag{35}$$

The simple zeros and poles of the meromorphic function in Eq. (35) are given by

$$z_n = \pm(2n+1/2), \quad \xi_p = \pm(2p-1/2), \quad p,n = \overline{1,\infty}; \tag{36}$$

$$M_\pm(\nu) = \frac{i2^{\pm\nu}\Gamma^2(\pm\nu/2+3/4)}{\Gamma(\pm\nu+1/2)(\pm\nu+1/2)}, \tag{37}$$

$M_+(\nu)$, $M_-(\nu)$ are split functions regular in overlap semi-planes $Re\nu > -1/2$, $Re\nu < 1/2$; $M_+(\nu) = M_-(-\nu) = O(\nu^{-1/2})$, if $|\nu| \to \infty$ in the regularity regions.

In this particular case, we reduce our problem to Equation (24) using the positive indices given by the expression (36), and then derive Equation (33), where the regularisation operators (29), (30) are formed using expressions (36), (37). The formulas for effective calculation of the matrix elements in Eq. (30), if $\gamma = \pi/2$, are presented in **Appendix B**.

6. RADIATION THROUGH THE SMALL CIRCULAR HOLE

Let us rewrite the basic ISLAE (33) by the way of

$$x_k^{(1)} = \sum_{q=1}^{\infty}\tau_{kq}\sum_{n=1}^{\infty}(a_{qn}-a_{qn}^{(11)})x_n^{(1)} - \sum_{q=1}^{\infty}\tau_{kq}\sum_{n=1}^{\infty}b_{qn}^{(11)}x_n^{(1)} + \sum_{q=1}^{\infty}\tau_{kq}f_q^{(1)}, \quad k = 1,2,3,\ldots \tag{38}$$

Let us simplify Equation (38). For this purpose we take into account the small dimensions of the hole ($|sc_1/2| \ll 1$). Thus, we apply the appropriate asymptotic expressions for modified Bessel and Macdonald functions [41] to estimate the known coefficients in Equation (38) and, neglecting the terms of order $|\rho_1/2|^2$ in the first double series, we immediately derive the approximate equation as

$$x_k^{(1)} + \sum_{q=1}^{\infty}\tau_{kq}\kappa_{\xi_q}(\rho_1,\rho_c)y_q = \sum_{q=1}^{\infty}\tau_{kq}f_q^{(1)}, \quad k = 1,2,3,\ldots. \tag{39}$$

Here

$$y_q = \sum_{n=1}^{\infty}\frac{x_n^{(1)}}{\xi_q+z_n}, \tag{40}$$

$$\kappa_{\xi_q}(\rho_1,\rho_c) = \begin{cases} \frac{2(\rho_1/2)^{2\xi_q}}{\Gamma(\xi_q)\Gamma(\xi_q+1)}\left[\frac{K_{\xi_q}(\rho_c)+2\rho_c K'_{\xi_q}(\rho_c)}{I_{\xi_q}(\rho_c)+2\rho_c I'_{\xi_q}(\rho_c)}\right], & \text{if } \xi_q \in \{\nu_p\}_{p=1}^{\infty}, \\ 0, \text{ if } \xi_q \in \{\mu_k\}_{k=1}^{\infty}. \end{cases} \tag{41}$$

$$f_{\xi_p}^{(1)} = -\sum_{n=1}^{\infty}A_n q(z_n,\gamma)\frac{(\rho_0/\rho_1)^{z_n}}{2z_n}\begin{cases} \frac{1}{\xi_p+z_n} + \frac{\kappa_{\xi_p}(\rho_1,\rho_c)}{\xi_p-z_n}, \text{ if } \xi_p \in \{\nu_p\}_{p=1}^{\infty}; \\ \frac{1}{\xi_p+z_n}, \text{ if } \xi_p \in \{\mu_p\}_{p=1}^{\infty}. \end{cases} \tag{42}$$

Let us introduce $\bar\rho_c = -i\omega_{\nu_p j}c\sqrt{\varepsilon\mu}$ $(p,j = 1,2,3,\ldots)$, where $\omega_{\nu_p j}$ is the real resonant frequency ($Im\omega_{\nu_p j} = 0$) of closed sphere-conical resonator, which corresponds to the resonant $TM_{\nu_p 0 j}$-mode and is determined from the solution of the transcendental equation as

$$I_{\nu_p}(\bar\rho_c) + 2\bar\rho_c I'_{\nu_p}(\bar\rho_c) = 0. \tag{43}$$

Let $\rho_c = \bar{\rho}_c + \Delta\rho_c$, ($\Delta\rho_c \equiv -i\Delta\omega_{\nu_p j} c\sqrt{\varepsilon\mu}$, $\Delta\omega_{\nu_p j} = Re\Delta\omega_{\nu_p j} + iIm\Delta\omega_{\nu_p j}$) and $|\Delta\rho_c| \ll 1$. Under these conditions, the further simplification of the ISLAE (39) leads to

$$x_k^{(1)} + \tau_{kp}\kappa_{\xi_p}(\rho_1, \rho_c) y_p = \tau_{kp} f_p^{(1)}, \quad \xi_p \in \{\nu_p\}_{p=1}^{\infty}; \quad k = 1, 2, 3, \ldots \quad . \tag{44}$$

The explicit solution of the ISLAE (44) looks as (see **Appendix C**).

$$x_k^{(1)} = \frac{\tau_{kp}\ [a_{\xi_p}^{+} + a_{\xi_p}^{-}\kappa_{\xi_p}(\rho_1, \rho_c)]}{1 + \frac{\kappa_{\xi_p}(\rho_1,\rho_c)}{2\xi_p\ [M_-^{-1}(\xi_p)]' M_+(\xi_p)}}, \quad k = 1, 2, 3, \ldots . \tag{45}$$

Here we assume that parameter ρ_c is very close to resonant value $\bar{\rho}_c$. The last one satisfies the transcendental Equation (43) for any selected indices ξ_p and j, with $\xi_p \in \{\nu_p\}_{p=1}^{\infty}$, $j = 1, 2, 3, \ldots$;

$$a_{\xi_p}^{\pm} = -\sum_{n=1}^{\infty} A_n q(z_n, \gamma) \frac{(\rho_0/\rho_1)^{z_n}}{2z_n(\xi_p \pm z_n)}.$$

Equating the denominator of the formula (45) to zero and assuming that $\rho_1 = -i\omega_{\nu_p j} c_1\sqrt{\varepsilon\mu}$, $|\rho_1/2| \ll 1$, we arrive at the expression for determination of $\Delta\rho_c = \Delta\rho_c^* \equiv -i\Delta\omega_{\nu_p j}^* c\sqrt{\varepsilon\mu}$ that gives the perturbation $\Delta\omega_{\nu_p j}^*$ of the resonant frequency $\omega_{\nu_p j}$ as

$$\Delta\omega_{\nu_p j}^* = \left(\frac{k_{\nu_p j} c_1}{2}\right)^{2\nu_p} \Phi(\nu_p). \tag{46}$$

Here $k_{\nu_p j} = \omega_{\nu_p j}\sqrt{\varepsilon\mu}$,

$$\Phi(\nu_p) = \frac{e^{-i\pi(\nu_p+1/2)}}{c\sqrt{\varepsilon\mu} M_+(\nu_p)[M_-^{-1}(\nu_p)]'\Gamma^2(\nu_p+1)} \left[\frac{K_{\nu_p}(\bar{\rho}_c) + 2\bar{\rho}_c K'_{\nu_p}(\bar{\rho}_c)}{\frac{d}{dx}[I_{\nu_p}(x) + 2xI'_{\nu_p}(x)]_{x=\bar{\rho}_c}}\right]. \tag{47}$$

This perturbation is caused by cutting of the conical vertex and appearance of the small circular hole in the sphere-conical resonator. From the correlations in Eqs. (46), (47), it follows that $\Delta\omega_{\nu_p j}^*$ is a complex value and depends on the truncated dimensionless radius ($k_{\nu_p j} c_1$), opening angle (γ), and resonant parameter of the closed sphere-conical resonator ($\bar{\rho}_c$). The problem of determination of the eigen frequencies for the two resonators connected through the small hole, using the different approximate techniques, was solved earlier in [6, 42, 43].

7. NUMERICAL CALCULATION

All characteristics of the scattered field are calculated by reduction of the ISLAE (33). The order of reduction has been chosen from the condition $N = |sc_1| + q$ with $q = (4 \div 10)$. Based on the solution of finite system of linear algebraic equations, we analyze the far-field characteristics for the sphere-conical structure **Q** with different geometrical parameters.

Let us introduce the far-field pattern $D(\theta) = \lim_{r\to\infty} |krH_\varphi(\theta)e^{-ikr}|$, where H_φ is the total magnetic field in region $D_2^{(2)}$. The presented curves here show the diffractive characteristics of the sphere-conical structure **Q** excited axially symmetrically by the radial electric dipole in region $D_2^{(2)}$. For calculation we use the source field representation given in Equation (9) with $p_0 k = 1/(4\pi)$ [A] and $Z = 1$ [Om].

Next, we numerically analyze the radiation of the TM-waves from the circular hole of the sphere-conical resonator at frequencies corresponding to the resonant frequencies of the closed resonator with the same parameters γ and c. Taking into account that the resonant frequencies/radiuses of the closed perfectly conducting sphere-conical resonator are positive and real, we determine them from the solution of Equation (43). For each value of ν_p, this equation allows obtaining the dimensionless parameter $kc = kc_{\nu_p j}$, where $c_{\nu_p j}$ is the resonance spherical radius, which corresponds to the resonant $TM_{\nu_p 0j}$-mode for closed sphere-conical volume. The examples of the resonant parameters for the closed sphere-conical resonator with the opening angle $\gamma = 50°$ are given in Table 1.

Here we are interested in the conditions for effective TM-modes radiation from the cavity through the hole. For analysis of the open sphere-conical cavity we compute the modulus of the magnetic

Table 1. Resonance parameters $kc_{\nu_p j}$ for closed sphere-conical resonator if $\gamma = 50°$.

ν_p \ $kc_{\nu_p j}$	$j = 1$	$j = 2$	$j = 3$	$j = 4$
$\nu_1 = 2.74004$	4.1558	7.7536	11.0408	14.2583
$\nu_2 = 6.31862$	8.0171	12.1750	15.7100	19.0845
$\nu_3 = 9.91202$	11.8250	16.3943	20.1492	23.6760

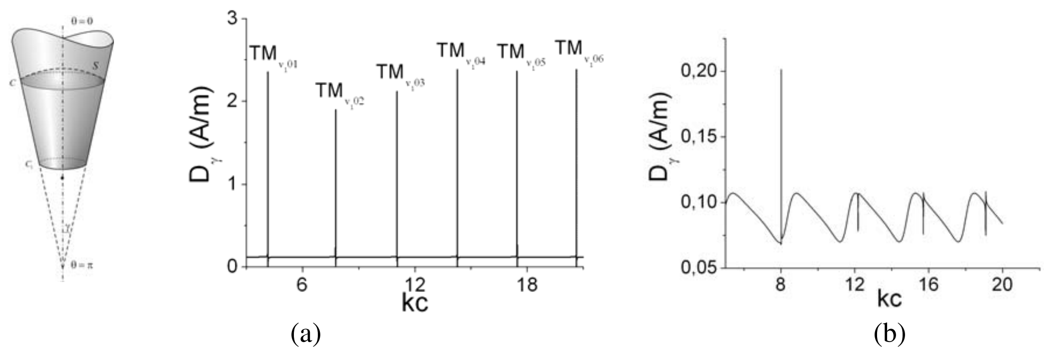


Figure 2. Dependences $D_\gamma \equiv D(\gamma + 0)$ on the parameter kc for $\gamma = 50°$ and $kr_0 = 0.5$; (a) $kc_1 = 1$; (b) $kc_1 = 4.5$.

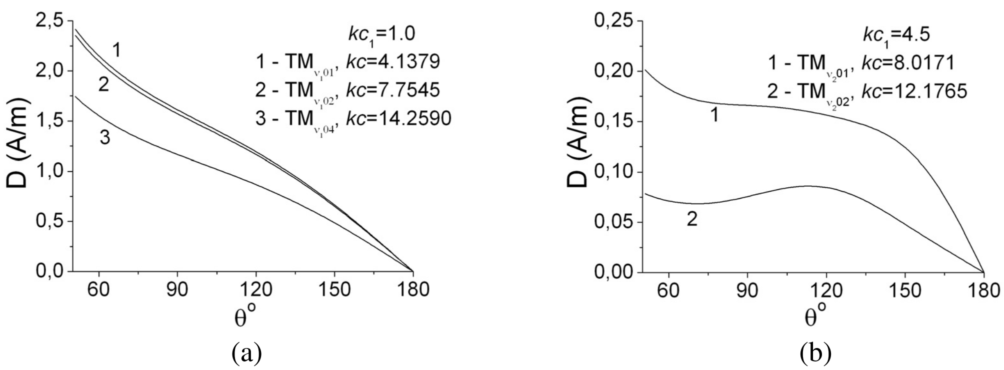
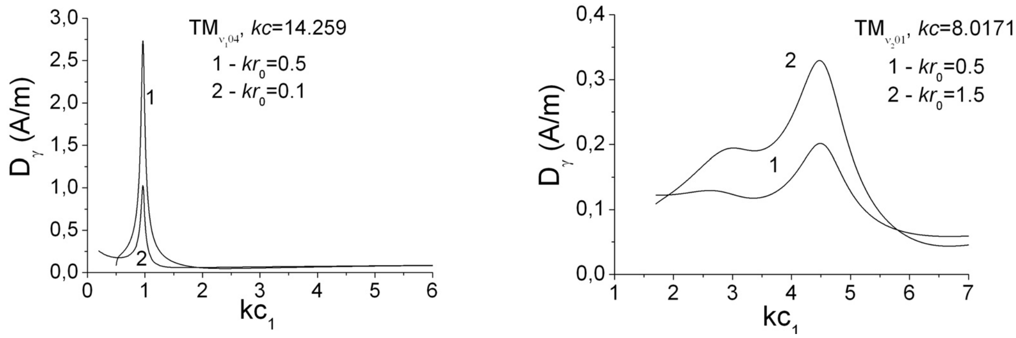


Figure 3. Dependencies of the far field on the type of resonant mode excitation in the sphere-conical cavity for $\gamma = 50°$, $kr_0 = 0.5$; kc corresponds to: (a) $TM_{\nu_1 0j}$-resonant modes; (b) $TM_{\nu_2 0j}$-resonant modes.

field component at the conical surface $\theta = \gamma + 0$ and $r \to \infty$ as the series of Legendre functions $P^1_{\mu_k - 1/2}(-\cos\theta)$.

Let us consider the sphere-conical volume $\mathbf{Q}$ (see Fig. 1(a)), where the cavity's hole is located near the virtual conical vertex and the dipole is placed on the conical axis outside of the cavity area. Fig. 2(a) shows the dependence $D_\gamma \equiv D(\gamma + 0)$ on kc for aperture angle $\gamma = 50°$ and the circular hole parameter $kc_1 = 1.0$. The sharp peaks on this figure are plotted with the step $\Delta(kc) = 0.0001$. We observe these peaks for $kc \to kc_{\nu_1 j}$ ($j = 1 \div 6$). This means that $TM_{\nu_1 0j}$-resonant modes, excited in the corresponding closed resonator, radiate effectively through the small hole (see Table 1).

As follows from Fig. 2(a) the resonant modes for higher indices ν_p ($p > 1$) that are excited in the closed sphere-conical resonator are radiated weakly through the hole. So, the sphere-conical volume $\mathbf{Q}$ with the small hole near the vertex radiates $TM_{\nu_1 0j}$-waves effectively and works as a mode's filter. The detailed analysis shows that the radiation efficiency through the circular hole depends essentially on the radius of the hole and dipole location. When the hole's radius kc_1 grows (see Fig. 2(b); $kc_1 = 4.5$), we



(a) (b)

Figure 4. Dependencies D_γ on the hole radius kc_1 for $\gamma = 50°$ and the different resonant modes excitation in the open cavity and source location kr_0; kc corresponds to: (a) $TM_{\nu_1 04}$ — the resonant mode; (b) $TM_{\nu_2 01}$ — the resonant mode.

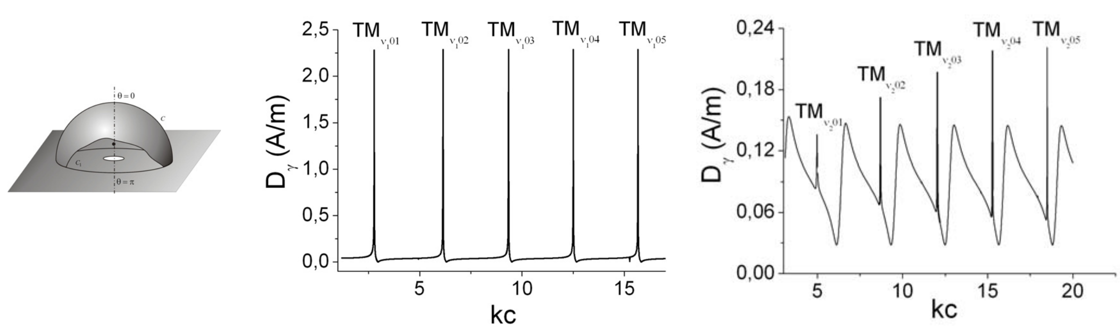
observe slight oscillations, as well as sharp peaks of the value D_γ. We also observe maxima of these slight oscillations for kc that are somewhat shifted from the resonant radiuses $kc_{\nu_1 j}$ of the closed resonator. The sharp peaks in Fig. 2(b) correspond to radiation through the slot of the resonant $TM_{\nu_2 0j}$-waves with $kc \to kc_{\nu_2 j}$ ($j = 1, 2, 3\ldots$, see Table 1). We see in this figure that only $TM_{\nu_2 01}$-mode is radiated effectively from the open cavity.

Also, we plot the far-field pattern dependencies on the type of resonant mode excitation in the sphere-conical cavity **Q** (see Fig. 3). As follows from Fig. 3(a) the excitation of $TM_{\nu_1 0j}$-resonant waves in the sphere-conical cavity forms the far field distribution that weakly depends on the resonant index j. This can be explained by dominant influences of the first mode, excited in the region $D_2^{(2)}$. Nevertheless, we observe such dependence for higher modes $TM_{\nu_2 0j}$ (see Fig. 3(b)) because they are radiated from the wider hole, and the scattered modes transformation by the conical edge becomes more effective.

To study the radiation properties of the open sphere-conical cavity we analyze the dependencies of

Table 2. Resonance parameters $kc_{\nu_p j}$ for closed sphere-conical resonator with $\gamma = 91°$.

ν_n \ $kc_{\nu_n j}$	$j = 1$	$j = 2$	$j = 3$	$j = 4$
$\nu_1 = 1.48273$	2.7240	6.0933	9.2920	12.4607
$\nu_2 = 3.46116$	4.9309	8.6728	12.0118	15.2601
$\nu_3 = 5.43931$	7.0750	11.1254	14.5925	17.9282



(a) (b)

Figure 5. Dependences D_γ on the parameter kc for $\gamma = 91°$ and $kr_0 = 0.5$; (a) $kc_1 = 1$; (b) $kc_1 = 3$.

the far field on the radius of the hole. We plot the curves for different positions of the dipole and different resonant modes excited in the sphere-conical volume (see Fig. 4). As follows from the comparison of the curves in Figs. 4(a), (b) the parameter kr_0 significantly influences the radiation intensity in the area of the main maximum.

Next, we analyze the TM-wave radiation through the circular hole in the hemispherical resonator. We obtain the open hemispherical resonator (see Fig. 1(b)) from the sphere-conical one (see Fig. 1(a)), if $\gamma \to 90°$. For further analysis we model the hemispherical resonator using the sphere-conical one with $\gamma = 91°$.

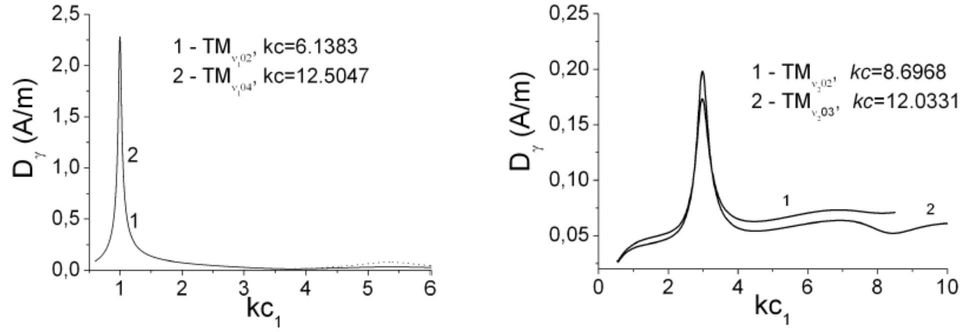
In Fig. 5(a), we observe the dependence D_γ on kc radiated through the hole in hemispherical resonator when $kc_1 = 1$. Sharp peaks in this figure correspond to the resonant values $kc \approx kc_{\nu_1 j}$ ($j = 1 \div 5$) for $TM_{\nu_1 0j}$-waves of the closed hemispherical resonator (see Table 2). As follows from this figure, the resonant modes for higher indices ν_p ($p > 1$), which are excited in the hemispherical resonator, do not radiate effectively through the small hole.

We also observe here that the growth of the parameter kc_1 (see Fig. 5(b); $kc_1 = 3$) leads to the slight oscillations with sharp peaks of the value $|H_\varphi|$. We observe maxima of these oscillations for kc that are shifted from the resonant values $kc_{\nu_1 j}$ of the closed resonator. This behavior is similar to the one we observe in the previous case. The sharp peaks in Fig. 5(b) correspond to the radiation through the hole of the resonant $TM_{\nu_2 0j}$-waves with $kc \approx kc_{\nu_2 j}$ ($j = 1, 2 \ldots$, see Table 2). As opposed to the sphere-conical resonator, here the radiation of the resonant modes $TM_{\nu_2 0j}$ through the hole is effective in whole observed diapason. Moreover, the radiation effectiveness of these modes increases with increasing of the mode index j.

Figure 6 shows the dependence of D_γ on the parameter kc_1 for different modes that are radiated from hemispherical cavity. We see that at certain dimensionless radius $b_0 = kc_1 \sin\gamma$ of the circular hole a significant increase of radiation for the selected resonant mode takes place. This radius depends on geometrical parameters kc and γ. We obtain $b_0 \approx 1.0$ and $b_0 \approx 3.0$ for $TM_{\nu_1 0j}$ and $TM_{\nu_2 0j}$-modes respectively, if $\gamma = 91°$. In the case of $\gamma = 50°$ (see Fig. 4) we obtain $b_0 \approx 0.77$ and $b_0 \approx 3.45$ for $TM_{\nu_1 0j}$- and $TM_{\nu_2 0j}$-modes.

Next, we investigate the influence of the parameter kc_1 on the near field of the sphere-conical cavity. In Fig. 7, we observe $|H_\varphi(\theta)|$ distribution which is plotted on the virtual spherical surface with the radius $r < c_1$. We compare the curves in Fig. 7(a). As follows from this comparison, $|H_\varphi(\theta)|$ decreases extremely, if the parameter kc_1 is slightly shifted from the resonant value (see Fig. 6(a)). The maximum of the field concentrates at $30° < \theta < 100°$, that is under the spherical cap of the cavity (see curves 1-3 in this figure). We also see the extreme decay of the field in the area lower the plane ($\theta > 90°$). Fig. 7(b) shows the near field behavior for higher resonant modes excited in the cavity (see Fig. 6(b)).

To verify the calculations, we test the satisfaction of the mode-matching conditions on the virtual spherical segment with the angle area $90° < \theta < 180°$ and for different means of the radius $r = c_1$. The corresponding data are shown in Fig. 7(b) (the comparison of curves 1 and 2, as well as 3 and 4). The behavior of these curves shows good agreement of the field's magnitude for all observation angles θ.



(a) (b)

Figure 6. D_γ as a function of kc_1 for $\gamma = 91°$ and $kr_0 = 0.5$; kc corresponds to: (a) $TM_{\nu_1 0j}$ — resonance mode; (b) $TM_{\nu_2 0j}$ — resonance mode.

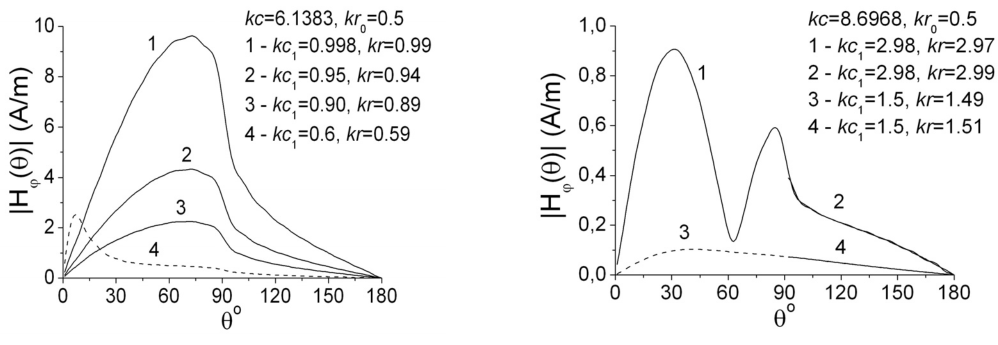


Figure 7. Near field distribution for $\gamma = 91°$: (a) $TM_{\nu_1 02}$-mode; (b) $TM_{\nu_2 02}$-mode.

8. CONCLUSIONS

The mode matching technique and the analytical regularisation procedure are developed for the solution of the canonical diffraction problem of axially symmetric excitation of sphere-conical cavity formed by the truncated semi-infinite cone and a spherical cap. With these techniques the diffraction problem is reduced to the ISLAE of the second kind, and the solution of which satisfies all the necessary conditions. The explicit solution of the ISLAE is obtained for the small hole of the cavity. Based on this solution the complex value of perturbation of the real resonant frequency of the closed sphere-conical volume caused by the small hole is determined. The transition to the hemispherical cavity is also obtained.

Numerical solution is used for examination of the radiation characteristics of two open sphere-conical cavities: the narrow cavity ($\gamma < \pi/2$) and the hemispherical one ($\gamma \to \pi/2$). We have analyzed the radiation of the TM-waves from the open end when the cavity is excited by the frequencies determined for closed sphere-conical resonator with the same parameters γ and c. The dimensionless truncated radiuseskc_1 that allow for the effective radiation of different resonant modes through the hole are found. The dependencies of near and far fields on geometrical parameters of the sphere-conical cavity are investigated.

APPENDIX A.

Taking into account the representation of the kernel functions in Eqs. (31), (32), we show the formulas for effective calculation of the elements of matrix operator in Eq. (30) as:

$$M'_-(z_n) = -\frac{\pi}{(z_n^2 - 1/4)[P_{z_n - 1/2}(\cos\gamma)]^2 M_+(z_n)};$$

$$\left\{[M_-(\xi_p)]^{-1}\right\}' = M_+(\xi_p)\frac{(\xi_p^2 - 1/4)}{\cos(\pi\xi_p)}\begin{cases} P_{\xi_p - 1/2}(-\cos\gamma)\frac{\partial}{\partial\xi}P_{\xi_p - 1/2}(\cos\gamma), \xi_p = \nu_p, \\ P_{\xi_p - 1/2}(\cos\gamma)\frac{\partial}{\partial\xi}P_{\xi_p - 1/2}(-\cos\gamma), \xi_p = \mu_p. \end{cases}$$

APPENDIX B.

Formulas for effective calculation of the elements of matrix operator in Eq. (30) for $\gamma = \pi/2$ look as:

$$M'_-(z_n) = i\frac{(\pi/2)^{3/2}\Gamma(n)}{\Gamma(n + 1/2)},\ z_n = 2n + 1/2,$$

$$\left\{[M_-(\xi_p)]^{-1}\right\}' = i\frac{(\pi/2)^{1/2}(p - 1/2)\Gamma(p)}{\Gamma(p + 1/2)},\ \xi_p = 2p - 1/2.$$

APPENDIX C.

Equation (44) leads to the correlation as:

$$x_n^{(1)}\tau_{kp} = x_k^{(1)}\tau_{np}. \tag{C1}$$

Here k, $n = 1, 2, 3, \ldots$; sub-index p determines ν_p for the transcendental Equation (43).

Taking into account the definition in Eq. (40)

$$y_p = \sum_{n=1}^{\infty} \frac{x_n^{(1)}}{\xi_p + z_n},$$

it follows from Eq. (C1) that

$$y_p = \frac{x_k^{(1)}}{\tau_{kp}} \sum_{n=1}^{\infty} \frac{\tau_{np}}{\xi_p + z_n}. \tag{C2}$$

Correlations in Eqs. (C2) and (44) lead to the solution as:

$$x_k^{(1)} = \frac{\tau_{kp} f_p^{(1)}}{1 + \kappa_{\xi_p}(\rho_1, \rho_c) b(\xi_p)}, \tag{C3}$$

where

$$b(\xi_p) = \sum_{n=1}^{\infty} \frac{\tau_{np}}{\xi_p + z_n}. \tag{C4}$$

Let us introduce a contour integral

$$J_p = \frac{1}{2\pi i} \int_{C_R} \frac{1}{(\xi_p + t)(t - \xi_p) M_-(t)} dt, \tag{C5}$$

where the circle C_R with radius $|t| = R$ envelopes the simple poles of the integrand at $t = z_n$ ($n = 1, 2, 3 \ldots$) and $t = -\xi_p$. The integrand in Eq. (C5) decays as $t^{-3/2}$, if $R \to \infty$. Next, using the residual theorem, it is found that

$$b(\xi_p) = \frac{1}{2\xi_p [M_-^{-1}(\xi_p)]' M_+(\xi_p)}. \tag{C6}$$

The expressions (C3) and (C6) give the approximate solution of the diffraction problem in the form of Eq. (45).

REFERENCES

1. Zhao, Y., Z. Zhang, and Z. Feng, "An electrically large metallic cavity antenna with circular polarization for satellite applications," *IEEE Antennas and Wireless Propagation Letters*, Vol. 10, 1461–1464, 2011.
2. Kumar, A., S. Sharma, and G. Singh, "Measurement of dielectric constant and loss factor of the dielectric material at microwave frequencies," *Progress In Electromagnetics Research*, Vol. 69, 47–54, 2007.
3. Gao, J., "Analytical formulas for the resonant frequency changes due to opening aperture on cavity walls," *Nuclear Instruments and Methods in Physics Research*, A311, 437–443, 1992.
4. Shang, K., L. Yan, K. Wen, Z. Guo, Y. Guo, W. Pan, and X. Luo, "Separation of resonance modes in nanoring resonator by a cascaded slot cavity," *Modern Physics Letters B*, Vol. 26, No. 23, 2012.
5. Ateeq, M., A. Shaw, L. Wang, and P. Dickson, "An innovative microwave cavity sensor for non-destructive characterisation of polymers," *Sensors and Actuators A*, Vol. 251, 156–166, 2016.
6. Bethe, H. A., "Theory of diffraction by small holes," *Phys. Rev.*, Vol. 66, 163–182, 1944.

7. Bouwkamp, C. J., "On Bethe's theory of diffraction by small holes," *Philips Res Rep.*, Vol. 5, 321–332, 1950.
8. Stevenson, A. F., "Theory of slots in rectangular waveguides," *J. Appl. Phys.*, Vol. 19, 24–38, January 1948.
9. Silver, S. and W. Saunders, "The external field produced by a slot in an infinite circular cylinder," *Journal of Applied Physics*, Vol. 21, 153–158, February 1950.
10. Galejs, J., "Admittance of a rectangular slot which is backed by a rectangular cavity," *IEEE Transactions on Antennas and Propagation*, Vol. 67D, No. 2, 119–126, 1963.
11. Omiya, M., T. Hikage, N. Ohno, K. Horiguchi, and K. Itoh, "Design of cavity-backed slot antennas using the finite-difference time-domain technique," *IEEE Transactions on Antennas and Propagation*, Vol. 46, No. 12, 1853–1858, December 1998.
12. Hirokawa, J., H. Arai, and N. Goto, "Cavity-backed wide slot antenna," *Proc. Inst. Elect. Eng.*, Vol. 136, pt. H, No. 1, 29–33, February 1989.
13. Umashankar, K., A. Taflove, and B. Beker, "Calculation and experimental validation of induced currents on coupled wires in an arbitrary shaped cavity," *IEEE Transactions on Antennas and Propagation*, Vol. 35, No. 11, 1248–1257, November 1987.
14. Wounchoum, P., D. Worasawate, C. Phongcharoenpanich, and M. Krairiksh, "A two-slot array antenna on a concentric sectoral cylindrical cavity excited by a coupling slot," *Progress In Electromagnetics Research*, Vol. 86, 135–154, 2008.
15. Kobayashi, K., "Some diffraction problems involving modified Wiener-Hopf geometries," *Analytical and Numerical Methods in Electromagnetic Wave Theory*, edited by M. Hashimoto, M. Idemen, O. A. Tretyakov, Science House Co., Ltd., Chapter 4, 147–228, 1993.
16. Kuryliak, D. B., K. Kobayashi, S. Koshikawa, and Z. T. Nazarchuk, "Wiener-Hopf analysis of the diffraction by circular waveguide cavities," *Journal of the Institute of Science and Engineering*, Vol. 10, 45–51, Tokyo, Chuo University, March 2004.
17. Kuryliak, D. B., K. Kobayashi, S. Koshikawa, and Z. T. Nazarchuk, "Wiener-Hopf analysis of the axial symmetric wave diffraction problem for a circular waveguide cavity," *International Workshop on Direct and Inverse Wave Scattering*, 2-672-81, Gebze, Turkey, 2000.
18. Vinogradov, S. S., P. D. Smith, and E. D. Vinogradova, *Canonical Problems in Scattering and Potential Theory. Part II: Acoustic and Electromagnetic Diffraction by Canonical Structures*, Chapman & Hall/CRC, Boca Raton, FL, 2002.
19. Shestopalov, V. P., *Summation Equations in the Modern Theory of Diffraction*, Naukova Dumka, Kyiv, 1983 (in Russian).
20. Ilchenko, M. E. and A. A. Trubin, *The Theory of Dielectric Resonators*, Lybid, Kyiv, 1993 (in Russian).
21. Kuryliak, D. B., "Axially symmetric electromagnetic wave diffraction from perfectly conducting finite conical shell in sectorial piecewise homogeneous dielectric medium," *Reports of National Academy of Sciences of Ukraine, A*, 31–34, 1987 (in Russian).
22. Kuryliak, D. B., "Symmetrical electromagnetic excitation of the piecewise homogeneous dielectric sphere with finite conical inclusion (E-, H- polarization)," *Izvestiya Vuzov. Electro-mechanics*, No. 1-2, 3–10, 1996 (in Russian).
23. Kuryliak, D. B., "Symmetrical electromagnetic excitation of the homogeneous dielectric sphere with finite conical inclusion (TM-, TE-waves)," *Izvestiya Vuzov. Radioelectronika*, Vol. 40, No. 2, 27–35, 1997 (in Russian).
24. Garcia-Gracia, H. and J. C. Gutiérrez-Vega, "Scalar wave scattering in spherical cavity resonator with conical channels," *J. Opt. Soc. Am. A*, Vol. 31, No. 2, 246–252, 2014.
25. Drobakhin, O. O., P. I. Zabolotny, and E. N. Privalov, "Resonant properties of microwave resonators in the form of a spherical sector," *Radioelektronika, Informatyka, Upravlinnia*, No. 2, 11–16, 2009 (in Russian).
26. Van't Hof, J. P. and D. D. Stancil, "Eigenfrequencies of a truncated conical resonator via the classical and Wentzel-Kramers-Brillouin methods," *Transactions on Microwave Theory and*

Techniques, Vol. 56, No. 8, 1909–1916, 2008.

27. Nesterenko, M. V., V. A. Katrich, Yu. M. Penkin, and S. L. Berdnik, "Analytical methods in theory of slot-hole coupling of electrodynamics volumes," *Progress In Electromagnetics Research*, Vol. 70, 79–174, 2007.
28. Barannik, A. A., S. A. Bunyaev, and N. T. Cherpak, "Scoop tapered quasi-optical resonator," *Technical Physics Letters*, Vol. 31, No. 19, 1–5, 2005 (in Russian).
29. Mayer, B., A. Reccius, and R. Knochel, "Conical cavity for surface resistance measurements of high temperature superconductors," *IEEE Transactions on Microwave Theory and Technlques*, Vol. 40. No. 2, 1992.
30. Ash, E. A. and G. Nichols, "Super-resolution aperture scanning microscope," *Nature*, Vol. 237, 510–512, 1972.
31. Drezet, A., J. C. Woehl, and S. Huant, "Diffraction by a small aperture in conical geometry: Application to metal-coated tips used in near-field scanning optical microscopy," *Physical Review E*, Vol. 65, 2002.
32. Kolodij, B. I. and D. B. Kuryliak, "Axially-symmetric wave diffraction problem for excitation of finite conical shell by radial electric dipole," *Report of National Academy of Sciences of Ukraine*, No. 12, 31–34, 1986 (in Ukrainian).
33. Kolodij, B. I. and D. B. Kuryliak, *Axially-symmetric Electromagnetic Wave Diffraction Problems for Conical Surfaces*, Naukova Dumka, Kyiv, 1995 (in Ukrainian).
34. Kuryliak, D. B., "Series equations with associated Legendre functions on the boundary of conical and spherical regions and their application in scalar problems of the diffraction theory," *Report of National Academy of Sciences of Ukraine*, No. 10, 70–78, 2000 (in Russian).
35. Kuryliak, D. B. and Z. T. Nazarchuk, *Analytical-numerical Methods in the Theory of Wave Diffraction on Conical and Wedge-shaped Surfaces*, Naukova Dumka, Kyiv, 2006 (in Ukrainian).
36. Kuryliak, D. B. and Z. T. Nazarchuk, "Convolution type operators for wave diffraction by conical structures," *Radio Science*, Vol. 43, RS4S03, doi: 10.1029/2007RS003792, 2008.
37. Kuryliak, D. B., Z. T. Nazarchuk, and V. O. Lysechko, "Diffraction of a plane acoustic wave from a finite soft (rigid) cone in axial irradiation," *Open Journal of Acoustics*, Vol. 5, No. 4, 193–206, 2015, http://dx.doi.org/10.4236/oja.2015.54015.
38. Kuryliak, D. B. and O. M. Sharabura, "Diffraction of axially-symmetric TM-wave from Bi-cone formed by finite and semi-infinite shoulders," *Progress In Electromagnetics Research B*, Vol. 68, 73–88, 2016.
39. Kuryliak, D. B., "Axially-symmetric field of electric dipole over truncated cone. I. Comparison between mode-matching technique and integral transformation method," *Radio Physics and Radio Astronomy*. Vol. 4, No. 2. 121–128, 1999; "Axially-symmetric field of electric dipole over truncated cone. II. Numerical Modeling," *Radio Physics and Radio Astronomy*, Vol. 5, No. 3, 284–290, 2000.
40. Kuryliak, D. B. and Z. T. Nazarchuk, "Development of the methods of analytical regularization in the theory of diffraction," *Materials Science*, Vol. 47, No. 2, 160–176, 2011.
41. Gradshteyn, I. S. and I. M. Ryzhik, *Tables of Integrals, Series and Products*, Dover, New York, 1972.
42. Agranovich, M. S., B. Z. Katsenelenbaum, A. N. Sivov, and N. N. Voitovich, *Generalized Method of Eigenoscillations in Diffraction Theory*, Wiley-VCH, Berlin, 1999.
43. Kiselev, A. A. and B. S. Pavlov, "The eigenvalues and eigenfunctions of Laplace operator with Neuman boundary conditions in the system of two connected resonators," *Theoretical and Mathematical Physics*, Vol. 100, No. 3, 354–366, 1994.