Progress In Electromagnetics Research M, Vol. 13, 145–156, 2010

COVARIANT CONSTITUTIVE RELATIONS, LANDAU DAMPING AND NON-STATIONARY INHOMOGENEOUS PLASMAS

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Abstract—Models of covariant linear electromagnetic constitutive relations are formulated that have wide applicability to the computation of susceptibility tensors for dispersive and inhomogeneous media. A perturbative framework is used to derive a linear constitutive relation for a globally neutral plasma enabling one to describe in this context a generalized Landau damping mechanism for non-stationary inhomogeneous plasma states.

1. INTRODUCTION

Constitutive relations are widely used when describing the behaviour of electromagnetic fields in continuous media. Although their domain of applicability is often determined experimentally, causality and locality play important roles in their theoretical foundations. Limitations arise due to the inherent non-linearity contained in the classical equations describing the coupling between the motion of individual particles or continuous charge distributions and a self-consistent electromagnetic field. For small disturbances perturbative linearization techniques are available and approximation schemes exist for calculating effective susceptibility tensors that arise from these constitutive relations. The effective constitutive relations that result from such schemes often rely for their validity on assumptions such as material homogeneity and non-relativistic perturbations about stationary configurations. This letter addresses some of the issues that arise when some of these assumptions are relaxed and the degree to which concepts such as Landau damping can be generalized for relativistic inhomogeneous plasmas [1–3].

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The *macroscopic* Maxwell equations can be written as the exterior system

$$dF = 0, \qquad \epsilon_0 d \star G = -\star \widetilde{J} \tag{1}$$

where F is the Maxwell 2-form, $G = \epsilon_0 F + \Pi$ the excitation 2-form and J the source current vector on a spacetime M, in terms of the Hodge map $\star$ and metric dual $\widetilde{J}$ associated with the spacetime metric g. In this form certain 4-current sources $-d\star\Pi$ are included in G with the remaining 4-currents denoted by J. A covariant constitutive model provides functional relations between G (or Π) and F and between J and F. Relative to any unit future-pointing timelike observer vector field U on M, the forms F and G define the frame dependent electromagnetic 1-forms $e^U = i_U F$, $b^U = i_U \star F$, $d^U = i_U G$ and $h^U = i_U \star G$ so that $F = \widetilde{U} \wedge e^U - \star(\widetilde{U} \wedge b^U)$ and likewise for G.

If one restricts to causal *linear* responses a natural covariant constitutive relation is given by the non-local expression

$$\Pi[F]_{ab}(x) = \frac{1}{4}\int_{y\in J^-(x)} \chi_{abcd}(x,y) F_{ed}(y) dy^{cdef} \tag{2}$$

where $\chi_{abcd}(x,y)$ is a two-point susceptibility kernel. The events x and y are given in arbitrary coordinates with summation over Latin indices from 0 to 3 and $dy^{cdef} = dy^c \wedge dy^d \wedge dy^e \wedge dy^f$. The causal structure has been imposed by requiring that $\chi_{abcd}(x,y) = 0$ if y does not lie in the past light cone, $J^-(x)$, of x. This constitutive relation can be used to model media which are spatially inhomogeneous and temporally non-stationary and is meaningful in spacetimes containing gravitation.

To facilitate the discussion and the use of two-point tensors introduce two copies of M, denoted M_X and M_Y, with generic points $x \in M_X$ and $y \in M_Y$ coordinated by $(x^0, \dots, x^3)$ and $(y^0, \dots, y^3)$ respectively. The values $\chi_{abcd}(x,y)$ denote the coordinate components of the 4-form field χ over the product manifold $M_X \times M_Y$ in the induced coordinates $(x^0, \dots, x^3, y^0, \dots, y^3)$:

$$\chi = \frac{1}{4}\chi_{abcd}\, dx^a \wedge dx^b \wedge dy^c \wedge dy^d. \tag{3}$$

In terms of χ and the projection $p_Y : M_X \times M_Y \to M_Y$, $p_Y(x,y) = y$ Equation (2) can be written

$$\Pi[F] = \int_{M_Y} \chi \wedge p_Y^\star F. \tag{4}$$

The tensor χ has 36 independent components since dx^{ab} and dy^{cd} are antisymmetric so $\chi_{abcd} = -\chi_{bacd} = -\chi_{abdc}$.

A special case of (2) arises in Minkowski spacetime. Being parallelizable it admits a family of *translation maps* $A_z : M \to M, x \mapsto A_z(x) = x + z$ for all events x in M. This induces the translation maps $B_z : M_X \times M_Y \to M_X \times M_Y$, $B_z(x, y) = (x + z, y + z)$. Imposing this translational symmetry on χ, i.e., $B_z^\star \chi = \chi$, Equation (2) can be written

$$\Pi_{\mathrm{I}}[F]_{ab}(x) = \int_{y \in J^-(x)} X_{abcd}(x - y) F_{ef}(y) dy^{cdef} \tag{5}$$

where $X_{abcd}(x - y) = \chi_{abcd}(x, y) = \chi_{abcd}(x + z, y + z)$ for any z. Since $\Pi_{\mathrm{I}}[A_z^\star F] = A_z^\star \Pi_{\mathrm{I}}[F]$ this relation describes spatially homogeneous and temporally stationary media and remains non-local in both space and time. Such a medium exhibits both *spatial and temporal dispersion* as follows: Minkowski spacetime admits preferred *global Lorentzian* coordinate systems, (x^0, x^1, x^2, x^3), with associated cobases (dx^0, dx^1, dx^2, dx^3) in which the components of the metric are diag $(-1, 1, 1, 1)$. If one defines for any scalar ϕ the Fourier transform

$$\widehat{\phi}(k) = \int_M e^{-ik \cdot x} \, \phi(x) \, dx^{0123}$$

with respect to such a coordinate system then $\widehat{\Pi_{\mathrm{I}}[F]}_{ab}(k) = \widehat{X}_{abcd}(k) \widehat{F}_{ef}(k) \epsilon^{cdef}$ in terms of the standard constant alternating symbol ϵ^{cdef}. Such a relation can give rise to dispersion in media.

If the bulk 4-velocity of a non-accelerating medium is V, with constant components in the above coordinate system, i.e., $\nabla V = 0$, a particular model for X in (5) is given by

$$\begin{aligned} \Pi_{I_a}[F]_{ab}(x) = & \int_{y \in J^-(x)} \mathcal{P}(x - y) \Big(i_V F \wedge \widetilde{V} \Big)_{ab} (y) dy^{0123} \\ & - \star_X \int_{y \in J^-(x)} \mathcal{M}(x - y) \Big(i_V \star_Y F \wedge \widetilde{V} \Big)_{ab} (y) dy^{0123} \end{aligned} \tag{6}$$

where $\mathcal{P}$ and $\mathcal{M}$ are polarization and magnetization susceptibility scalars respectively. The Fourier transform of (6) yields the simple constitutive relations for a spatially and temporally dispersive homogeneous isotropic medium: $\widehat{d_a^V}(k) = (\epsilon_0 + \widehat{\mathcal{P}}(k)) \widehat{e_a^V}(k)$ and $\widehat{h_a^V}(k) = (\mu_0^{-1} + \widehat{\mathcal{M}}(k)) \widehat{b_a^V}(k)$. If V is not inertial ($\nabla V \neq 0$) then (6) is not a special case of (5) and its Fourier transform, although local in k, is not of the form above.

For media that lack spatial dispersion the history of the medium may give rise to *temporal dispersion* alone. This can be expressed geometrically in terms of tensor transport along the integral curves $C_x : \mathbb{R} \to M$ of the 4-velocity field V of the medium. If these curves are each parameterized by proper time τ let $\Phi_{\tau}^{\hat\tau}(x)$ be a map that transports tensors at $C_x(\tau)$ to tensors at $C_x(\hat\tau)$ along each integral curve of V. Natural choices of transport maps include Lie, parallel (with respect to some spacetime connection ∇) and Fermi-Walker transport. Different choices of $\Phi_{\tau}^{\hat\tau}(x)$ correspond to different electromagnetic responses of the medium to the disposition of the integral curves of V in the spacetime history of the medium. If $Y(z)$ denotes a tensor field mapping 2-forms at z to 2-forms at z, a constitutive relation for a spatially inhomogeneous medium may be written

$$\Pi_{\mathbb{II}}[F](C_x(\tau)) = \int_{-\infty}^{\tau} Y^{\Phi}(\tau,\hat\tau,x)\Big(\Phi_{\hat\tau}^{\tau}(x)\big(F(C_x(\hat\tau))\big)\Big)d\hat\tau \qquad (7)$$

where $Y^{\Phi}(\tau,\hat\tau,x) = \Phi_{\tau-\hat\tau}^{\tau}(x)\big(Y(C_x(\tau-\hat\tau))\big)$ is a tensor at $C_x(\tau)$. This is another special case of (2). Since

$$\Pi_{\mathbb{II}}\left[F^{\Phi}\right]\big(C_x(\hat\tau)\big) = \Phi_{\tau}^{\hat\tau}(x)\Big(\Pi_{\mathbb{II}}[F]\big(C_x(\tau)\big)\Big)$$

where $F^{\Phi}(C_x(\hat\tau)) = \Phi_{\tau}^{\hat\tau}(x)\big(F(C_x(\tau))\big)$ the medium is said to be stationary with respect to the transport map and hence V and (7) is valid in any spacetime. This generalizes the notion of a *temporally stationary* medium in a spacetime with timelike Killing vectors. The temporal dispersive properties of the medium are best defined with respect to a modified Fourier transform that remains valid in a general spacetime and employs the transport map along the family of curves describing the history of the medium. For any tensor field α and curve C_x define, at the event $C_x(0)$, the tensor:

$$\overline{\alpha}(\omega,x) = \int_{-\infty}^{\infty} e^{-i\omega\tau}\Phi_{\tau}^{0}(x)\big(\alpha(C_x(\tau))\big)d\tau.$$

Then constitutive relation $\overline{\Pi_{\mathbb{II}}[F]}(\omega,x) = \overline{Y}(\omega,x)\big(\overline{F}(\omega,x)\big)$ describes an anisotropic, spatially inhomogeneous but temporally dispersive medium.

If V is geodesic (i.e., $\nabla_V V = 0$) and $\Phi_{\hat\tau}^{\tau}(x)$ describes *parallel* transport then a particular model for Y in (7) is given by

$$\Pi_{\mathbb{II}_a}[F](C_x(\tau)) = \int_{-\infty}^{\tau}\mathcal{P}(\tau-\hat\tau,x)\Big(\Phi_{\hat\tau}^{\tau}(x)\left(i_V F\wedge\widetilde{V}\right)(C_x(\hat\tau))\Big)d\hat\tau$$
$$-\star\int_{-\infty}^{\tau}\mathcal{M}(\tau-\hat\tau,x)\Big(\Phi_{\hat\tau}^{\tau}(x)\left(i_V\star F\wedge\widetilde{V}\right)(C_x(\hat\tau))\Big)d\hat\tau \quad (8)$$

where $\mathcal{P}$ and $\mathcal{M}$ are spatially inhomogeneous polarization and magnetization susceptibility scalars respectively. This describes non-magneto-electric, spatially inhomogeneous but temporally dispersive media with constitutive relations $\overline{d^V}(\omega, x) = (\epsilon_0 + \overline{\mathcal{P}}(\omega, x))\overline{e^V}(\omega, x)$ and $\overline{h^V}(\omega, x) = (\mu_0^{-1} + \overline{\mathcal{M}}(\omega, x))\overline{b^V}(\omega, x)$.

The historic covariant constitutive relations proposed by Minkowski belong to a class of *local linear* relations on spacetime of the form

$$\Pi_{\mathbb{III}}[F](x) = \mathcal{Z}(x)\big(F(x)\big) \tag{9}$$

where the tensor $\mathcal{Z}(x)$ maps 2-forms at events x to 2-forms at x. These constitutive relations are often used or derived for the premetric formulation of electromagnetism [4, 5]. They are also used in the eikonal approximation or with the assumption that only one frequency is involved [6].

For a medium with bulk 4-velocity field V this tensor may be chosen so that $d^V(x) = (\epsilon_0 + \mathcal{P}(x))e^V(x)$ and $h^V(x) = (\mu_0^{-1} + \mathcal{M}(x))b^V(x)$ valid for inhomogeneous and non-stationary isotropic media in an arbitrary spacetime. However in a Minkowski background their Fourier transforms for non-constant $\mathcal{P}$ and $\mathcal{M}$ yield non-local constitutive relations among the Fourier components of the electromagnetic fields so do not describe normal dispersive continua.

The different constitutive relations (5)–(9) above are applicable to the phenomenological description of *linear* media that exhibit temporal and/or spatial dispersion but rely on some knowledge of the electromagnetic response of systems in either inertial or co-moving reference frames. However such relations do not encompass the effective susceptibility that arises when one applies a perturbative analysis to processes involving non-stationary, spatially inhomogeneous plasmas [7–10]. Such processes are not uncommon in astrophysical applications or in regimes where instabilities arise [11–13] from inhomogeneities such as those in laser-plasma systems. In these situations the *microscopic* Maxwell system for a neutral plasma, $dF = 0$, $\epsilon_0 d{\star}F = -{\star}\widetilde{J}$, can be coupled with equations for the charged sources J in order to effect a linearization. Since $d \star \widetilde{J} = 0$ it is convenient to write $\star\widetilde{J} = -d \star \Pi$ and $\epsilon_0 F = G - \Pi$, to cast the system into the form used above for neutral polarizable media. Writing Π as a functional of F in the form (2), enables one to derive an effective constitutive relation for a globally neutral plasma.

2. THE COVARIANT SUSCEPTIBILITY KERNEL FOR A NON-STATIONARY, SPATIALLY INHOMOGENEOUS PLASMA

Consider a macroscopically neutral plasma composed of particles labelled by the species index $\lfloor\alpha\rceil$, mass $m^{\lfloor\alpha\rceil}$ and charge $q^{\lfloor\alpha\rceil}$, described dynamically by the coupled relativistic (collisionless) Maxwell-Vlasov equations for F and the one-particle probability distributions $f^{\lfloor\alpha\rceil}(x,v)$ in an arbitrary (background) gravitational field:

$$W^{\lfloor\alpha\rceil}(f^{\lfloor\alpha\rceil}) = 0, \tag{10}$$

$$dF = 0 \qquad \text{and} \qquad \epsilon_0 d \star F = - \star \widetilde{J} \tag{11}$$

with the total current 1-form $\tilde{J}(x) = g_{ab} J^b(x) dx^a$ given by

$$J^b(x) = -\sum_{\lfloor\alpha\rceil} q^{\lfloor\alpha\rceil} \int \frac{v^b |\det g|^{1/2}}{v_0} f^{\lfloor\alpha\rceil}(x,v) dv^{123} \tag{12}$$

and Liouville vector field $W^{\lfloor\alpha\rceil}$

$$W^{\lfloor\alpha\rceil}(x,v) = v^a \frac{\partial}{\partial x^a} + \left(-\Gamma^\nu{}_{ef}(x) v^e v^f + \frac{q^{\lfloor\alpha\rceil}}{m^{\lfloor\alpha\rceil}} F_{ef}(x) g^{\nu e} v^f \right) \frac{\partial}{\partial v^\nu} \tag{13}$$

with summation over Greek indices from 1 to 3. The function $v^0(x, v^1, v^2, v^3)$ is a solution of $v^a v^b g_{ab}(x) = -1$.

A linear constitutive relation arises from a perturbation of this system about a background (zeroth-order) spatially inhomogeneous and temporally non-stationary solution, $F_0(x)$, $f_0^{\lfloor\alpha\rceil}(x,v)$. The standard perturbation expansion about such a solution yields a linear system for f_1 and F_1 in terms of f_0 and F_0 that can be solved in principle by the method of characteristics [14]. This yields a first order system of integro-differential equations for F_1:

$$dF_1 = 0, \qquad d \star F_1 = -d \star \Pi_1[F_1] \tag{14}$$

where Π_1 is a linear functional depending on f_0 and the solutions to the zeroth-order Lorentz force equations

$$\nabla_{\dot{C}^{\lfloor\alpha\rceil}} \dot{C}^{\lfloor\alpha\rceil} = \frac{q^{\lfloor\alpha\rceil}}{m^{\lfloor\alpha\rceil}} \widetilde{i_{\dot{C}^{\lfloor\alpha\rceil}} F_0} \tag{15}$$

after elimination of f_1. This gives rise to a class of solutions χ_1 to (4) where Π, χ are replaced by Π_1, χ_1 respectively.

If one employs standard inertial coordinates in a Minkowski spacetime and a zeroth-order electromagnetic field $F_0 = 0$, in zeroth-order all particles move along straight line time-like geodesics independent of $\lfloor\alpha\rceil$ and

$$\chi_1^{\lfloor\alpha\rceil}(x,y) = \frac{q^{\lfloor\alpha\rceil 2}}{m^{\lfloor\alpha\rceil}} \frac{f_0^{\lfloor\alpha\rceil}(y,\hat{u})}{4\hat{u}_0 \hat{\tau}^2} g^{\mu c} \hat{u}^b \epsilon_{cbih} \big(2dx_{0\mu} + \epsilon^{d\sigma jk} \epsilon_{\mu\nu\sigma} \hat{u}^\nu \hat{u}_d dx_{jk} \big) \wedge dy^{ih}$$

where $\hat{\tau}(x,y) = (-g(x-y,\, x-y))^{1/2}$ and $\hat{u}(x,y) = (x-y)/\hat{\tau}(x,y)$ which is manifestly not a function of $x - y$ alone.

3. LANGMUIR MODES

The general solution to the second equation in (14) is given by $\epsilon_0 \star F_1 = -\star\Pi_1[F_1] + d\beta$ where β is an arbitrary 1-form. We define the generalized Langmuir sector to contain particular solutions satisfying

$$\epsilon_0 F_1 = -\Pi_1[F_1]. \tag{16}$$

This then reduces to the standard perturbative solution describing longitudinal plasma oscillations about a *stationary* f_0 in a *spatially homogeneous* plasma.

3.1. A Stationary, Spatially Inhomogeneous Plasma in Minkowski Spacetime

In [15] Bernstein, Greene and Kruskal (BGK) gave a set of solutions to the non-relativistic 1-dimensional stationary Maxwell-Vlasov equation for a number of species. There has been much debate in the literature [16] as to whether Landau damping leads to a stationary BGK solution [17]. This has been challenged by new simulations [18, 19]. Looking at the perturbations of the BGK solutions may give new insight into this debate.

In 1-dimension the non-relativistic Maxwell-Vlasov system may be written for a particle distribution $f_0^{\lfloor\alpha\rceil}(x,v)$ and a electric potential $\phi_0(x)$

$$v\frac{\partial f_0^{\lfloor\alpha\rceil}}{\partial x} - \frac{q^{\lfloor\alpha\rceil}}{m^{\lfloor\alpha\rceil}}\frac{\partial\phi_0}{\partial x}\frac{\partial f_0^{\lfloor\alpha\rceil}}{\partial v} = 0 \tag{17}$$

$$\epsilon_0\frac{\partial^2\phi_0}{\partial x^2} = -\sum_{\lfloor\alpha\rceil} q^{\lfloor\alpha\rceil}\int_{-\infty}^{\infty} f_0^{\lfloor\alpha\rceil}(x,v)dv \tag{18}$$

A class of solution for (17) are given by

$$f_0^{\lfloor\alpha\rceil}(x,v) = \hat{f}_0^{\lfloor\alpha\rceil}\left(\frac{1}{2}m^{\lfloor\alpha\rceil}v^2 + q^{\lfloor\alpha\rceil}\phi_0(x)\right) \tag{19}$$

In [15], two methods are given to solve (18), either by prescribing $\hat{f}_0^{\lfloor\alpha\rceil}$ for all species $\lfloor\alpha\rceil$ and hence solving for ϕ_0 or by prescribing ϕ_0 and $\hat{f}_0^{\lfloor\alpha\rceil}$ for all except one species and solving for that species.

We may use the susceptibility kernel to derive the equation to the perturbed Maxwell-Vlasov about a BGK mode. Since the BGK solutions are stationary we can look for perturbations which are at a fixed frequency: $F_1 = E_\omega(x)e^{-i\omega t}dt \wedge dx$. This solves (16) if $E_\omega(x)$

satisfies the integral equation

$$E_\omega(x) = -\sum_{\lfloor\alpha\rceil} \frac{q^{\lfloor\alpha\rceil 2}}{\epsilon_0 m^{\lfloor\alpha\rceil}} \int_{\tau=-\infty}^{0} d\tau \int_{v=-\infty}^{\infty} dv \hat{f}_0^{\lfloor\alpha\rceil} \left(\frac{1}{2} m^{\lfloor\alpha\rceil} v^2 + q^{\lfloor\alpha\rceil}\phi_0(x)\right) \times e^{-i\omega\tau} E_\omega(\hat{y}^{\lfloor\alpha\rceil}) \frac{\partial \hat{y}^{\lfloor\alpha\rceil}}{\partial v} \tag{20}$$

where $\hat{y}^{\lfloor\alpha\rceil}(t,x,v)$ is the solution to Lorentz force equation

$$\frac{\partial^2 \hat{y}^{\lfloor\alpha\rceil}}{\partial t^2}(t,x,v) = \frac{q^{\lfloor\alpha\rceil}}{m^{\lfloor\alpha\rceil}} \frac{d\phi_0}{dx}\left(\hat{y}^{\lfloor\alpha\rceil}(t,x,v)\right) \qquad \text{with initial conditions}$$

$$\hat{y}^{\lfloor\alpha\rceil}(0,x,v) = x \qquad \text{and} \qquad \frac{\partial \hat{y}^{\lfloor\alpha\rceil}}{\partial t}(0,x,v) = v$$

The class of unperturbed BGK solutions $(\hat{f}_0^{\lfloor\alpha\rceil}, \phi_0)$ such that there exist solutions to (20) is an open question. Except for the homogeneous case, it may be necessary to look for numerical solutions.

3.2. A Neutral Non-stationary, Spatially Inhomogeneous Plasma in Minkowski Spacetime

Consider the situation where $F_0 = 0$ and planar inhomogeneities arise from the following zeroth-order non-stationary spatially inhomogeneous solution to the Maxwell-Vlasov system: (10)–(13)

$$f_0^{\lfloor\text{el}\rceil}\left(t,\xi,x^2,x^3,u,v^2,v^3\right) = f_0^{\lfloor\text{ion}\rceil}\left(t,\xi,x^2,x^3,u,v^2,v^3\right) = h\left(\xi - \frac{u\xi}{(1+u^2)^{1/2}}, u\right)\delta\left(v^2\right)\delta\left(v^3\right). \tag{21}$$

If

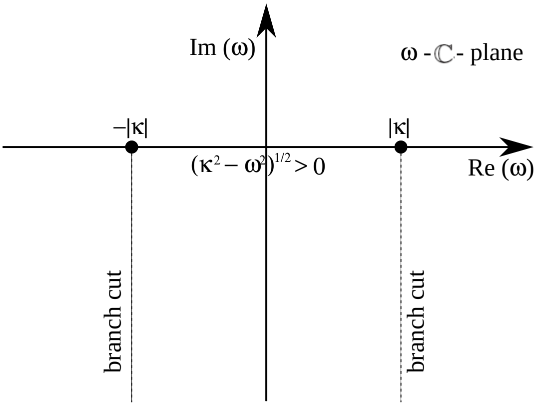
$$h(\xi,u) = n^{\lfloor\text{ion}\rceil}(\xi) A^{\lfloor\text{ion}\rceil}(\xi) \exp\left(-\frac{m^{\lfloor\text{ion}\rceil}(1+u^2)^{1/2}}{k_B T^{\lfloor\text{ion}\rceil}(\xi)}\right)$$

where $A^{\lfloor\text{ion}\rceil}(\xi)$ normalizes (21), then $f^{\lfloor\text{ion}\rceil}$ initially at $t = 0$ represents a distribution of ions where, at each spatial point ξ, the velocities belong to the 1-dimensional Maxwell-Jüttner distribution, but where the temperature $T^{\lfloor\text{ion}\rceil}(\xi)$ and the number density of ions $n^{\lfloor\text{ion}\rceil}(\xi)$, depend on position. It follows from (21) that $f^{\lfloor\text{el}\rceil}$ also initially represents a position dependent Maxwell-Jüttner distribution where $n^{\lfloor\text{el}\rceil}(\xi) = n^{\lfloor\text{ion}\rceil}(\xi)$ and $T^{\lfloor\text{el}\rceil}(\xi) = T^{\lfloor\text{ion}\rceil}(\xi) m^{\lfloor\text{el}\rceil}/m^{\lfloor\text{ion}\rceil}$. After the initial moment, the ions and electrons drift according to (21) and velocities do not remain in the Maxwell-Jüttner distributions. Alternatively (21) might describe a plasma composed of particles and anti-particles.

In the theory of a homogeneous plasma one has the solution $F_1 = dt \wedge d\xi \widehat{E}(\omega,\kappa) e^{-i\omega t + i\kappa\xi}$ provided ω and κ satisfy a transcendental dispersion relation. This relation contains an integral that is potentially singular. The Landau prescription circumvents this singularity by complexifying ω and defining an analytic continuation for the integral in the complex ω plane. In an inhomogeneous plasma there is no such time harmonic solution or associated algebraic dispersion relation between ω and κ. We therefore propose solving (16) with a longitudinal field F_1 represented as

$$F_1(t,\xi) = dt \wedge d\xi \int_{\omega=-\infty}^{\infty} d\omega \int_{\kappa=-\infty}^{\infty} d\kappa \; \widehat{E}(\omega,\kappa) e^{-i\omega t + i\kappa\xi}.$$

In this case, the Landau dispersion relation is replaced by an integral equation for $\widehat{E}(\omega,\kappa)$. This equation contains a double integral that requires analytic continuation in the complex ω plane for its definition. If one restricts to modes with real κ there are now two singular branch points in the ω plane at $\omega = \pm|\kappa|$. To define an analytic continuation from the upper-half ω plane the associated cuts are disposed along the half-lines $\{\omega = -|\kappa| - i\lambda,\ \lambda > 0\}$ and $\{\omega = |\kappa| - i\lambda,\ \lambda > 0\}$ (see figure).



The integral equation for $\widehat{E}(\omega,\kappa)$ is now given by

$$\begin{aligned}
\widehat{E}(\omega,\kappa) &= \frac{\mathcal{Q}_0}{2\pi}\int_{-\infty}^{\infty} P(\omega,\kappa,\hat{\kappa})\,d\hat{\kappa} \\
&\qquad \text{if} \quad \mathrm{Im}(\omega) > 0 \quad \text{or} \quad |\mathrm{Re}(\omega)| > |\kappa|, \\
\widehat{E}(\omega,\kappa) &= \frac{\mathcal{Q}_0}{2\pi}\int_{-\infty}^{\infty} P(\omega,\kappa,\hat{\kappa})\,d\hat{\kappa} - i\mathcal{Q}_0\int_{-\infty}^{\infty} R(\omega,\kappa,\hat{\kappa})\,d\hat{\kappa} \\
&\qquad \text{if} \quad \mathrm{Im}(\omega) < 0 \quad \text{and} \quad |\mathrm{Re}(\omega)| \le |\kappa|,
\end{aligned} \tag{22}$$

$$\widehat{E}(\omega,\kappa) = \frac{\mathcal{Q}_0}{2\pi}\int_{-\infty}^{\infty} P(\omega,\kappa,\hat{\kappa})\,d\hat{\kappa} - \frac{i\mathcal{Q}_0}{2}\int_{-\infty}^{\infty} R(\omega,\kappa,\hat{\kappa})\,d\hat{\kappa}$$
$$\text{if}\quad \mathrm{Im}(\omega)=0 \quad\text{and}\quad |\mathrm{Re}(\omega)|<|\kappa|$$

where $\mathcal{Q}_0 = \sum_{\lfloor\alpha\rceil} q^{\lfloor\alpha\rceil 2}/\epsilon_0 m^{\lfloor\alpha\rceil}$, the principal value integral is given by

$$P(\omega,\kappa,\hat{\kappa}) = \int_{-|\kappa-\hat{\kappa}|}^{|\kappa-\hat{\kappa}|} \frac{\widehat{E}\,(\omega+\kappa',\hat{\kappa})(\kappa-\hat{\kappa})}{(\omega\kappa-\omega\hat{\kappa}-\kappa'\kappa)^2}\,\widehat{h}\left(\kappa-\hat{\kappa},\,\frac{\kappa'}{\left((\kappa-\hat{\kappa})^2-\kappa'^2\right)^{1/2}}\right)d\kappa'$$

and the residue by

$$R(\omega,\kappa,\hat{\kappa}) = \frac{|\kappa-\hat{\kappa}|}{\kappa|\kappa|}\frac{\partial\widehat{E}}{\partial\omega}\left(\frac{\omega\hat{\kappa}}{\kappa},\hat{\kappa}\right)\widehat{h}\left(\kappa-\hat{\kappa},\,\frac{s_\kappa s_{\kappa-\hat{\kappa}}\omega}{(\kappa^2-\omega^2)^{1/2}}\right)$$
$$-\frac{\kappa}{(\kappa^2-\omega^2)^{3/2}}\widehat{E}\left(\frac{\omega\hat{\kappa}}{\kappa},\hat{\kappa}\right)\frac{\partial\widehat{h}}{\partial u}\left(\kappa-\hat{\kappa},\,\frac{s_\kappa s_{\kappa-\hat{\kappa}}\omega}{(\kappa^2-\omega^2)^{1/2}}\right).$$

Here $s_\kappa = \kappa/|\kappa|$ and $\widehat{h}(\kappa,u) = \int_{s=-\infty}^{\infty} e^{-i\kappa s}h(s,u)ds$. The square root $(\kappa^2-\omega^2)^{1/2}$ is defined so that for $\omega\in\mathbb{R}$, $|\omega|<|\kappa|$ and with the branch cuts given in the figure then $(\kappa^2-\omega^2)^{1/2}>0$.

For real κ these integral equations can be analyzed numerically [20] in the different domains in the ω plane and wave instability is associated with solutions for which $\mathrm{Im}(\omega)>0$. Although the nature of Landau damping ($\mathrm{Im}(\omega)<0$) in the presence of inhomogeneities is clearly more complicated that analogous damping in homogeneous plasmas, the results above indicate how the mechanism depends on the nature of the initial state and analytic continuation in the complex ω plane.

4. CONCLUSION

Hitherto the analysis of Landau damping in inhomogeneous or non-stationary plasmas has relied mainly on numerical analysis of the Maxwell-Vlasov system. This is often computationally expensive and time consuming. Motivated by our introduction of the covariant constitutive relations (4), we have reduced this system to the numerical analysis of a particular class of integral Equations (20) and (22).

In case 3.1, we have applied our new technique to a perturbation of the BGK system. The resulting analysis generalizes the notion of Landau damping to spatially inhomogeneous but stationary plasmas. In case 3.2, we have been able to interpret the wave-plasma interaction for a spatially inhomogeneous and non-stationary plasma in terms of the analytic structure of the kernel in the integral Equation (22). This generalizes the approach adopted by Landau in his description of the interaction for homogeneous stationary plasmas.

In both cases, a numerical analysis of these integral equations may offer a more efficient means to elucidate characteristics of the wave-plasma interaction for plasmas which may be non-stationary or spatially inhomogeneous.

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