

**A COMMON THEORETICAL BASIS FOR
PRECONDITIONED FIELD INTEGRAL EQUATIONS
AND THE SINGULARITY EXPANSION METHOD**

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Abstract—It is demonstrated that there is a common theoretical basis for the Singularity Expansion Method (SEM) and stabilized, preconditioned electric field and magnetic field integral equations (EFIE, MFIE) defining radiation and scattering from a closed perfect electric conductor in a homogeneous medium.

An operator relation termed the Calderon preconditioner links the MFIE and EFIE, based on the fundamental Stratton-Chu integral representations for the problem geometry. This preconditioner is known to stabilize the ill-posed first kind EFIE, yielding the Modified EFIE (MEFIE). The same preconditioner has been applied to the weakly singular MFIE kernel, giving a Modified MFIE (MMFIE), the equation then being solved using the Fredholm determinant theory. Since this analytical integral theory is the foundation of the SEM, it follows that the Calderon preconditioner enables stabilized and common SEM representations to be defined for both the MEFIE and MMFIE.

For a finite-sized object admitting only pole singularities, the solution of the preconditioned EFIE and MFIE is equivalent to the frequency-domain SEM solution. The common SEM representation differs only in the coupling coefficient terms. Coupling coefficients for the MFIE are known, however, explicit formulations for the EFIE, and the modified coupling coefficients for the MEFIE and MMFIE are new contributions.

1. INTRODUCTION

The Singularity Expansion Method (SEM) has been the subject of more than 100 papers since its first exposure in the literature in the early 1970s. It is based on the analytic properties of the electromagnetic response as a function of the two-sided Laplace transform variable s , complex frequency. Singularities of the Laplace Transform are used to characterize the electromagnetic response of a structure to incident radiation or a driving source, in both the time and complex frequency domains [1–3].

A recent resurgence in interest in the SEM has occurred, reinforcing the popularity of the method for both scattering and radiation applications. For example, a target discrimination application requires characterization of perfectly conducting radar targets in the resonance region, based on natural pole selection derived from the frequency-domain SEM formulation [4]. Chauveau [5] developed a unified analysis of antenna radiation for both the frequency- and time-domain.

The SEM is known to be sensitive to noisy data in its application to the radar target identification/discrimination problem [4, 6]. Singular value decomposition methods, such as the Matrix Pencil Method [7] have recently been used to minimize the effects of minor fluctuations (improve tolerance to noisy data) in excitation or field strength on the accurate solution of poles and residues. Many recent papers treat SEM as an abstract procedure for approximation of the response of objects to incident radiation, in certain cases referring to Model-Based Parameter Estimation (MBPE), an abstraction of SEM [8–10]. In many cases, the underlying integral equations may be responsible for creating the instabilities or sensitive numerical behavior. Fredholm integral equations of the first kind, such as the Electric Field Integral Equation (EFIE) are known to be ill-posed equations, and by definition sensitive to perturbations [11]. The projection of the infinite-dimensional EFIE into a finite-dimensional (matrix) subspace by a discretization technique is also problematic. As the discretization interval reduces, an unbounded increase in condition number of the matrix can occur [12]. Preconditioning techniques for boundary integral equations have been developed by several authors, and allow a stable solution to the integral equation to be sought.

In this paper it is shown that the mathematical foundation of the Singularity Expansion Method (SEM), which was previously based on Mariani's work on the Magnetic Field Integral Equation (MFIE) [13, 14], can be established from the preconditioned EFIE. The preconditioning approach used by Adams [15–18] for the MEFIE, and following the

method of Harrington and Mautz [19] for an MCFIE, is examined as a viable method for stabilizing the ill-posed nature of the EFIE. It is demonstrated that the solution of the preconditioned EFIE is equivalent to the frequency-domain SEM solution, for a finite-sized object admitting only pole singularities.

The Calderon Projectors, inextricably linking the EFIE and MFIE formulations, provide a preconditioning relation that when applied to the EFIE (yielding Adams MEFIE [16]) enable both second kind Fredholm integral equations to be solved using Carlemans method, giving a justifiable basis for a valid SEM approximation. Therefore, a frequency domain solution/characterization of the electromagnetic response of a structure is justified when approximated by an SEM pole-residue form. Section 4 explores the preconditioning of the EFIE and Section 5 the application to the solution of the boundary integral equations. Common representations are considered in Section 6, leading to the abstract time- and frequency domain SEM expansions and the MBPE form. The explicit formulation of coupling coefficients for the 4 cases of the EFIE, MFIE, MMFIE and MEFIE are also given in Section 6, with detailed derivations performed in the Appendix.

2. BOUNDARY INTEGRAL EQUATION FORMULATIONS

Consider a perfectly conducting closed body of arbitrary shape and finite extent, with surface S , embedded in a homogeneous medium. We use the Stratton-Chu representation, a direct approach to the solution of the inhomogeneous vector wave equation based on application of the 2nd vector Green's theorem.

The total electric field in the space D_+ , external to surface S is $\mathbf{E}(\mathbf{r}) = \mathbf{E}^i(\mathbf{r}) + \mathbf{E}^S(\mathbf{r})$, $\mathbf{r} \in D_+$. For notational ease, we use $\mathbf{E}_- \equiv \mathbf{E}^i$ and $\mathbf{E}_+ \equiv \mathbf{E}^S$ for certain combined forms. Similarly, for magnetic field $\mathbf{H}(\mathbf{r})$. For Hölder-continuous $\mathbf{n} \cdot \mathbf{E}_\pm$, and tangential derivatives of $\mathbf{n} \times \mathbf{E}_\pm$ that are Hölder continuous on S (a condition stronger than mere continuity), where $\mathbf{n} \equiv \mathbf{n}(\mathbf{r})$ is the outward-directed normal at \mathbf{r} , in the limit $\mathbf{r} \rightarrow S^\pm$ we write the Stratton-Chu representation

$$\begin{aligned} \oint_S \{i\omega\mu g(\mathbf{r}, \mathbf{r}')[\mathbf{n}(\mathbf{r}') \times \mathbf{H}(\mathbf{r}')] + \nabla' g(\mathbf{r}, \mathbf{r}')[\mathbf{n}(\mathbf{r}') \cdot \mathbf{E}(\mathbf{r}')] \\ - \nabla' g(\mathbf{r}, \mathbf{r}') \times [\mathbf{n}(\mathbf{r}') \times \mathbf{E}(\mathbf{r}')] \} ds' = \mathbf{E}(\mathbf{r}) - \frac{1}{2} \mathbf{E}^i(\mathbf{r}) \end{aligned} \quad (1)$$

where \oint denotes a Cauchy principal value integral. Hölder continuity imposes smoothness conditions on the problem that may be limiting

in certain engineering applications. The scalar free-space Green's function is defined as $g(\mathbf{r}, \mathbf{r}') = \exp(ik|\mathbf{r} - \mathbf{r}'|)/4\pi|\mathbf{r} - \mathbf{r}'|$. For the homogenous medium the wavenumber is $k = \omega(\mu\epsilon)^{1/2}$, where μ is the permeability and ϵ the permittivity of the medium. Characteristic impedance, $Z = 1/Y = \sqrt{\mu/\epsilon}$. \mathbf{r} and \mathbf{r}' are position vectors, at source and field points, respectively. Following [20], note that $\mathbf{E}(\mathbf{r}') = (iZ/k)[\nabla' \times \mathbf{H}(\mathbf{r}')]]$ and the identity $\mathbf{n}(\mathbf{r}') \cdot [\nabla' \times \mathbf{H}(\mathbf{r}')] = -\nabla'_t \cdot [\mathbf{n}(\mathbf{r}') \times \mathbf{H}(\mathbf{r}')]]$. ∇'_t denotes the surface nabla operator with respect to source coordinates \mathbf{r}' . Applying the tangential electric field boundary conditions for a perfect conductor, $\mathbf{n} \times \mathbf{E} = 0$, expression (1) reduced to Maues form [21], is the Electric Field Integral Equation (EFIE),

$$\begin{aligned} (i\omega\mu)\mathbf{n}(\mathbf{r}) \int_S \left\{ g(\mathbf{r}, \mathbf{r}') [\mathbf{n}(\mathbf{r}') \times \mathbf{H}(\mathbf{r}')] - \frac{1}{k^2} \nabla' g(\mathbf{r}, \mathbf{r}') \nabla'_t \cdot [\mathbf{n}(\mathbf{r}') \times \mathbf{H}(\mathbf{r}')] \right\} ds' \\ = -\mathbf{n}(\mathbf{r}) \times \mathbf{E}^i(\mathbf{r}) \end{aligned} \quad (2)$$

Similarly, following Stratton-Chu for deriving the magnetic field equivalent,

$$\begin{aligned} \int_S \left\{ -i\omega\epsilon g(\mathbf{r}, \mathbf{r}') [\mathbf{n}(\mathbf{r}') \times \mathbf{E}(\mathbf{r}')] + \frac{i\omega\epsilon}{k^2} \nabla' g(\mathbf{r}, \mathbf{r}') [\mathbf{n} \cdot \mathbf{H}(\mathbf{r}')] \right. \\ \left. - \nabla' g(\mathbf{r}, \mathbf{r}') \times [\mathbf{n}(\mathbf{r}') \times \mathbf{H}(\mathbf{r}')] \right\} ds' = \mathbf{H}(\mathbf{r}) - \frac{1}{2} \mathbf{H}^i(\mathbf{r}) \end{aligned} \quad (3)$$

Applying Maxwells equations and the surface Nabla operator as above, Maues form of the Magnetic Field Integral Equation follows

$$-\mathbf{n}(\mathbf{r}) \times \int_S \nabla' g(\mathbf{r}, \mathbf{r}') \times [\mathbf{n}(\mathbf{r}') \times \mathbf{H}(\mathbf{r}')] ds' = \frac{1}{2} \mathbf{n}(\mathbf{r}) \times \mathbf{H}(\mathbf{r}) - \mathbf{n}(\mathbf{r}) \times \mathbf{H}^i(\mathbf{r}) \quad (4)$$

For a perfect electric conductor, operator representations in terms of unknown tangential surface current $\mathbf{J}(\mathbf{r}) = [Z\mathbf{n}(\mathbf{r}) \times \mathbf{H}(\mathbf{r})]$ of the integral equations (2) and (4) can be used (in shorthand notation) [16, 22],

$$T\mathbf{J} = -\mathbf{n} \times \mathbf{E}^i = \mathbf{M}^i \quad (5)$$

$$\left(\frac{1}{2} I + K \right) \mathbf{J} = Z\mathbf{n} \times \mathbf{H}^i = \mathbf{J}^i \quad (6)$$

where Z is the intrinsic impedance and I the identity operator. Using an indirect (layer ansatz or source) approach [21], we can show that

an equivalent MFIE can be written, namely

$$\left(\frac{1}{2}I - K\right) \mathbf{J} = \mathbf{M}^i \quad (7)$$

Hsiao et al. [22] compiled tangential representations of (1) and (3) in matrix form as

$$\begin{pmatrix} I/2 \mp K & \pm T \\ \mp T & I/2 \mp K \end{pmatrix} \begin{pmatrix} \mathbf{n} \times \mathbf{E}_{\pm} \\ Z\mathbf{n} \times \mathbf{H}_{\pm} \end{pmatrix} = \begin{pmatrix} \mathbf{n} \times \mathbf{E}_{\pm} \\ Z\mathbf{n} \times \mathbf{H}_{\pm} \end{pmatrix} \quad (8)$$

whereby the matrix term on the left, the Calderon projector, projects the tangential components of the boundary values of the interior and exterior solutions onto themselves. Through a series of algebraic manipulations, following Hsiao et al. [21],

$$\mp \begin{pmatrix} I/4 - K^2 + T^2 & (KT + TK) \\ -(TK + KT) & I/4 - K^2 + T^2 \end{pmatrix} \begin{pmatrix} \mathbf{J} \\ \mathbf{M} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \quad (9)$$

For a zero coefficient matrix, two operator relations follow,

$$TK = -KT \quad (10)$$

and

$$\frac{1}{4}I - K^2 = -T^2 \quad (11)$$

which we term the Calderon preconditioning relation. We elaborate on its application in Sections 4 and 5.

It is well-known that operator $K : L^2(S) \rightarrow L^2(S)$ in (6) is compact [22, p. 346], and hence bounded [23, p. 140]. Since $KK^* \neq K^*K$, thanks to the $\exp(ik|\mathbf{r} - \mathbf{r}'|)/(4\pi|\mathbf{r} - \mathbf{r}'|)$ term in the scalar free space Green's function $g(\mathbf{r}, \mathbf{r}')$, operator K is also nonselfadjoint and nonnormal. The identity operator in (6) is bounded, but not compact. The relevance of compact operators is that they have useful properties in the "forward direction", but are problematic in the "reverse direction". The inverse of a compact and invertible operator is unbounded. The compact-plus-identity operator, like (6), behaves well due to the noncompact identity operator.

The (complete) Hilbert space of Lebesgue-square-integrable functions, $L^2(S)$ in $S \subset \mathbb{R}^2$, is the appropriate function space for this application. Therefore, $\mathbf{J}^i \in L^2(S)$ and the MFIE domain is $D_K = \{\mathbf{J} : \mathbf{J}(\mathbf{r}), K\mathbf{J}(\mathbf{r}) \in L^2(S), \mathbf{r} \neq \mathbf{r}'\}$.

The Cauchy singular integral in the EFIE is more difficult to characterize in function space. Per Dolph [24], integral $T : C^{1+\lambda}(S) \rightarrow$

$C^{1+\lambda}(S)$ if $S \in C^2$, where $C^{1+\lambda}$ is the space of continuous functions with continuous first derivative, i.e., Hölder continuous with exponent λ . C^2 is the space of twice continuously differentiable functions. Since not every bounded sequence will converge in $C^{1+\lambda}(S)$, T is not compact. This can be useful; if T is invertible, it has a bounded inverse. However, as Dolph noted, regularization is required before the Fredholm Alternative can be invoked to establish uniqueness. Since T is not bounded from below, the inverse is also not continuous; as \mathbf{M}^i varies, $T^{-1}\mathbf{M}^i$ does not vary continuously. This problem of ill-posedness is addressed in Section 4.

In the sections to follow, it will be shown that the Calderon preconditioner converts the EFIE to a second kind equation in K^2 (or T^2) which operates on $L^2(S)$. As discussed above, the resulting compact-plus-identity operator behaves well in both forward and reverse directions.

3. SINGULARITY EXPANSION METHOD

The SEM method, first presented by Baum [1], found its origins in two places, the theoretical analyses of canonical problems to determine natural oscillations [25–28] and experimental observations of induced currents and scattered fields seemingly describable by exponentially damped sinusoidal oscillations [29, 30]. Michalski [31] compiled a comprehensive bibliography of the SEM literature through 1981. As noted in Section 1, there continues to be interest in the application of SEM to target identification, discrimination, wideband antenna characterization, transient analysis [32] and transfer function estimation [8].

The SEM applies to fields, as well as current and charge densities on objects, as this body of literature has proven. In symmetric product notation, a frequency domain representation of the integral equation (6) for an excitation $\mathbf{I}(\mathbf{r}, s)$ is

$$\langle \Gamma(\mathbf{r}, \mathbf{r}'; s_\alpha); \mathbf{J}(\mathbf{r}', s) \rangle = \mathbf{I}(\mathbf{r}, s) \quad (12)$$

where $\Gamma(\mathbf{r}, \mathbf{r}'; s_\alpha)$ is the kernel of the integral equation, evaluated at $s = s_\alpha$. SEM builds on the concept of natural frequencies and natural modes corresponding to “free oscillations.” The nontrivial solutions of the homogeneous equations for the scattering problem, defined in terms of an impedance (dyadic) operator, $\Gamma(\mathbf{r}, \mathbf{r}'; s_\alpha)$ are natural and coupling mode vectors, $\mathbf{N}_\alpha, \mathbf{C}_\alpha$.

$$\langle \Gamma(\mathbf{r}, \mathbf{r}'; s_\alpha); \mathbf{N}_\alpha(\mathbf{r}') \rangle = 0 \quad (13)$$

and

$$\langle \mathbf{C}_\alpha(\mathbf{r}); \Gamma(\mathbf{r}, \mathbf{r}'; s_\alpha) \rangle = 0 \quad (14)$$

These relations exist at a specific frequency, s_α , termed the natural frequency. Marin [3, 13] proved for the MFIE case that the inverse operator is analytic, except for a finite number of poles, permitting a pole-residue approximation. Using the Mittag-Leffler representation, Taylor and Laurent series expansions of (12) near $s = s_\alpha$ yield the frequency domain SEM representation.

$$\mathbf{J}(\mathbf{r}, s) = \sum_{\alpha} \frac{\Psi_{\alpha}(s) \mathbf{N}_{\alpha}(\mathbf{r})}{(s - s_{\alpha})^{m_{\alpha}}} + \mathbf{J}_{np}(\mathbf{r}, s) \quad (15)$$

The series term consists of pole singularities, of multiplicity m_α ($m_\alpha = 1$ for the PEC case) and corresponding residue terms. $\Psi_\alpha(s)$ is the coupling coefficient, defining the strength of the natural oscillation in terms of the object and incident wave parameters. The second term, $\mathbf{J}_{np}(\mathbf{r}, s)$, is the nonpole term, typically corresponding to the entire function contribution arising from the Mittag-Leffler expansion. Time domain solutions clearly follow, with appropriate inverse Laplace/Fourier Transformations enabling transient solutions to be determined. The concept of series expansions of singularities is elementary; of more substance is the underlying theory that motivates the validity of these expansions.

There is a significant body of literature dedicated to SEM analysis, with the basic foundations presented by Baum [2]; the analyses of Marin [13, 14] provided the key underlying mathematical theory. Dolph [24] observed that many of the papers on SEM were difficult to interpret mathematically, since neither the properties of the integral operators, nor the space in which solutions are sought were specified. While this observation was made more than 30 years ago, the SEM is still applied in abstract form by many authors. The use of the EFIE in particular admits room for further analysis.

4. PRECONDITIONING EFIE

The first kind integral equation is known to be ill-posed, in the sense of Hadamard. Formally, for function spaces X and Y , Kress [11] defines a well-posed problem for integral equation $LI = e$: if for any $e \in Y$, the problem has a unique solution $I \in X$, such that

$$\|I\|_X \leq C \|e\|_Y \quad (16)$$

for a constant $C > 0$. This implies that small variations in e result in limited excursions in I . By (16), $\|L^{-1}e\| \leq C\|e\|$, and $\|LI\| \geq (1/C)\|I\|$ for all $I \in X$; we can conclude that L is bounded below. As a result, it possesses a continuous (bounded) inverse that depends continuously on e . As established in the prior section, the second kind compact-plus-identity MFIE operator has a bounded inverse and is therefore well-posed. The EFIE, on the other hand is not. This fact has been exploited in several different approaches, including the Method of Analytical Regularization (MAR) [33], also known as Nosich' semi-inversion technique [34], and the method of Burton and Miller [35]. The latter is particularly interesting, being applied to the scalar scattering problem and Singular Function Expansions [36, 37].

Adams [16] and Hsiao [21] presented insightful analyses into the ill-posedness of the EFIE operator, in contrast to its MFIE counterpart. The EFIE imposes a boundary condition on magnetic currents in terms of electric sources, an impedance-type mapping, i.e., operation $T\mathbf{J}$ maps the electric current \mathbf{J} into a magnetic current \mathbf{M} . The MFIE maps electric current into electric current. Recent developments [15–18] have shown that the impedance-type mapping effected by T can be stabilized by using an admittance pre-multiplier, a preconditioning approach. The net effect is that the preconditioning causes the first kind operator T to be converted to a second kind operator, mapping electric current into electric current, as is the case with the MFIE.

Pre-multiplying the EFIE (5) by T yields $T^2\mathbf{J} = T\mathbf{M}^i$. It has been demonstrated [15] that the application of the composite operator T^2 , as opposed to “just” T , provides a well-posed formulation of the EFIE, a result of the smoothing action of the integral operator. Operator T^2 maps the ill-posed EFIE into a second-kind form, giving Adams' MEFIE (Modified EFIE) [16]. To differentiate various preconditioners, we use the term “Calderon Preconditioner” in this text, in reference to its origin.

The numerical implementation of MEFIE for certain canonical problems and details of the discretization of the operator product, the T^2 operator, based on Helmholtz decompositions are considered in [16, 18, 38, 39]. For the Helmholtz decomposition of the EFIE operator as follows:

$$T = T_s + T_h \quad (17)$$

$$T_s\mathbf{J} = i\omega\mu \int g(\mathbf{r}, \mathbf{r}')\mathbf{J}(\mathbf{r}')ds' \quad (18)$$

$$T_h\mathbf{J} = -\frac{i\omega\mu}{k^2} \int \nabla'g(\mathbf{r}, \mathbf{r}')\nabla'_t \cdot \mathbf{J}(\mathbf{r}')ds' \quad (19)$$

Cauchy singular operator T is decomposed into T_s , a smoothing

operator and T_h , a hypersingular operator (given the differential operator acting on the singular Green's function). The Helmholtz decomposition of T^2 as $(T_s T_s + T_s T_h + T_h T_s + T_h T_h)$ can be used to understand how discretization can maintain stabilization properties. The square of the hypersingular operator T_h needs to be removed, i.e., $T_h^2 = 0$ to ensure a stable (well-conditioned) solution. The composite operators that remain have been demonstrated to smooth the resulting second kind equation.

For Method of Moments discretization, the projection of operator T^2 into a finite dimensional subspace can result in $T_h^2 \neq 0$. In [18], subspace mapping considerations were used to construct a discretization using low-order divergence conforming RWG (Rao-Wilton-Glisson) basis elements that maintained a zero hypersingular-squared operator. The resulting solution requires 7 matrices to be used, and therefore carries significant computational burden.

5. SOLVING MFIE AND EFIE

Marin presented a comprehensive analysis of the solution to the MFIE based on the Fredholm Determinant theory and Carleman's method [13, 14, 40]. This work is recognized to have given the SEM a stronger mathematical foundation. The method is reviewed in this section and extended to the EFIE case as well, by first preconditioning the EFIE per operator relation (11), yielding the MEFIE. It will also be revealed that Marin's approach includes a preconditioning of the MFIE, for different reasons than those of the EFIE. We term this form the Modified MFIE (MMFIE).

Through the identical Calderon preconditioner, both the (M)EFIE and (M)MFIE can be shown to be stable integral equations with solutions that can be approximated by a Singularity Expansion Method (SEM), giving series of poles and residues in the frequency domain form and series of damped exponentials in time domain form.

As discussed in Section 2, the compact-plus-identity MFIE has desirable properties, namely being well-posed with a bounded inverse, i.e., readily solvable. To apply Carleman's method, based on the Fredholm Determinant theory, an additional property is required of the operator, namely to be Hilbert-Schmidt. While the compact-plus-identity operator equation is readily solvable, this additional property is needed to ensure that the recursive series expansions of the Fredholm Determinant method apply and that the series converges. For integral operator $K : L^2(S) \rightarrow L^2(S)$ to be Hilbert-Schmidt it requires a finite

norm-squared, defined by

$$\|K\|^2 \leq \int_S \int_S |k(\mathbf{x}, \mathbf{y})|^2 ds_x ds_y < \infty \quad (20)$$

where surface $S \subset \mathbb{R}^2$, and $k(\mathbf{x}, \mathbf{y})$ is the integral kernel corresponding to operator K . While K in the MFIE is bounded and compact, it does not satisfy (20) and therefore is not Hilbert-Schmidt [41, pp. 162–165]; it can be shown that operator K^2 and T^2 are. It is noteworthy that Hilbert-Schmidt operators are compact, but that the reverse does not apply [42].

Multiplying both sides of (6) by $(1/2 I - K)$, the tangential current density is the solution of the equation

$$(-1/4I + K^2)\mathbf{J} = [(-1/2I + K)]\mathbf{J}^i \quad (21)$$

which we refer to as the Modified MFIE (MMFIE). Multiplying both sides of (5) by T and applying (11), we have the Modified EFIE (MEFIE)

$$(-1/4I + K^2)\mathbf{J} = T\mathbf{M}^i \quad (22)$$

The solutions of the MMFIE and MEFIE are therefore

$$\mathbf{J} = (-1/4I + K^2)^{-1} [(-1/2I + K)\mathbf{J}^i] \quad (23)$$

$$\text{and } \mathbf{J} = (-1/4I + K^2)^{-1} [T\mathbf{M}^i] \quad (24)$$

We briefly (and formally) explore the Fredholm approach to solving these integral equations, requiring similar recursive determinant terms. Consider a one-dimensional (in space) “structure” in some medium such that $S = [u_1, u_n]$ with n equispaced subdivisions. There is a scalar current distribution on S written in matrix notation $\mathbf{x} = [x(u_1), x(u_2), \dots, x(u_n)]$ and a modified excitation $\mathbf{y} = [y(u_1), y(u_2), \dots, y(u_n)]$. We have already established that the same integral kernels are used for both MMFIE and MEFIE and use a common matrix approximation $Z = 4K^2$. Further, the kernel is an $n \times n$ matrix $\mathbf{Z} = \mathbf{Z}(u_i, u_j)$ with $i, j = 1, 2, \dots, n$. The differential length in our integral equation is approximated by the subdivision of width δ_n . The infinite-dimensional MxFIE integral equation (c.f. Eq. (21) or (22)) is therefore approximated by the finite-dimensional matrix equation

$$\mathbf{x} = \mathbf{y} + \lambda\delta_n\mathbf{Z}\mathbf{x} \quad (25)$$

which has a unique solution if determinant

$$d_n(\lambda) = \det(\mathbf{I} - \lambda\delta_n\mathbf{Z}) \neq 0 \quad (26)$$

Consistent with Cramer's theorem, this solution is

$$\mathbf{x} = \frac{1}{d_n(\lambda)} \text{adj}(\mathbf{I} - \lambda\delta_n\mathbf{Z})\mathbf{y} \quad (27)$$

The determinant can be expanded in polynomial terms, which in the limit as $n \rightarrow \infty$ is

$$\begin{aligned} d(\lambda) = \lim_{n \rightarrow \infty} d_n(\lambda) &= 1 - \lambda \int_S Z(u, u) du \\ &+ \frac{\lambda^2}{2!} \int_S \int_S \begin{vmatrix} Z(u, u) & Z(u, v) \\ Z(v, u) & Z(v, v) \end{vmatrix} du dv \\ &- \frac{\lambda^3}{3!} \int_S \int_S \int_S \begin{vmatrix} Z(u, u) & Z(u, v) & Z(u, w) \\ Z(v, u) & Z(v, v) & Z(v, w) \\ Z(w, u) & Z(w, v) & Z(w, w) \end{vmatrix} du dv dw + \dots \end{aligned} \quad (28)$$

The adjoint is determined as usual by using the minors of the matrix. We can therefore derive a similar expression for $D_\lambda(s, t)$. For this one-dimensional example, the solution in the limit is [43, pp. 67–68]

$$x(u) = y(u) + \lambda \int_S H_\lambda(u, v) y(v) dv = y(u) + \frac{\lambda}{d(\lambda)} \int_S D_\lambda(u, v) y(v) dv \quad (29)$$

If $d(\lambda) \neq 0$, λ is a regular value of $Z(u, v)$ and the resolvent $H_\lambda(u, v)$ is given by:

$$H_\lambda(u, v) = \frac{D_\lambda(u, v)}{d(\lambda)} \quad (30)$$

The successive approximation method used in the determinant theory shows that the solution can be written

$$x(u) = y(u) + \sum_{m=1}^{\infty} \lambda^m \int_S Z_m(u, v) y(v) dv \quad (31)$$

where $Z_m(u, v)$ is defined recursively,

$$Z_m(u, v) = \int_S Z(u, w) Z_{m-1}(w, v) dw \quad (32)$$

For the second kind integral in (25), it is the integral kernel operator Z that needs to be Hilbert-Schmidt. The identity operator does not contribute to the series convergence, instead giving the first term on the

right hand side of (31). Therefore, the identity-plus-compact operator solved using the determinant theory requires the compact operator be Hilbert-Schmidt to ensure convergence of the solution.

By the Fredholm Determinant theory and Carleman's method, we can define the modified Fredholm determinant, $\delta(\lambda)$, and the modified first Fredholm minor of $Z(u, v)$, denoted $D_\lambda(u, v)$

$$d(\lambda) = \sum_{n=0}^{\infty} d_n \lambda^n \quad (33)$$

$$D_\lambda(u, v) = \sum_{n=0}^{\infty} D_n(u, v) \lambda^n \quad (34)$$

Smithies [43, pp. 65–105] showed that the series were convergent for all complex λ ; Carleman's contribution was in proving that these expansions applied under the sole assumption that the operator Z was Hilbert-Schmidt. We can extend this one-dimensional approach to the solution of the general MEFIE and MMFIE in terms of modified Fredholm minors and determinants; we now consider functions in $L^2(S)$ where S is now a surface in a three-dimensional space. The validity of this extension from a scalar case to a higher-dimensional function space was established by Marin in the appendix of his paper, based on a technique developed by Goursat [44, pp. 152–165]. For the MMFIE the solution is written in the form

$$\mathbf{J} = \left(I + \frac{D}{d} \right) \left[\left(-\frac{1}{2}I + K \right) \mathbf{J}^i \right] \quad (35)$$

where $D \equiv D(\mathbf{r}, \mathbf{r}'; s)$ is the modified first Fredholm minor and $d \equiv d(s)$ the modified Fredholm determinant with complex frequency, s and we set $\lambda = 1$. The tangential current density solution for the MEFIE can be written in the same way as earlier,

$$\mathbf{J} = \left(I + \frac{D}{d} \right) [T\mathbf{M}^i] \quad (36)$$

Clearly, the $(-1/2I + K)$ preconditioner applied to the MFIE can be interpreted as the Calderon preconditioner $(-1/4I + K^2)$ applied to a modified excitation term, namely $(-1/2I + K)\mathbf{J}^i$ versus \mathbf{J}^i . The Calderon preconditioner, creating the MEFIE and MMFIE forms, establishes a common mathematical basis; following the approach of Marin, this leads to an SEM formulation, as demonstrated in the next section. We also examine the details of the Fredholm minor and determinant terms for both MEFIE and MMFIE forms, leading to this result.

6. THE COMMON SEM SOLUTION

The resolvent operator $d(s)^{-1}D(\mathbf{r}, \mathbf{r}'; s)$ is common to both MMFIE and MEFIE with recursive Fredholm minor and determinant terms having been generated by recursive integrals using the analytical Fredholm integral theory (and forming a Neumann series). Formally, the modified determinant and minor are [14, 43, pp.71–101, and 45, pp. 257–286]

$$d(s) = \sum_{n=0}^{\infty} d_n(s) \tag{37}$$

$$D(\mathbf{r}, \mathbf{r}'; s) = \sum_{n=0}^{\infty} D_n(\mathbf{r}, \mathbf{r}'; s) \tag{38}$$

where

$$d_n(s) = \frac{(-1)^n}{n!} \int_{S_{\tau_n}} \dots \int_{S_{\tau_1}} \begin{vmatrix} 0 & Z(\tau_1, \tau_2) & \dots & Z(\tau_1, \tau_n) \\ Z(\tau_2, \tau_1) & 0 & \dots & Z(\tau_2, \tau_n) \\ \dots & \dots & \dots & \dots \\ Z(\tau_n, \tau_1) & Z(\tau_n, \tau_2) & \dots & 0 \end{vmatrix} d\tau_1 d\tau_2 \dots d\tau_n \tag{39}$$

and

$$D_n(\mathbf{r}, \mathbf{r}'; s) = \frac{(-1)^n}{n!} \int_{S_{\tau_n}} \dots \int_{S_{\tau_1}} \begin{vmatrix} Z(s, t) & Z(s, \tau_1) & \dots & Z(s, \tau_n) \\ Z(\tau_1, t) & 0 & \dots & Z(\tau_1, \tau_n) \\ \dots & \dots & \dots & \dots \\ Z(\tau_n, t) & Z(\tau_n, u_1) & \dots & 0 \end{vmatrix} d\tau_1 d\tau_2 \dots d\tau_n \tag{40}$$

$d_0(s) = 1$ and $D_0(\mathbf{r}, \mathbf{r}'; s) = 4K^2(\mathbf{r}, \mathbf{r}'; s)$, where

$$4K^2 \mathbf{J} = 4 \int_S \int_S K(\mathbf{r}, \mathbf{r}''; s) K(\mathbf{r}'', \mathbf{r}'; s) \cdot \mathbf{J}(\mathbf{r}') ds'' ds' \tag{41}$$

with

$$K\mathbf{J} = -\frac{\mathbf{n}(\mathbf{r})}{4\pi} \times \int_S \nabla' g(\mathbf{r}, \mathbf{r}') \times \mathbf{J} ds' \tag{42}$$

We exclude the singular point at $\mathbf{r} = \mathbf{r}'$. These formal representations are of limited value in their application to engineering problems, however, can be suitably approximated due to their interesting properties as functions in the complex plane.

The analytic Fredholm theorem states that for the family of compact operators $A(s)$ on an open, connected subset of the complex plane S ; either $(I - A(s))$ is nowhere invertible in S or $(I - A(s))^{-1}$ is meromorphic in S [24, 46]. In a finite region of the complex plane, there are a finite number of poles. $D(\mathbf{r}, \mathbf{r}'; s)$ is an operator-valued analytic function of s , and $d(s)$ an analytic function of s ; both are convergent for all s [43, pp. 30–31]. It follows that $(-1/4I + K^2)^{-1}$ is an analytic operator valued function, except at the zeros of $d(s)$, where it has poles. The Mittag-Leffler theorem asks the question: is a function uniquely specified by its singular points and the coefficients of its Laurent series? The theorem statement is essentially that one can always construct a meromorphic function $f(z)$ with principal parts $G_n(z)$ of the Laurent expansion at an infinite sequence of prescribed poles provided that the sequence of poles approaches infinity [47]. Any such function can be written as

$$f(z) = \phi(z) + \sum_1^{\infty} (G_n(z) + Q_n(z)) \quad (43)$$

where $\phi(z)$ is an entire function and $\{Q_n(z)\}$ are polynomials that guarantee the convergence of the expansion. For the scattering problem, the entire function contributions are required to ensure convergence of the series for the early time/high-frequency components. This accounts for transient effects during the interval in which the object is responding to the leading edge of an incident field traversing it. Baum grouped the polynomial summands and $\phi(z)$ together into a single entire function term, appended to the SEM pole series. Pearson raised concerns regarding convergence due to separating polynomial terms from their respective poles [47], making this approach questionable for the early time case. Two classes of coefficients for accommodating the entire function are detailed below.

As demonstrated in the Appendix, the class 2 coupling coefficients for both the MFIE and EFIE forms can be derived by Laurent and Taylor series expansions around $s = s_n$. The tangential current distribution therefore follows for these two cases as

$$\mathbf{J}^{e,m}(\mathbf{r}, s) = \sum_{\alpha} \left[\frac{\Psi_{\alpha}^{e,m}(s) \mathbf{N}_{\alpha}^{e,m}(\mathbf{r})}{(s - s_{\alpha})^{m_{\alpha}}} + \mathbf{J}_{\alpha}^{e,m}(\mathbf{r}, s) \right] + \mathbf{J}_{np}^{e,m}(\mathbf{r}, s) \quad (44)$$

$$\text{with } \Psi_{\alpha}^{e,m}(s) = \frac{\langle \mathbf{C}_{\alpha}^{e,m}(\mathbf{r}); \mathbf{I}^{e,m}(\mathbf{r}, s) \rangle}{\langle \mathbf{C}_{\alpha}^{e,m}(\mathbf{r}); \Gamma_{1_{\alpha}}^{e,m}(\mathbf{r}, \mathbf{r}'); \mathbf{N}_{\alpha}^{e,m}(\mathbf{r}') \rangle} \quad (45)$$

where we use the e, m superscript to denote the EFIE and MFIE forms, respectively. Excitation terms are given by $\mathbf{I}^e(\mathbf{r}, s) = \mathbf{M}^i(\mathbf{r}, s)$ and

$\mathbf{I}^m(\mathbf{r}, s) = \mathbf{J}^i(\mathbf{r}, s)$. $\Gamma_{1\alpha}^{e,m}(\mathbf{r}, \mathbf{r}')$ is the derivative in s of the applicable kernel (c.f. Eqs. (2) and (4)). The coupling coefficients are stated explicitly in (A19) through (A23). Comparing to (43), the $\mathbf{J}_\alpha^{e,m}(\mathbf{r}, s)$ term is the polynomial entire function ensuring convergence of the series, and $\mathbf{J}_{mp}^{e,m}(\mathbf{r}, s)$ corresponding to entire function $\phi(z)$.

The solution of the MMFIE coincides with the solution of the MFIE for all values of s for which the inverse operators exist. By the analytic Fredholm theorem; $(1/2I - K)^{-1}$ and $(-1/4I + K^2)^{-1}$ are analytic operator-valued functions of s , except at finite $s = s_n$ where they have the same poles. The coupling coefficients, as determined by Marin and Baum, are based on the “original” integral equation, namely the MFIE. It’s denominator is a function of the MFIE kernel. The EFIE coupling coefficient, $\Psi_\alpha^e(s)$ in (45), still contains a singular integral and may possibly present numerical instability. Modified coupling coefficients can be derived that are based on series expansions of the modified kernels (after application of the preconditioner).

Let $P(\mathbf{r}, \mathbf{r}'; s) \equiv [-1/4I + K^2](\mathbf{r}, \mathbf{r}'; s)$ denote the preconditioning operator. It’s derivative with respect to s , evaluated at s_α is

$$P_{1\alpha}(\mathbf{r}, \mathbf{r}') = \left. \frac{\partial}{\partial s} P(\mathbf{r}, \mathbf{r}'; s) \right|_{s=s_\alpha} \quad (46)$$

As detailed in the Appendix, the class 2 modified coupling coefficients for the MMFIE and MEFIE are

$$\Phi_\alpha^m(s) = \frac{\langle \mathbf{U}_\alpha(\mathbf{r}); [-1/2I + K]_\alpha(\mathbf{r}, \mathbf{r}'); \mathbf{J}_\alpha^i(\mathbf{r}, s) \rangle}{\langle \mathbf{U}_\alpha(\mathbf{r}); P_{1\alpha}(\mathbf{r}, \mathbf{r}'); \mathbf{V}_\alpha(\mathbf{r}') \rangle} \quad (47)$$

$$\Phi_\alpha^e(s) = \frac{\langle \mathbf{U}_\alpha(\mathbf{r}); T_\alpha(\mathbf{r}, \mathbf{r}'); \mathbf{M}_\alpha^i(\mathbf{r}, s) \rangle}{\langle \mathbf{U}_\alpha(\mathbf{r}); P_{1\alpha}(\mathbf{r}, \mathbf{r}'); \mathbf{V}_\alpha(\mathbf{r}') \rangle} \quad (48)$$

where $[-1/2I + K]_\alpha(\mathbf{r}, \mathbf{r}') \equiv [-1/2I + K](\mathbf{r}, \mathbf{r}'; s)|_{s=s_\alpha}$ and $T_\alpha(\mathbf{r}, \mathbf{r}') = T(\mathbf{r}, \mathbf{r}'; s)|_{s=s_\alpha}$. As demonstrated in the Appendix, the natural modes and coupling vectors are derived from the homogeneous solution of the integral equations (c.f. (A25) and (A26)) and their adjoints at $s = s_\alpha$. Since the LHS of both of these equations is the same, the natural modes of the MEFIE and MMFIE are equal, denoted $\mathbf{U}_\alpha(s)$. The same applies to the coupling vectors, $\mathbf{V}_\alpha(s)$. This differs from the EFIE and MFIE coupling coefficients where natural mode and coupling vectors are not common (the superscripts in (45) denote the different term for EFIE and MFIE case).

For the class 2 coefficients in (45), the frequency dependence of the $\Psi_\alpha^{e,m}(s)$ comes from $\mathbf{I}^{e,m}(\mathbf{r}, s)$. In the time domain, this corresponds to smoothing out the rise time of the α -th pole by convolution [3]. For

incident radiation or an applied source, Baum introduced the “turn-on time” t' at which the pole series is allowed to begin contributing to the representation of the surface current induced on a scattering object. If the turn-on time is chosen later than the time at which the actual response begins, then the entire function contribution must “fill the gap” between the time that the response begins and the time that the pole series contributions are allowed to contribute to the representation [3, 47]. The EFIE and MFIE class 1 coefficient is thus

$$\Psi_\alpha^{e,m}(s_\alpha) = \frac{\langle \mathbf{C}_\alpha^{e,m}(\mathbf{r}); \mathbf{I}_{0_\alpha}^{e,m}(\mathbf{r}) \rangle}{\langle \mathbf{C}_\alpha^{e,m}(\mathbf{r}); \Gamma_{1_\alpha}^{e,m}(\mathbf{r}, \mathbf{r}'); \mathbf{N}_\alpha^{e,m}(\mathbf{r}) \rangle} e^{(s_\alpha - s)t'} \quad (49)$$

where the current term $\mathbf{I}_{0_\alpha}^{e,m}(\mathbf{r})$ is evaluated at the pole. The same expression holds for the modified coupling coefficients, $\Phi_\alpha^{e,m}(s_\alpha)$. Various asymptotic techniques have been suggested for handling the early time contributions, including physical optics methods [48] and Geometrical Theory of Diffraction [49]. The entire function is not needed for the late-time description due to the early time effects having subsided. Class 2 coupling coefficients are more complicated to calculate than class 1 coefficients, however give smoother early-time results for a finite number of poles when included in the numerical summation, due to the smoother rise of resulting pole terms in the time domain [50].

Application of Laplace or Fourier Transformation, as required, gives the late-time form equivalent of (44) [3],

$$\mathbf{J}(\mathbf{r}, t) = u(t - t') \sum_\alpha \Psi_\alpha^{e,m}(s_\alpha) \mathbf{N}_\alpha^{e,m}(\mathbf{r}) e^{s_\alpha t} \quad (50)$$

where $u(t - t')$ is a Heaviside unit step function at $t = t'$.

Since operator $(-1/4I + K^2)$ applies to both the MMFIE and MEFIE case, with different modified excitation terms, the only difference between the MEFIE-based solution and the MMFIE-based one is the coupling coefficient, $\Psi_\alpha^{e,m}(s_\alpha)$. The solution is stable, in the sense of Hadamard, with the ill-posed first kind EFIE removed by the Calderon preconditioning, and both MEFIE and MMFIE forms having bounded norm-square on L^2 .

In many applications, an abstracted form is used, where the numerator is denoted $R(s_\alpha)$ such that

$$\mathbf{J}(\mathbf{r}, s) = \sum_\alpha \frac{R(s_\alpha)}{(s - s_\alpha)} = \frac{b_0 + b_1 s + \dots + b_n s^n}{a_0 + a_1 s + \dots + a_d s^d} \quad (51)$$

Applications in the frequency domain employ this reduced order model to approximate some system response, or transfer function by poles and

residues (numerator and denominator coefficients, b_k, a_k). The order of the rational function polynomials, or the commensurate number of poles and residues, are examined in other documents [9, 51, 52]. Model-based parameter estimation techniques arise from this abstract form [53].

Most recent (last 15 years) applications of the SEM do not typically calculate coupling coefficients in terms of the complex underlying recursive determinant integrals. Instead, the abstracted form is used, and poles and residues found using standard techniques of Prony, Cauchy and Newton-Raphson. Prony's methods, in the time or frequency domain are most popular. Sensitivity to input time or frequency samples have been handled by the modified LS-Prony [54] and TLS-Prony [55] methods, and other techniques based on the use of the singular value decomposition [56]. The Matrix Pencil Method (MPM) [7] has also been used for applications in low signal-to-noise ratio environments. Iterative search methods based on Newton-Raphson methods are also applied [57].

7. CONCLUSION

The integral equations for radiation and scattering from a perfectly conducting object in a homogeneous medium were stated based on the Stratton-Chu representations as EFIE and MFIE operator representations. The Calderon projectors were used to expose the linkage between the operator EFIE and MFIE, through operator relation $(1/4I - K^2) = -T^2$. When applied to the EFIE, giving Adams' MEFIE, this operator preconditions the ill-posed first kind equation, mapping it to a second kind form. It is also the same operator applied by Marin in the mathematical foundations for the SEM.

Tangential current density, the solution for both MEFIE and MMFIE, is defined by the Calderon preconditioner $((-1/4I + K^2)^{-1})$ acting on a modified incident excitation term. The resolvent operators for the MMFIE and MEFIE are identical, with modified Fredholm minors and determinants specified in terms of recursive integral representations. The second kind equations can be solved using Carleman's method (following Marin) leading to a SEM formulation. A common SEM form was shown to exist, with MEFIE- and MMFIE-specific coupling coefficients.

Through this "preconditioning" relation, both the MEFIE and MMFIE can be shown to be stable integral equations with solutions that can be approximated by a Singularity Expansion Method (SEM), giving series of poles and residues in the frequency domain form and series of damped exponentials in time domain form. Abstract time and

frequency domain methods based on Prony's method or Model-Based Parameter Estimation (MBPE) are therefore valid approaches.

A common representation was given for both EFIE and MFIE coupling coefficients, and the more complex modified coupling coefficients for the MMFIE and MEFIE defined. An analysis and comparison of the various coupling coefficients and the trades inherent in the additional complexity versus the improved numerical conditioning would be most interesting. Additional work in this area would be beneficial in answering questions about validity of EFIE coupling coefficients, as well as practical application of modified coupling coefficients.

APPENDIX A.

In this Appendix, a generic form of coupling coefficient will be derived that is consistent with the MFIE approach in the literature today. Thereafter, forms specific to the EFIE will be defined, and a new modified coupling coefficient applicable to MEFIE and MMFIE derived.

The starting point for these derivations is the homogeneous equations for the scattering problem, defined in terms of an impedance operator, $\Gamma(\mathbf{r}, \mathbf{r}'; s)$ and natural mode and coupling vectors, $\mathbf{N}_\alpha(\mathbf{r})$ and $\mathbf{C}_\alpha(\mathbf{r})$.

$$\langle \Gamma(\mathbf{r}, \mathbf{r}'; s_\alpha); \mathbf{N}_\alpha(\mathbf{r}') \rangle = 0 \quad (\text{A1})$$

and

$$\langle \mathbf{C}_\alpha(\mathbf{r}); \Gamma(\mathbf{r}, \mathbf{r}'; s_\alpha) \rangle = 0 \quad (\text{A2})$$

where $\Gamma(\mathbf{r}, \mathbf{r}'; s_\alpha)$ is the kernel of the either the EFIE or MFIE, defined in Section 2, evaluated at $s = s_\alpha$. For an excitation $\mathbf{I}(\mathbf{r}, s)$, the inhomogeneous equation is

$$\langle \Gamma(\mathbf{r}, \mathbf{r}'; s); \mathbf{J}(\mathbf{r}', s) \rangle = \mathbf{I}(\mathbf{r}, s) \quad (\text{A3})$$

Expanding this integral equation near $s = s_\alpha$ using the Taylor series formula,

$$\Gamma(\mathbf{r}, \mathbf{r}'; s) = \sum_{m=0}^{\infty} (s - s_\alpha)^m \times \frac{1}{m!} \frac{\partial^m}{\partial s^m} \Gamma(\mathbf{r}, \mathbf{r}'; s) \Big|_{s=s_\alpha} \quad (\text{A4})$$

$$\mathbf{I}(\mathbf{r}, s) = \sum_{m=0}^{\infty} (s - s_\alpha)^m \times \frac{1}{m!} \frac{\partial^m}{\partial s^m} \mathbf{I}(\mathbf{r}, s) \Big|_{s=s_\alpha} \quad (\text{A5})$$

Let

$$\Gamma_{m\alpha}(\mathbf{r}, \mathbf{r}') = \frac{1}{m!} \frac{\partial^m}{\partial s^m} \Gamma(\mathbf{r}, \mathbf{r}'; s) \Big|_{s=s_\alpha} \quad (\text{A6})$$

$$\text{and } \mathbf{I}_{m\alpha}(\mathbf{r}) = \frac{1}{m!} \frac{\partial^m}{\partial s^m} \mathbf{I}(\mathbf{r}, s) \Big|_{s=s_\alpha} \quad (\text{A7})$$

Assuming that there is a singularity at $s = s_\alpha$, corresponding to a natural frequency, a Laurent series expansion is required. The general form, for some $z, z_0 \in \mathbb{C}$ is

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \quad (\text{A8})$$

defined in the immediate neighbourhood of z_0 , a region of the form $0 < |z - z_0| < R$, where $R \in \mathbb{C}$. Separating the pole in the Mittag-Leffler series expansion at s_α ,

$$\mathbf{J}(\mathbf{r}, s) = \frac{\Psi_\alpha(s) \mathbf{N}_\alpha(\mathbf{r})}{(s - s_\alpha)^{m_\alpha}} + \mathbf{J}_\alpha(\mathbf{r}, s) \quad (\text{A9})$$

The second term, $\mathbf{J}_\alpha(\mathbf{r}, s)$, corresponds to the entire function contribution (c.f. Eq. (44)). Using Baum's assumptions of a single pole approximation, $m_\alpha = 1$.

Substituting (A4), (A5) and (A9) into (A3),

$$\begin{aligned} & \left\langle \sum_{m=0}^{\infty} (s - s_\alpha)^m \times \frac{1}{m!} \frac{\partial^m}{\partial s^m} \Gamma(\mathbf{r}, \mathbf{r}'; s) \Big|_{s=s_\alpha} ; \frac{\Psi_\alpha(s) \mathbf{N}_\alpha(\mathbf{r}')}{s - s_\alpha} + \mathbf{J}'(\mathbf{r}, s) \right\rangle \\ &= \sum_{m=0}^{\infty} (s - s_\alpha)^m \times \frac{1}{m!} \frac{\partial^m}{\partial s^m} \mathbf{I}(\mathbf{r}, s) \Big|_{s=s_\alpha} \end{aligned} \quad (\text{A10})$$

Therefore,

$$\begin{aligned} & \left\langle [(s - s_\alpha)^0 \Gamma_{0\alpha}(\mathbf{r}, \mathbf{r}') + (s - s_\alpha) \Gamma_{1\alpha}(\mathbf{r}, \mathbf{r}') + \dots] ; \frac{\Psi_\alpha(s) \mathbf{N}_\alpha(\mathbf{r}')}{s - s_\alpha} + \mathbf{J}'(\mathbf{r}, s) \right\rangle \\ &= (s - s_\alpha)^0 \mathbf{I}_{0\alpha}(\mathbf{r}) + (s - s_\alpha) \mathbf{I}_{1\alpha}(\mathbf{r}, \mathbf{r}') + \dots \end{aligned} \quad (\text{A11})$$

Combining terms corresponding to $(s - s_\alpha)^{-1}$,

$$\langle \Gamma_{0\alpha}(\mathbf{r}, \mathbf{r}'); \Psi_\alpha(s) \mathbf{N}_\alpha(\mathbf{r}') \rangle = 0 \quad (\text{A12})$$

Combining terms in $(s - s_\alpha)^0$,

$$\langle \Gamma_{1\alpha}(\mathbf{r}, \mathbf{r}'); \Psi_\alpha(s) \mathbf{N}_\alpha(\mathbf{r}') \rangle + \langle \Gamma_{0\alpha}(\mathbf{r}, \mathbf{r}'); \mathbf{J}'(\mathbf{r}', s) \rangle = \mathbf{I}_{0\alpha}(\mathbf{r}) \quad (\text{A13})$$

Left operating by $\mathbf{C}_\alpha(\mathbf{r})$, the second term disappears, since

$$\langle \mathbf{C}_\alpha(\mathbf{r}); \Gamma(\mathbf{r}, \mathbf{r}'; s_\alpha) \rangle = 0 \quad (\text{A14})$$

$$\text{and} \quad \Gamma_{0\alpha}(\mathbf{r}, \mathbf{r}') = \Gamma(\mathbf{r}, \mathbf{r}'; s_\alpha) \quad (\text{A15})$$

Therefore,

$$\Psi_\alpha(s_\alpha) = \frac{\langle \mathbf{C}_\alpha(\mathbf{r}); \mathbf{I}_{0\alpha}(\mathbf{r}) \rangle}{\langle \mathbf{C}_\alpha(\mathbf{r}); \Gamma_{1\alpha}(\mathbf{r}, \mathbf{r}'); \mathbf{N}_\alpha(\mathbf{r}') \rangle} \quad (\text{A16})$$

This defines the coupling coefficient at s_α . Following [58], the coupling coefficient as a function of all s can be calculated using the preceding method, but not performing the Taylor expansion in (A5). The class 2 coupling coefficient is so defined as

$$\Psi_\alpha(s) = \frac{\langle \mathbf{C}_\alpha(\mathbf{r}); \mathbf{I}(\mathbf{r}, s) \rangle}{\langle \mathbf{C}_\alpha(\mathbf{r}); \Gamma_{1\alpha}(\mathbf{r}, \mathbf{r}'); \mathbf{N}_\alpha(\mathbf{r}') \rangle} \quad (\text{A17})$$

For the MFIE, the natural modes and coupling vectors are $\mathbf{N}_\alpha^m(\mathbf{r})$ and $\mathbf{C}_\alpha^m(\mathbf{r})$ and the class 2 coupling coefficient defined by

$$\Psi_\alpha^m(s) = \frac{\langle \mathbf{C}_\alpha^m(\mathbf{r}); \mathbf{I}(\mathbf{r}, s) \rangle}{\langle \mathbf{C}_\alpha^m(\mathbf{r}); \Gamma_{1\alpha}^m(\mathbf{r}, \mathbf{r}'); \mathbf{N}_\alpha^m(\mathbf{r}') \rangle} \quad (\text{A18})$$

where the denominator term is given by

$$\begin{aligned} \langle \mathbf{C}_\alpha^m(\mathbf{r}); \Gamma_{1\alpha}^m(\mathbf{r}, \mathbf{r}'); \mathbf{N}_\alpha^m(\mathbf{r}') \rangle = & (-4\pi c)^{-1} \int_S \int_S \mathbf{C}_\alpha^m(\mathbf{r}) [\mathbf{n}(\mathbf{r}) \\ & \times \nabla \{ \exp(-s_n R/c) \} \times \mathbf{N}_\alpha^m(\mathbf{r}')] dS' dS \end{aligned} \quad (\text{A19})$$

with $R = |\mathbf{r} - \mathbf{r}'|$ and the numerator

$$\langle \mathbf{C}_\alpha^m(\mathbf{r}); \mathbf{I}(\mathbf{r}, s) \rangle = \int_S \mathbf{C}_\alpha^m(\mathbf{r}) \mathbf{I}(\mathbf{r}, s) dS \quad (\text{A20})$$

For the EFIE, the coupling coefficient is

$$\Psi_\alpha^e(s) = \frac{\langle \mathbf{C}_\alpha^e(\mathbf{r}); \mathbf{I}(\mathbf{r}, s) \rangle}{\langle \mathbf{C}_\alpha^e(\mathbf{r}); \Gamma_{1\alpha}^e(\mathbf{r}, \mathbf{r}'); \mathbf{N}_\alpha^e(\mathbf{r}') \rangle} \quad (\text{A21})$$

For $g(\mathbf{r}, \mathbf{r}') = (4\pi R)^{-1} \exp(-sR/c)$, we have $\nabla' g(\mathbf{r}, \mathbf{r}') = g(\mathbf{r}, \mathbf{r}') [s/c + 1/R] \hat{\mathbf{R}}(\mathbf{r}, \mathbf{r}')$ where unit vector $\hat{\mathbf{R}}(\mathbf{r}, \mathbf{r}') = \mathbf{R}(\mathbf{r}, \mathbf{r}')/R$. The denominator term is thus

$$\langle \mathbf{C}_\alpha^e(\mathbf{r}); \Gamma_{1\alpha}^e(\mathbf{r}, \mathbf{r}'); \mathbf{N}_\alpha^e(\mathbf{r}') \rangle = s_\alpha \mu \int_S \int_S \left[(-4\pi c)^{-1} \exp(-s_\alpha R/c) \mathbf{N}_\alpha^e(\mathbf{r}') \right. \\ \left. + (c^2/s_\alpha^2) g(\mathbf{r}, \mathbf{r}') [s_\alpha/c + 1/R] \hat{\mathbf{R}}(\mathbf{r}, \mathbf{r}') \nabla'_t \cdot \mathbf{N}_\alpha^e(\mathbf{r}') \right] \mathbf{C}_\alpha^e(\mathbf{r}) dS' dS \quad (\text{A22})$$

As before, the numerator is

$$\langle \mathbf{C}_\alpha^e(\mathbf{r}); \mathbf{I}(\mathbf{r}, s) \rangle = \int_S \mathbf{C}_\alpha^e(\mathbf{r}) \mathbf{I}(\mathbf{r}, s) dS \quad (\text{A23})$$

The common class 2 coupling coefficient representation is

$$\Psi_\alpha^{e,m}(s_\alpha) = \frac{\langle \mathbf{C}_\alpha^{e,m}(\mathbf{r}); \mathbf{I}^{e,m}(\mathbf{r}, s) \rangle}{\langle \mathbf{C}_\alpha^{e,m}(\mathbf{r}); \Gamma_{1\alpha}^{e,m}(\mathbf{r}, \mathbf{r}'); \mathbf{N}_\alpha^{e,m}(\mathbf{r}') \rangle} \quad (\text{A24})$$

Following the same method as earlier, modified coupling coefficients for the MMFIE can be derived,

$$\langle (-1/4I + K^2); \mathbf{J} \rangle = \langle (-1/2I + K); \mathbf{J}^i \rangle \quad (\text{A25})$$

and the MEFIE

$$\langle (-1/4I + K^2); \mathbf{J} \rangle = \langle T; \mathbf{M}^i \rangle \quad (\text{A26})$$

Let $P(\mathbf{r}, \mathbf{r}'; s) \equiv (-1/4I + K^2)(\mathbf{r}, \mathbf{r}'; s)$ denote the preconditioning operator. It's derivative with respect to s , evaluated at s_α is

$$P_{1\alpha} = \left. \frac{\partial}{\partial s} P(\mathbf{r}, \mathbf{r}'; s) \right|_{s=s_\alpha} \quad (\text{A27})$$

The natural modes and coupling vectors are derived from the homogeneous solution of (A25) and (A26) and their adjoints at $s = s_\alpha$. Since the LHS of both of these equations is the same, the natural modes of the MEFIE and MMFIE are equal, denoted $\mathbf{U}_\alpha(\mathbf{s})$. The same applies to the coupling vectors, $\mathbf{V}_\alpha(\mathbf{s})$. Derived in the same manner as earlier in this Appendix, the class 2 modified coupling coefficients for the MMFIE and MEFIE are

$$\Phi_\alpha^m(s) = \frac{\langle \mathbf{U}_\alpha(\mathbf{r}); [-1/2I + K]_\alpha(\mathbf{r}, \mathbf{r}'); \mathbf{J}_\alpha^i(\mathbf{r}, s) \rangle}{\langle \mathbf{U}_\alpha(\mathbf{r}); P_{1\alpha}(\mathbf{r}, \mathbf{r}'); \mathbf{V}_\alpha(\mathbf{r}') \rangle} \quad (\text{A28})$$

$$\Phi_\alpha^e(s) = \frac{\langle \mathbf{U}_\alpha(\mathbf{r}); T_\alpha(\mathbf{r}, \mathbf{r}'); \mathbf{M}_\alpha^i(\mathbf{r}, s) \rangle}{\langle \mathbf{U}_\alpha(\mathbf{r}); P_{1\alpha}(\mathbf{r}, \mathbf{r}'); \mathbf{V}_\alpha(\mathbf{r}') \rangle} \quad (\text{A29})$$

where $[-1/2I + K]_{\alpha}(\mathbf{r}, \mathbf{r}') \equiv [-1/2I + K](\mathbf{r}, \mathbf{r}'; s)|_{s=s_{\alpha}}$ and $T_{\alpha}(\mathbf{r}, \mathbf{r}') = T(\mathbf{r}, \mathbf{r}'; s)|_{s=s_{\alpha}}$. For the class 2 coefficients in (45), the frequency dependence of the $\Psi_{\alpha}^{e,m}(s)$ comes from $\mathbf{I}^{e,m}(\mathbf{r}, s)$. Discussion of the application of these modified coupling coefficients is presented in Section 6.

ACKNOWLEDGMENT

The author wishes to thank Professor George W. Hanson for his many useful comments, suggestions, probing questions and thorough review of this paper.

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