Progress In Electromagnetics Research C, Vol. 5, 21–33, 2008

DETERMINATION OF THE FREQUENCY-AMPLITUDE RELATION FOR NONLINEAR OSCILLATORS WITH FRACTIONAL POTENTIAL USING HE'S ENERGY BALANCE METHOD

S. S. Ganji †, D. D. Ganji, and S. Karimpour

Department of Civil and Mechanical Engineering
Babol University of Technology
P.O. Box 484, Babol, Iran

Abstract—A He's Energy balance method (EBM) is used to calculate the periodic solutions of nonlinear oscillators with fractional potential. Some examples are given to illustrate the effectiveness and convenience of the method. We find this EBM works very well for the whole range of initial amplitudes, and the excellent agreement of the approximate frequencies and periodic solutions with the Exact or other analytical solutions has been demonstrated and discussed. Comparison of the result obtained using this method with that obtained by Exact or other analytical solutions reveal that the EBM is very effective and convenient and can therefore be found widely applicable in engineering and other science.

1. INTRODUCTION

Nonlinear oscillator models have been widely used in many areas of physics and engineering and are of significant importance in mechanical and structural dynamics for the comprehensive understanding and accurate prediction of motion. The study of nonlinear oscillators is of interest to many researchers and various methods of solution have been proposed. Surveys of the literature with numerous references, and useful bibliographies, have been given by Nayfeh [1], Mickens [2], Jordan and Smith [3] and more recently by He [4].

† Also with Department of Civil and Transportation Engineering, Azad Islamic University of Science and Research Branch, Tehran, Iran and Department of Civil and Structural Engineering, Semnan University , Semnan, Iran.

Various approaches, including the Non-preservative methods [4], homotopy perturbation method [5–9], Lindstedt- Poincaré method [10–12], parameter-expansion method [13–16], Parameterized perturbation method [17, 18], multiple scale method [19–22], and the harmonic balance method (HBM) [23–26] have been developed to study the nonlinear oscillators.

Recently, some approximate variational methods, including approximate energy method [27–31], variational iteration method [32–36] and variational approach [37–40], to solution, bifurcation, limit cycle and period solutions of nonlinear equations have been given much attention. Among these methods, the EBM is considered to be one of powerful methods capable of handling strongly nonlinear behaviors and, it can converge to an accurate periodic solution for smooth nonlinear systems.

The main objective of this paper is to approximately solve nonlinear oscillators with fractional potential by applying the Energy balance method (EBM), and to compare the approximate frequency obtained with the exact one and with other approximate frequency obtained applying the variational approach solution [4] to the same nonlinear oscillators. As we can see, the results presented in this Letter reveal that the method is very effective and convenient for nonlinear oscillators with fractional terms.

2. DESCRIPTION OF ENERGY BALANCE METHOD

In the present paper, we consider a general nonlinear oscillator in the form [29]:

$$u'' + f(u(t)) = 0 \tag{1}$$

in which u and t are generalized dimensionless displacement and time variables, respectively. Its variational principle can be easily obtained:

$$J(u) = \int_0^t \left(-\frac{1}{2}u'^2 + F(u) \right) dt \tag{2}$$

Where $T = 2\pi/\omega$ is period of the nonlinear oscillator, $F(u) = \int f(u)du$.

Its Hamiltonian, therefore, can be written in the form:

$$H = \frac{1}{2}u'^2 + F(u) + F(A) \tag{3}$$

Or:

$$R(t) = \frac{1}{2}u'^2 + F(u) - F(A) = 0 \tag{4}$$

Oscillatory systems contain two important physical parameters, i.e., the frequency ω and the amplitude of oscillation, A. So let us consider such initial conditions:

$$u(0) = A, \qquad u'(0) = 0 \tag{5}$$

We use the following trial function to determine the angular frequency ω:

$$u(t) = A\cos(\omega t) \tag{6}$$

Substituting (6) into u term of (4), yield:

$$R(t) = \frac{1}{2}\omega^2 A^2 \sin^2 \omega t + F(A\cos\omega t) - F(A) = 0 \tag{7}$$

If, by chance, the exact solution had been chosen as the trial function, then it would be possible to make R zero for all values of t by appropriate choice of ω. Since Eq. (5) is only an approximation to the exact solution, R cannot be made zero everywhere. Collocation at $\omega t = \pi/4$ gives:

$$\omega = \sqrt{\frac{2(F(A) - F(A\cos\omega t))}{A^2 \sin^2 \omega t}} \tag{8}$$

Its period can be written in the form:

$$T = \frac{2\pi}{\sqrt{\dfrac{2(F(A) - F(A\cos\omega t))}{A^2 \sin^2 \omega t}}} \tag{9}$$

3. NUMERICAL EXAMPLES

Example 1. To illustrate the basic procedure of the present method, we consider an $u^{1/3}$ force nonlinear oscillator [4]:

$$\frac{d^2u}{dt} + \varepsilon u^{1/3} = 0, \tag{10}$$

with the initial conditions

$$u(0) = A, \qquad \frac{du}{dt}(0) = 0. \tag{11}$$

For this problem, $f(u) = u^{1/3}$ and $F(u) = \frac{3}{4}\varepsilon u^{4/3}$.

Its variational principle can be easily obtained:

$$J(u) = \int_0^t \left(-\frac{1}{2}u'^2 + \frac{1}{3}\varepsilon u^{4/3}\right) dt \tag{12}$$

Its Hamilton, therefore, can be written in the form

$$H=\frac{1}{2}u'^2+\frac{3}{4}\varepsilon u^{4/3}=\frac{3}{4}\varepsilon A^{4/3}, \tag{13}$$

or

$$\frac{1}{2}u'^2+\frac{3}{4}\varepsilon u^{4/3}-\frac{3}{4}\varepsilon A^{4/3}=0, \tag{14}$$

In Eqs. (13) and (14) the kinetic energy (E) and potential energy (T) can be respectively expressed as $E = u'^2/2$, $T = 3\varepsilon u^{4/3}/4$. Throughout the oscillation, it holds that $H = E + T =$ constant.

Substituting (6) into (14), we obtain :

$$R(t)=\frac{1}{2}A^2\omega^2\sin^2\omega t+\frac{3}{4}\varepsilon(A\cos\omega t)^{4/3}-\frac{3}{4}\varepsilon A^{4/3}=0, \tag{15}$$

We obtain the following result:

$$\omega=\frac{1}{2}\frac{\sqrt{\left(-6\varepsilon\left(A\cos\omega t\right)^{4/3}+6\varepsilon A^{4/3}\right)}}{A\sin\omega t}, \tag{16}$$

with $T=\frac{2\pi}{\omega}$, yield:

$$T=\frac{4\pi A\sin\omega t}{\sqrt{(-6\varepsilon(A\cos\omega t)^{4/3}+6\varepsilon^{4/3})}}. \tag{17}$$

If we collocate at $\omega t=\pi/4$, we obtain:

$$\omega=\frac{\sqrt{6\left(-(2^{1/3})\varepsilon A^{4/3}+2A^{4/3}\right)}}{2A}, \tag{18}$$

with $T=\frac{2\pi}{\omega}$, yield:

$$T=\frac{4\pi A}{\sqrt{6\left(-(2^{1/3})\varepsilon A^{4/3}+2A^{4/3}\right)}}. \tag{19}$$

Example 2. Consider a more complex example in form [4]:

$$u''+au+bu^3+cu^{1/3}=0,\quad u(0)=A,\quad u'(0)=0, \tag{20}$$

For this problem, $f(u)=au+bu^3+cu^{1/3}$ and $F(u)=a\frac{u^2}{2}+b\frac{u^4}{4}+3\varepsilon\frac{u^{4/3}}{4}$.

Its variational form reads

$$J(u)=\int_0^t\left(-\frac{1}{2}u'^2+a\frac{u^2}{2}+b\frac{u^4}{4}+3\varepsilon\frac{u^{4/3}}{4}\right)dt \tag{21}$$

Substituting $u(t) = A\cos\omega t$ into (21) and with $T = \frac{2\pi}{\omega}$, we obtain the following results:

$$\begin{aligned} R(t) \;=\; & \frac{1}{2}A^2\omega^2\sin^2\omega t + \frac{1}{2}aA^2\cos^2\omega t + \frac{1}{4}bA^4\cos^4\omega t \\ & + \frac{3}{4}cA^{4/3}\cos^{4/3}\omega t - \frac{\alpha A^2}{2} - \frac{bA^4}{4} - \frac{3cA^{4/3}}{4} = 0, \end{aligned} \tag{22}$$

$$\omega = \frac{\sqrt{4aA^2\sin^2\omega t + 4bA^4\sin^2\omega t - 2bA^4\sin^4\omega t - 6c(A\cos\omega t)^{4/3} + 6cA^{4/3}}}{2A\sin\omega t}, \tag{23}$$

$$T = \frac{4\pi A\sin\omega t}{\sqrt{4aA^2\sin^2\omega t + 4bA^4\sin^2\omega t - 2bA^4\sin^4\omega t - 6c(A\cos\omega t)^{4/3} + 6cA^{4/3}}}. \tag{24}$$

From (23), (24) and $\omega t = \pi/4$, we have:

$$\omega = \frac{\sqrt{4aA^2 + 3bA^4 - 6c2^{1/3}A^{4/3} + 12cA^{4/3}}}{2A}, \tag{25}$$

with $T = \frac{2\pi}{\omega}$, yield:

$$T = \frac{4\pi A}{\sqrt{4aA^2 + 3bA^4 - 6c2^{1/3}A^{4/3} + 12cA^{4/3}}}. \tag{26}$$

If we collocate at $\omega t = \pi/4$ and with $T = \frac{2\pi}{\omega}$, we obtain:

$$\omega \;=\; \frac{\sqrt{A^2 + 2\gamma\left(\sqrt{4+2A^2} - 2\sqrt{1+A^2}\right)}}{A}, \tag{27}$$

$$T \;=\; \frac{2\pi A}{\sqrt{A^2 + 2\gamma\left(\sqrt{4+2A^2} - 2\sqrt{1+A^2}\right)}}. \tag{28}$$

4. DISCUSSION OF EXAMPLES

The exact frequency ω_{ex} for *Example 1*, governed by Eq. (10) can be derived as shown in Eq. (29) and the variational approach frequency ω_{va} for *Example 2*, governed by Eq. (20) can be derived as shown in Eq. (30) [4].

$$\omega_{ex} \;=\; \frac{1.070451\varepsilon^{1/3}}{A^{1/3}}, \tag{29}$$

$$\omega_{va} \;=\; \sqrt{a + \frac{3}{4}bA^2 + 1.15959526696393cA^{-2/3}},\;\; a = b = c \neq 0. \tag{30}$$

The corresponding analytical approximation results of above examples, are tabulated in Tables 1, 2, 3, 4, and 5.

Table 1. Comparison between analytical and the Exact solutions for *Example 1*, when $\varepsilon = 10.0, 1.0$.

A	$\varepsilon = 0.1$			$\varepsilon = 1.0$		
	ω	ω_{ex}	$\lvert\omega - \omega_{ex}\rvert / \omega_{ex}$	ω	ω_{ex}	$\lvert\omega - \omega_{ex}\rvert / \omega_{ex}$
0.1	0.7178240235	0.7292897765	0.5938 %	2.269958874	2.306216768	1.572 %
0.5	0.4197860355	0.4264912487	0.5938 %	1.327480002	1.348683748	1.572 %
1	0.3331843972	0.3385063283	0.5938 %	1.053621576	1.070451000	1.572 %
5	0.1948474174	0.1979597017	0.5938 %	0.6161616353	0.6260035423	1.572 %
10	0.1571207194	0.1580592379	0.5938 %	0.4890478141	0.4968593409	1.572 %
50	0.09188475413	0.09243360305	0.5938 %	0.2859968965	0.2905651053	1.572 %
100	0.07292897765	0.07336459936	0.5938 %	0.2269958874	0.2306216768	1.572 %
500	0.04264912487	0.04290387799	0.5938 %	0.1327480002	0.1348683748	1.572 %
1000	0.03385063283	0.03405283051	0.5938 %	0.1053621576	0.1070451000	1.572 %

Table 2. Comparison between analytical and the Exact solutions for *Example 1*, when $\varepsilon = 10.0, 100.0$.

A	$\varepsilon = 10.0$			$\varepsilon = 100.0$		
	ω	ω_{ex}	$\lvert\omega - \omega_{ex}\rvert / \omega_{ex}$	ω	ω_{ex}	$\lvert\omega - \omega_{ex}\rvert / \omega_{ex}$
0.1	7.178240235	7.292897765	1.572 %	22.69958874	23.06216768	1.572 %
0.5	4.197860355	4.264912487	1.572 %	13.27480002	13.48683748	1.572 %
1	3.331843972	3.385063283	1.572 %	10.53621576	10.70451000	1.572 %
5	1.948474174	1.979597017	1.572 %	6.161616353	6.260035423	1.572 %
10	1.546504978	1.571207194	1.572 %	4.890478141	4.968593409	1.572 %
50	0.9044015966	0.9188475413	1.572 %	2.859968965	2.905651053	1.572 %
100	0.7178240235	0.7292897765	1.572 %	2.269958874	2.306216768	1.572 %
500	0.4197860355	0.4264912487	1.572 %	1.327480002	1.348683748	1.572 %
1000	0.3331843972	0.3385063283	1.572 %	1.053621576	1.070451000	1.572 %

Table 3. Comparison between EBM and the Variational approach for *Example 2*, when $a = b = c = 1$.

A	$a = b = c = 1$	
	ω	ω_{va}
0.1	2.481977697	2.527818119
0.5	1.717469987	1.740184688
1	1.691188466	1.705753577
5	4.486608425	4.488493734
10	8.731504325	8.732114705
50	43.31375987	43.31380196
100	86.60861115	86.06862442
500	433.0138769	433.0138778
1000	866.0259875	866.0259878

Table 4. Comparison between EBM and the Variational approach for *Example 2*, when $a = c = 1$, $b = 10$.

A	$a = c = 1$, $b = 10$	
	ω	ω_{va}
0.1	2.495538677	2.541134480
0.5	2.153416624	2.171576097
1	3.100019100	3.107988943
5	13.74334949	13.74396508
10	27.40784254	27.40893700
50	136.9345895	136.9346028
100	273.8631986	273.8632027
500	1369.306765	1369.306765
1000	2738.612972	2738.612972

Table 5. Comparison between EBM and the Variational approach for *Example 2*, when $a = b = 1$, $c = 10$.

A	$a = b = 1$, $c = 10$	
	ω	ω_{va}
0.1	7.248077875	7.404805496
0.5	4.336995685	4.426615803
1	3.584854846	3.653211282
5	4.852478915	4.869882959
10	8.853907480	80859925071
50	43.32225689	43.32267763
100	86.61128835	86.61142093
500	433.0140600	433.0140692
1000	866.0260450	866.0260481

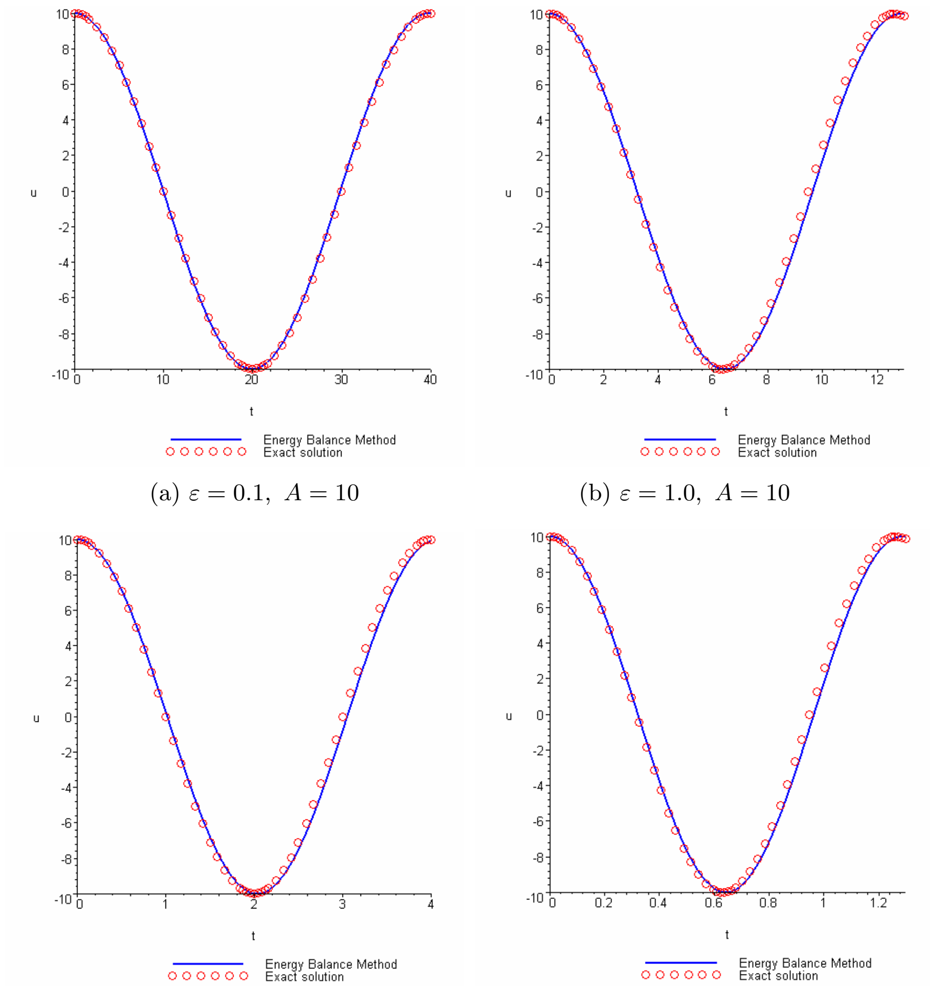
In *Example 1*, it is seen from the Table. 1, that the error percentage of EBM is 0.5983% for $\varepsilon = 0.1$. So from the Tables 1 and 2, the error percentage of the EBM are 1.572% for $\varepsilon = 1.0, 10.0, 100.0$.

For *Example 2*, In case $a = 1$, $c = 0$, Eq. (20) reduces to the well-known Duffing equation, and its approximate frequency reads:

$$\omega = \frac{2\pi A}{\sqrt{1 + 3/4bA^2}}. \tag{31}$$

With the Exact solution is:

$$\omega_{ex} = \frac{2}{\pi\sqrt{1 + bA^2}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k\sin^2 x}}, \quad a = c = 0, \tag{32}$$



(a) $\varepsilon = 0.1,\ A = 10$ (b) $\varepsilon = 1.0,\ A = 10$

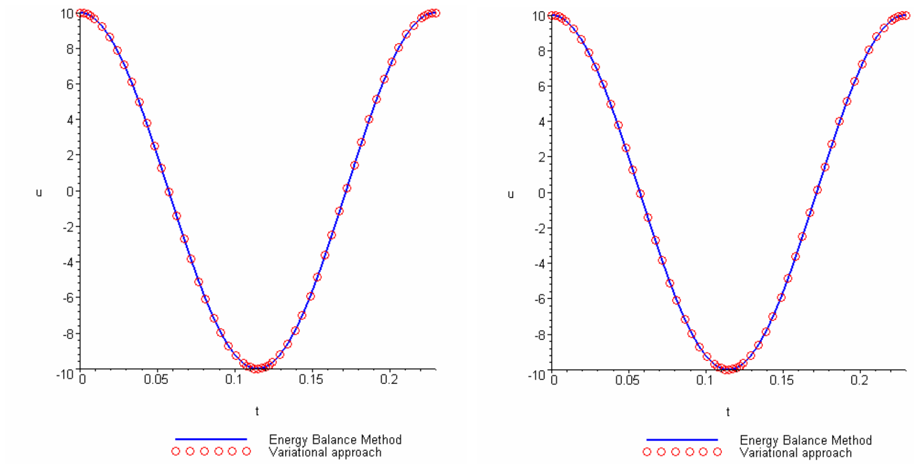
(c) $\varepsilon = 10.0,\ A = 10$ (d) $\varepsilon = 100,\ A = 10$

Figure 1. Comparison of the approximate solution (EBM) with the Exact solution for *Example 1*.

Where $k = 0.5bA^2/(1 + bA^2)$.

What is rather surprising about the remarkable range of validity of (31) with is that the approximate frequency, Eq. (31), as $b \to \infty$ is also of high accuracy.

$$\lim_{b\to\infty} \frac{\omega_{ex}}{\omega} = \frac{\sqrt{3}}{\pi} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - 0.5\sin^2 x}} = 0.9294. \tag{33}$$



(a) $\alpha = c = 1,\ b = 10,\ A = 10$ (b) $\alpha = c = 1,\ b = 10,\ A = 10$

Figure 2. Comparison of the approximate solution (EBM) with the Variational approach solution for *Example 2*.

Therefore, for any value of $b > 0$, it can be easily proved that the maximal relative error of the frequency (31) is less than 7.6%, i.e., $|\omega - \omega_{ex}|/\omega_{ex} < 7.6\%$.

In case $a = 0,\ c = 0$, Eq. (20) becomes

$$u'' + bu^3 = 0, \tag{34}$$

Its frequency, then, reads

$$\omega = \sqrt{\frac{3}{4}bA^2} = 0.866b^{1/2} \tag{35}$$

Its exact frequency [4] is $\omega_{ex} = 0.8472b^{1/2}A$. Therefore, its accuracy reaches 2.2%. In case $a = b = 0$, Eq. (20) turns out to be Eq. (10).

To further illustrate and verify the accuracy of this approximate analytical approach for *Example 1*, comparison of this analytical method with the Exact solution are presented in Figs. 1(a)–(d), for $\varepsilon = 0.1, 1.0, 10.0, 100.0$.

Figs. 2(a), (b) represent the corresponding displacement $u(t)$ in *Example 2*, for $A = 10,\ a = b,\ c = 10$. Apparently, it is confirmed that the analytical approximations show excellent agreement with the exact or other analytical approximation solutions.

5. CONCLUSION

We used a very simple but effective method (EBM) for nonlinear oscillators. The method consists of a combination of He's variational approach, to determine frequency and amplitude of the system. These examples have shown that the approximate analytical solutions are in excellent agreement with the corresponding exact solutions. The method can be easily extended to any nonlinear oscillator without any difficulty. Moreover, the present work can be used as paradigms for many other applications in searching for periodic solutions of nonlinear oscillations and so can be found widely applicable in engineering and science.

REFERENCES

1. Nayfeh, A. H., *Perturbation Methods*, Wiley-Interscience, New York, 1973.
2. Mickens, R. E., *Oscillations in Planar Dynamic Systems*, Scientific, Singapore, 1966.
3. Jordan, D. W. and P. Smith, *Nonlinear Ordinary Differential Equations*, Clarendon Press, Oxford, 1987.
4. He, J. H., "Non-perturbative methods for strongly nonlinear problems," Dissertation.de-Verlag im Internet GmbH, 2006.
5. He, J. H., "Homotopy perturbation method for bifurcation on nonlinear problems," *International Journal of Non-Linear Science and Numerical Simulation*, Vol. 6, 207–208, 2005.
6. Ganji, D. D. and A. Sadighi, "Application of He's homotopy-prturbation method to nonlinear coupled systems of reaction-diffusion equations," *Int. J. Nonl. Sci. and Num. Simu*, Vol. 7, No. 4, 411–418, 2006.
7. He, J. H., "The homotopy perturbation method for nonlinear oscillators with discontinuities," *Applied Mathematics and Computation*, Vol. 151, 287–292, 2004.
8. Beléndez, A., C. Pascual, S. Gallego, M. Ortuño, and C. Neipp, "Application of a modified He's homotopy perturbation method to obtain higher-order approximations of an force nonlinear oscillator," *Physics Letters A*, Vol. 371, 421–426, 2007.
9. Alizadeh, S. R. S., G. Domairry, and S. Karimpour, "An approximation of the analytical solution of the linear and nonlinear integro-differential equations by homotopy perturbation method," *Acta Applicandae Mathematicae*, doi: 10.1007/s10440-008-9261-z.

10. He, J. H., "Modified Lindstedt-Poincare methods for some strongly nonlinear oscillations, Part I: Expansion of a constant," *International Journal Non-linear Mechanic*, Vol. 37, 309–314, 2002.
11. He, J. H., "Modified Lindstedt-Poincare methods for some strongly nonlinear oscillations. Part III: Double series expansion," *International Journal Non-linear Science and Numerical Simulation*, Vol. 2, 317–320, 2001.
12. Özis, T. and A. Yildirim, "Determination of periodic solution for a $u^{1/3}$ force by He's modified Lindstedt-Poincare method," *Journal of Sound and Vibration*, Vol. 301, 415–419, 2007.
13. Wang, S. Q. and J. H. He, "Nonlinear oscillator with discontinuity by parameter-expansion method," *Chaos & Soliton and Fractals*, Vol. 35, 688–691, 2008.
14. He, J. H., "Some asymptotic methods for strongly nonlinear equations," *International Journal Modern Physic B*, Vol. 20, 1141–1199, 2006.
15. Wang, S. Q. and J. H. He, "Nonlinear oscillator with discontinuity by parameter-expanding method," *Chaos, Solitons & Fractals*, Vol. 29, 108–113, 2006; Vol. 35, 688–691, 2008.
16. Shou, D. H. and J. H. He, "Application of parameter-expanding method to strongly nonlinear oscillators," *International Journal of Nonlinear Sciences and Numerical Simulation*, Vol. 8, No. 1, 121–124, 2007.
17. He, J. H., "Some new approaches to Duffing equation with strongly and high order nonlinearity (II) parameterized perturbation technique," *Communications in Nonlinear Science and Numerical Simulation*, Vol. 4, 81–82, 1999.
18. He, J. H., "A review on some new recently developed nonlinear analytical techniques," *International Journal of Nonlinear Science and Numerical Simulation*, Vol. 1, 51–70, 2000.
19. Rong, H. W., X. D. Wang, W. Xu, and T. Fang, "Saturation and resonance of nonlinear system under bounded noise excitation," *Journal of Sound and Vibration*, Vol. 291, 48–59, 2006.
20. Das, S. K., P. C. Ray, and G. Pohit, "Free vibration analysis of a rotating beam with nonlinear spring and mass system," *Journal of Sound and Vibration*, Vol. 301, 165–188, 2007.
21. Okuizumi, N. and K. Kimura, "Multiple time scale analysis of hysteretic systems subjected to harmonic excitation," *Journal of Sound and Vibration*, Vol. 272, 675–701, 2004.
22. Marathe, A., "Anindya Chatterjee, wave attenuation in nonlinear

periodic structures using harmonic balance and multiple scales," *Journal of Sound and Vibration*, Vol. 289, 871–888, 2006.

23. Gottlieb, H. P. W., "Harmonic balance approach to limit cycles for nonlinear jerk equations," *Journal of Sound and Vibration*, Vol. 297, 243–250, 2006.
24. Yuan, Z. W., F. L. Chu, and Y. L. Lin, "External and internal coupling effects of rotor's bending and torsional vibrations under unbalances," *Journal of Sound and Vibration*, Vol. 299, 339–347, 2007.
25. Wiener, N., *Nonlinear Problems in Random Theory*, Wiley, New York, 1958.
26. Penga, Z. K., Z. Q. Langa, S. A. Billingsa, and G. R. Tomlinson, "Comparisons between harmonic balance and nonlinear output frequency response function in nonlinear system analysis," *Journal of Sound and Vibration*, Vol. 311, 56–73, 2008.
27. He, J. H., "Determination of limit cycles for strongly nonlinear oscillators," *Physic Review Letter*, Vol. 90, 174–181, 2006.
28. Ganji, S. S., S. Karimpour, D. D. Ganji, and Z. Z. Ganji, "Periodic solution for strongly nonlinear vibration systems by energy balance method," *Acta Applicandae Mathematicae*, doi: 10.1007/s10440-008-9283-6.
29. He, J. H., "Preliminary report on the energy balance for nonlinear oscillations," *Mechanics Research Communications*, Vol. 29, 107–118, 2002.
30. Özis, T. and A. Yildirim, "Determination of the frequency-amplitude relation for a Duffing-harmonic oscillator by the energy balance method," *Computers and Mathematics with Applications*, Vol. 54, 1184–1187, 2007.
31. Porwal, R. and N. S. Vyas, "Determination of the frequency-amplitude relation for a Duffing-harmonic oscillator by the energy balance method," *Computers and Mathematics with Applications*, Vol. 54, 1184–1187, 2007.
32. He, J. H., "Variational iteration method — A kind of nonlinear analytical technique: Some examples," *Int J Nonlinear Mech.*, Vol. 34, 699–708, 1999.
33. Rafei, M., D. D. Ganji, H. Daniali, and H. Pashaei, "The variational iteration method for nonlinear oscillators with discontinuities," *Journal of Sound and Vibration*, Vol. 305, 614–620, 2007.
34. He, J. H. and X. H. Wu, "Construction of solitary solution and compaction-like solution by variational iteration method," *Chaos,*

Solitons & Fractals, Vol. 29, 108–113, 2006.

35. Varedi, S. M., M. J. Hosseini, M. Rahimi, and D. D. Ganji, "He's variational iteration method for solving a semi-linear inverse parabolic equation," *Physics Letters A*, Vol. 370, 275–280, 2007.
36. Hashemi, S. H. A., K. N. Tolou, A. Barari, and A. J. Choobbasti, "On the approximate explicit solution of linear and non-linear non-homogeneous dissipative wave equations," *Istanbul Conferences*, Torque, accepted, 2008.
37. He, J. H., "Variational approach for nonlinear oscillators," *Chaos, Solitons and Fractals*, Vol. 34, 1430–1439, 2007.
38. Ganji, S. S., D. D. Ganji, H. Babazadeh, and S. Karimpour, "Variational approach method for nonlinear oscillations of the motion of a rigid rod rocking back and cubic-quintic Duffing oscillators," *Progress In Electromagnetics Research M*, Vol. 4, 23–32, 2008.
39. Wu, Y., "Variational approach to higher-order water-wave equations," *Chaos & Solitons and Fractals*, Vol. 32, 195–203, 2007.
40. Xu, L., "Variational approach to solitons of nonlinear dispersive equations," *Chaos, Solitons & Fractals*, Vol. 37, 137–143, 2008.