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ON THE PULSE VELOCITY IN ABSORBING AND NONLINEAR MEDIA AND PARALLELS WITH THE QUANTUM MECHANICS

I. Gurwich

Life Science Research Israel, Ltd., Israel

Abstract–A novel definition of pulse propagation velocity is introduced. It is shown that the present definition does not lead to confusing results such as complex velocity or velocity exceeding the light velocity in the vacuum. Also shown are the parallels of this definition to the classical and quantum mechanics conceptions. Using the present definition reveals certain analogies between electromagnetic pulse propagation in the classical physics and de Broglie wave-packets propagation in the quantum mechanics, thus adding support to its validity.

1. INTRODUCTION

The problems of pulse propagation in absorbing media have been discussed over the years. The main point here is that the conception of the group velocity introduced by Lord Rayleigh [1] for dispersive media: $\boldsymbol{\nu}_g = \partial\omega/\partial\mathbf{k}$, where ω is the angular frequency and $\mathbf{k}$ is the wave vector, fails in the case where a medium possesses strong absorption. First, the above definition leads formally to a complex velocity [2, 3]. Using only the real part (as has been suggested by Suchy [2]) may lead to ν_g being greater than the light velocity in the vacuum or even being negative [4, 5]. Sommerfeld and Brillouin [6] have undertaken the fundamental study of "signal propagation" in a medium with arbitrary dielectric properties. Using the technique of contour integration in the complex ω-plane, they showed that in any medium a signal propagates with the velocity ("signal velocity") $\nu_s < c$. In their analysis they used a sine wave, step-switched at the moment $t = 0$ and then interminable. Their results close the question from one point: there is no physical inconsistency in the fundamental theory. Thus, the problem is in the correct and sensible definition of *velocity.* Group velocity inevitably includes also "the velocity of the wave-packet deformation" and thus does not determine the motion of the object (pulse), therefore it is not a "signal velocity". But what is a strict definition of the signal velocity? Sommerfeld and Brillouin [6] have no answer, and the discussion remains open [7–18]. There are several attempts to this end, which are now discussed.

It is worth mentioning first the results of Bolda, Chiao and Garrison [19]. They prove that *in any medium there is a frequency for which the group velocity is superluminal, infinite or negative.* This fact immediately entails modifying the group velocity conception in order to adopt it to wave packet propagation in absorbing media.

Suchy's first attempt [2] of using only the real part of $\mathbf{k}$ for calculating the group velocity was later criticized both by himself [12] and by Censor [13]. Modifications were made by Suchy [14] by using the saddle point analysis, and by Suchy and Censor who developed the real ray tracing formalism [10, 12, 15, 16] finally formulated by Censor [17]. Both these approaches involve complex frequency and hence certain special requirements to the pulse shape allowing for the correct introduction of the real velocity. At the same time, the authors did not check if the velocity they introduced ever remains less than c.

Another modification was suggested by Muschietti and Dum [18]. Proceeding again from the saddle point analysis and using a moving frame, they have constructed a velocity (they call it *real group velocity*) as linear combination $\mathrm{Re}(d\omega/dk) - a\mathrm{Im}(d\omega/dk)$, where both derivatives are taken at a certain value of k, which can vary with time (*despite the medium being homogeneous*). Essentially that their real coefficient a is determined by the pulse shape. This can be considered as an attempt to overcome the local character of the canonical group velocity definition (in my opinion the latter fact is the real source of the problems associated with group velocity). However the whole construction looks to artificial and has a somewhat vague physical meaning. The critical question of possible superluminal value of this velocity has not been studied by the authors.

Recently Sonnenschein, Rutkevich, and Censor [11] undertook a detailed study of the group velocity role in pulse propagation in absorbing media. They have shown that group velocity can be used for describing long (and short) time propagation.

In the present study the novel definition of the velocity ν_{pul} is introduced. It adequately describes the propagation of an electromagnetic pulse in a manner, which allows avoiding the paradoxical results of complex group velocity, or $\nu_{pul} > c$. In order to evade further confusion, the term "pulse velocity" is used.

We intend to proceed from a mechanical analogy — non-rigid body motion. The body velocity (as a whole) is defined canonically as the velocity of its mass center. However, if, say, we have a wax piece moving in hot air, this piece is permanently deformed and its "mass center" shifts. Despite this, the velocity of this permanently shifting imaginary point (wherein, possibly, there is no mass located) characterizes the motion of this piece as a whole. This approach has a certain advantage compared with the "signal velocity" considered by Brillouin and Sommerfeld [6], since the latter is to depend on a signal registration method (i.e., on the moment the system decides that it has received a signal). The novel definition introduced here is determined only by the physical nature of the pulse.

However, there is another justification. It is very interesting and important to note that the conception of the mass center has the mathematical similarity to the quantum mechanics definition of the average value $\overline{f}$ corresponding to the operator $\hat{f}$:

$$\overline{f} = \left\langle \Psi | \hat{f} | \Psi^* \right\rangle .$$

Both these reasons justify the present approach. It is shown that the pulse velocity introduced in the present study is free from the problems mentioned above and thus has clear physical substance. The physical parallels between electromagnetic pulse and De Broglie wave-packet propagation are discussed.

It is necessary to note that attempts to introduce an average (in a certain sense) velocity had already been undertaken in the seventies [20, 21]. Smith [20] introduced a so-called "centrovelocity". It describes "motion of the *temporal* center of gravity... of the wave packet". The author has not declared any special aims associated with absorbing media, hence has not studied the problems faced here. It can be seen that despite the seeming similarity, Smith's definition and the one introduced here are not identical. In particular, Smith's centrovelocity has no direct physical parallels with mechanics and quantum mechanics. Seven years later, Bloch [21] introduced a new velocity. He considered the cross-correlation between the initial and propagating pulse. Bloch's velocity describes the motion of the cross-section peak. This approach appears useful in practical terms, however its theoretical analysis seems problematic (and the author does not provide it, restricting himself by numerical illustrations). Moreover, the general physical validity of this velocity is rather vague. In my opinion the velocity of an object (here wave packet) is to be determined only by its present state but not by its history (initial state). All this works in favour of the velocity introduced here.

An additional purpose of this study is to include nonlinear effects in the analysis. Nonlinear harmonic generation acts for a certain harmonic as an effective source of the energy (as a product of the interaction with other harmonics) or as absorption of the energy (which is utilized for production of other harmonics). From this point of view the nonlinear effects can be considered as an effective absorption or medium activity (see, [22]).

The problem of calculating the new defined velocity is discussed separately.

2. THE GENERAL THEORY

Consider a one-dimensional electromagnetic pulse propagated in y-direction. Let its spectrum $F(\omega_0 + \Omega, y) = F(\Omega, y)$, i.e., ω_0 be its central (carrying) frequency. Further, functions attributed to the electric field will be indexed by $_E$ and those attributed to the magnetic field by $_H$. Thus, in the space-time coordinates its electric field is described in the complex form as

$$f_E(t, y) = e^{i(\omega_0 t - \tilde{k}_0 y)} \int F_E(\Omega, y) e^{i[\Omega t - K(\Omega) y]} d\Omega \tag{1}$$

The real electric field is expressed then as $E(t, y) = 1/2\,[f_E(t, y) + f_E^*(t, y)]$. The y-dependence F_E-function indicates the possible nonlinear interaction in the medium leading to variations of the pulse spectrum (see, e.g., [22]). It is assumed that F is a slowly varying function of y as compared with $e^{-i\tilde{k}_0 y}$. The wave vector corresponding to a central frequency ω_0 is generally complex

$$\tilde{k}_0 = k_0 - i\beta \tag{2}$$

where $\beta > 0$. Function $K(\Omega)$ containing dispersion characteristics can be represented as

$$K(\Omega) = s\Omega + R(\Omega) \tag{3}$$

where $1/s = \nu_g$ is the canonically defined group velocity (parameter s we be called further *group mobility*). Note that $K(\Omega)$ does not contain y-dependence, this infers that the dispersion equation is formulated either in the linear approximation or for a steady state (see, [23]).

We define here the pulse velocity as the velocity of the "mass"-center. The "mass"-center of the pulse is defined as

$$y_e = \frac{\displaystyle\int y|f(t,y)|^2 dy}{\displaystyle\int |f(t,y)|^2 dy} \tag{4}$$

and thus the pulse velocity is

$$\nu_{pul} = \frac{dy_e}{dt} \tag{5}$$

This suggested definition appears to be rational, and can be interpreted directly as the velocity of the pulse "mass" being concentrated at a single point — the "mass" center (if $f = \sqrt{\rho}$ and ρ is the mass density). On the other hand, it corresponds strictly to the quantum mechanics definition $\overline{\nu} = \langle\Psi|\hat{\nu}|\Psi^*\rangle$ if function $f(t,y)$ in (4) is normalized to 1. In order to keep this analogy, we prefer form (4) to the proper expression containing $|f(t,y)|$. The field function f corresponds to the quantum mechanical wave function Ψ. Note that we do not specify here whether f-field used in (4) is electric or magnetic one. This point should be discussed, because generally both these fields do not provide the same value of velocity (this indicates the fact that different velocities can be attributed to electromagnetic pulse propagation). Indeed, according to the Maxwell equation (in the MKSA system)

$$\nabla \times \mathbf{E} = -\frac{\partial(\mu\mathbf{H})}{\partial t} \tag{6}$$

we have

$$F_H(\Omega, y) = \sqrt{\frac{\varepsilon}{\mu}} F_E(\Omega, y) + i\frac{1}{\mu\omega_0(1+\Omega/\omega_0)}\frac{\partial}{\partial y} F_E(\Omega, y) \tag{7}$$

where ε and μ parameters are extracted from the relations $\mathbf{B} = \mu\mathbf{H}$ and $k = \omega\sqrt{\mu\varepsilon}$. Thus generally $f_H(t, y)$ is not proportional to $f_E(t, y)$. This determines the difference between the pulse velocity obtained from (4), (5) by using the electric field (ν_{pE}), and that obtained by using the magnetic field (ν_{pH}). There is neither an inconsistency nor a special deep meaning here. These two velocities describe the motion of the proper field distribution and the both have to be less than c. Only appearance of the pulse (its front point) has to be registered identically. This is in fact so, according to Sommerfeld's reasoning [6] discussed below. However, this point carries certain vagueness in our discussion. In order to obviate this, note that in a linear medium, velocities determined according **E**- and **B**-fields, and **D**- and **H**-fields will coincide. Thus, taking $F = \sqrt{\varepsilon}F_E$, or $F = \sqrt{\mu}F_H$ the same velocity is obtained, at least in a linear medium (in a linear medium the derivative of F_E in (7) drops; the situation where the medium possesses nonlinearity is discussed below). Moreover, in a medium without dissipation, definition (4) determines the energy center (generally, we avoid using energy parameters because of the problems in defining the electromagnetic energy in lossy media, see Landau, Lifshitz, and Pitaevskii [25]). Thus, in such a case, velocity, defined by (4), (5) with the proper choosing of F-function *determines the energy flux velocity.* This consideration clarifies the physical interpretation of the definition introduced, and weighs in favor of the following choice of the spectral function: $F = \sqrt{\varepsilon}F_E$ or $F = \sqrt{\mu}F_H$.

Note from the very beginning, that ν_{pul} defined by relations (4), (5) is always real, and thus the first problem is automatically dropped.

In a simple case of a linear medium without losses, i.e., $F(\Omega, y) = F(\Omega)$, and $\tilde{k}_0$, s and $R(\Omega)$ are real, the integration in (4) performed in the interval $y \in [-\infty, \infty]$ results in $\nu_{pul} = 1/s$ for pulses *sufficiently wide.* Indeed,

$$|f(t, y)|^2 = \iint F(\Omega)F^*(\Omega_1)e^{i[t-sy)(\Omega-\Omega_1)-yR(\Omega)+yR(\Omega_1)]}d\Omega d\Omega_1 \tag{8}$$

Perform now integration with respect to y. Taking into account that $\Omega = \Omega_1$ converts function $F(\Omega, \Omega_1) = R(\Omega) - R(\Omega_1)$ into zero and thus the latter can be represented as $F(\Omega, \Omega_1) = (\Omega - \Omega_1) R_1(\Omega, \Omega_1)$, where $R_1(\Omega, \Omega_1)$ is an even function with respect to exchanging $\Omega \leftrightarrow \Omega_1$, we have

$$\begin{aligned}\int y|f(t,y)|^2 dy &= \int y \iint F(\Omega)F^*(\Omega_1)e^{i[t-sy-yR_1(\Omega,\Omega_1)](\Omega-\Omega_1)}d\Omega d\Omega_1 dy \\ &= \iint F(\Omega)F^*(\Omega_1)\left\{\int ye^{i[t-sy-yR_1(\Omega,\Omega_1)](\Omega-\Omega_1)}dy\right\}d\Omega d\Omega_1 \\ &= \iint \left\{\frac{1}{s_1^2}\int \xi_1 e^{i\xi_1(\Omega-\Omega_1)}d\xi_1 - \frac{t}{s_1^2}\int e^{i\xi_1(\Omega-\Omega_1)}d\xi_1\right\} \\ &\quad \cdot F(\Omega)F^*(\Omega_1)d\Omega d\Omega_1 \qquad (9)\end{aligned}$$

where the new coordinates $\xi = t - sy$, $\xi_1 = \xi - yR_1(\Omega, \Omega_1) = t - [s + R_1(\Omega, \Omega_1)]y$ are introduced, and $s_1 = s + R_1(\Omega, \Omega_1)$; in other words $\xi = constant$ determines a point moving with the group velocity (and similarly for ξ_1). In the same manner

$$\int |f(t,y)|^2 dy = -\iint \left\{\frac{1}{s_1}\int e^{i\xi_1(\Omega-\Omega_1)}d\xi_1\right\} F(\Omega)F^*(\Omega_1)d\Omega d\Omega_1 \quad (10)$$

where the left hand side does not actually depend on time, despite its formal indication as an argument of the field function f. Inserting this into (5), one sees that the first term in the braces of (9) is dropped by derivation with respect to t. Thus, for pulses with spectra so narrow that the contribution of $R_1(\Omega, \Omega_1)$ in (3) can be ignored in the pulse spectrum band, we obtain $\nu_{pul} = 1/s$, i.e., in such a case velocity determined by relation (4), (5) coincides with the group velocity (and thus with the energy flux velocity as has been noted above). *Generally, the pulse velocity defined by (4), (5) turns out to be determined also by the high terms in the power expansion of dispersion relation $K(\Omega)$, and by the pulse spectrum.* The fact that group velocity provides only an approximate description of pulse propagation even in a linear medium without losses, is well known (see, e.g., Stratton [24]). Here we have found the proper correction, which is actually negligible, if the pulse spectrum is sufficiently narrow.

Consider now a case where a medium possesses losses. Parameter β is now not zero, and relation (3) has to be written as

$$K(\Omega) = s_R\Omega + is_J\Omega + R(\Omega) + iJ(\Omega) \qquad (3.1)$$

and

$$\beta - s_J\Omega - J(\Omega) > 0 \tag{11}$$

in the whole frequency range $(\beta - s_J\Omega - J(\Omega) < 0$ in a certain frequency interval indicates the medium amplifying activity).

Designating

$$\begin{aligned} s_{1R} &= s_R + R_1(\Omega, \Omega_1) \\ s_{1J} &= s_J + \frac{J(\Omega) + J(\Omega_1)}{\Omega + \Omega_1} \end{aligned} \tag{12}$$

we can write

$$\begin{aligned} \int y|f(t,y)|^2 dy = \frac{1}{2} e^{\frac{2\beta}{s_{1R}}t} \iint \Bigg\{ &\frac{1}{s_{1R}^2} \int \xi_1 e^{\frac{2\beta - s_{1J}\Omega_\Sigma}{s_{1R}}\xi_1} e^{i\xi_1\Omega_d} d\xi_1 \\ &- \frac{t}{s_{1R}^2} \int e^{\frac{2\beta - s_{1J}\Omega_\Sigma}{s_{1R}}\xi_1} e^{i\xi_1\Omega_d} d\xi_1 \Bigg\} e^{\frac{s_{1J}\Omega_\Sigma}{s_{1R}}t} \\ &\cdot F\left(\frac{\Omega_\Sigma + \Omega_d}{2}\right) F^*\left(\frac{\Omega_\Sigma - \Omega_d}{2}\right) d\Omega_\Sigma d\Omega_d \end{aligned} \tag{13}$$

where $\Omega_\Sigma = \Omega + \Omega_1$, $\Omega_d = \Omega - \Omega_1$, and $y = 0$ is *presumed as a starting space point for the pulse.* Assuming additionally that the pulse has a finite space extension for any time instance t (condition, which is ever satisfied actually), one can obtain the following relation

$$\nu_{pul} = \frac{1}{\overline{s}_{1R}} \left[1 - \frac{D}{(t - \overline{\xi})^2} \left(\overline{\overline{\xi}} - \overline{\xi} \right) \right] \approx \frac{1}{\overline{s}_{1R}} \left[1 - \frac{D}{t^2} \left(\overline{\overline{\xi}} - \overline{\xi} \right) \right] \tag{14}$$

valid for large enough time moments t. Here $\overline{\xi}$ and $\overline{\overline{\xi}}$ are the proper mean values of ξ and constant D is determined by the pulse shape (see Appendix A).

Thus, one can conclude that *for rather wide (but ever finitely extended) pulses, the asymptotic (with respect to time moment t) relation*

$$\nu_{pul} \propto \frac{1}{\overline{s}_{1R}} \tag{15}$$

is valid, i.e., the real mean group velocity $\nu_g = 1/\overline{s}_{1R}$ represents the asymptotic pulse velocity (compare with the results of Sonnenshine, Rutkevich and Censor [11]). Relation (15) is valid independently of the f-function choice, thus the velocities determined by different f-functions coincide asymptotically. Note that in the case of a narrow

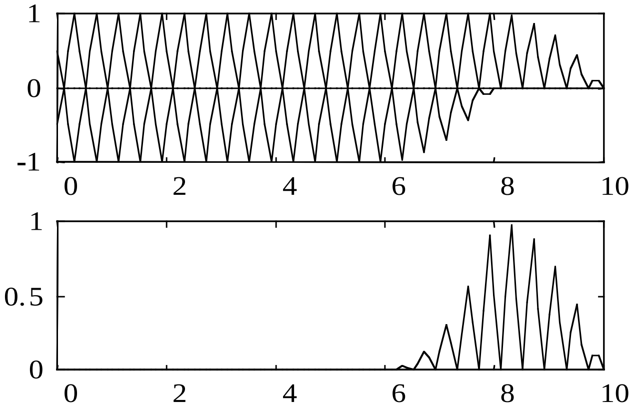

Figure 1. A wave-packet as a superposition of two sine wave, step-switched at different moments.

pulse spectrum and the normal dispersion, the relation $\overline{s}_{1R} = s_R$ takes place. However in the critical case of abnormal dispersion, functions of $R_1(\Omega, \Omega_1)$ and $J(\Omega)$ in (9) cannot be ignored, therefore $\overline{s}_{1R}$ contains an average contribution of high terms in the power expansion of $K(\Omega)$, which prevents the catastrophic result $\nu_{pul} > c$ in (15).

Relation (1) involved in (4) and (5) is similar to that used by Sommerfeld and Brillouin [6] in their analysis of the "front velocity" of an unterminated wave. Their reasoning shows that the front velocity (describing the appearance of the wave front at a certain space point, i.e., detection of an infinitely weak field) is equal to c in an arbitrary medium. In a linear medium, a pulse can be represented as the superposition of two unterminated waves having different time start-points (Fig. 1).

The first wave front determines the appearance of the pulse. Since the front velocities $\nu_{f1} = c$, the conclusion $\nu_{pul} \leq c$ holds for the pulse velocity defined by (5), *but after a certain time interval* only. In the initial period (its duration is determined by the pulse width), the "mass-center" can migrate along the pulse, and thus the previous arguments still do not allow for concluding $\nu_{pul} < c$. *We have to prove that according to the present definition, condition* $\nu_{pul} < c$ *holds for any time, otherwise it does not have an evident advantage compared with the velocity of the pulse maximum.*

In order to prove this, turn again to relation (13). First, write the dispersion equation in the form

$$k^2 = \mu(\varepsilon_R + i\varepsilon_J)\omega^2 \tag{16}$$

and

$$\begin{aligned} \varepsilon_R &= \varepsilon_R^0 + \frac{d\varepsilon_R}{d\omega}_{|\omega=\omega_0}\Omega + O(\Omega^2) = \varepsilon_R^0 + \varepsilon_{R1}\Omega \\ \varepsilon_J &= \varepsilon_J^0 + \frac{d\varepsilon_J}{d\omega}_{|\omega=\omega_0}\Omega + O(\Omega^2) = \varepsilon_J^0 + \varepsilon_{J1}\Omega \end{aligned} \tag{17}$$

Using (2), (3) and (12), we can write

$$\beta = \mu\varepsilon_J^0 \frac{\omega_0^2}{2k_0}$$

$$s_{1J} = \varepsilon_{J1} \frac{\omega_0 \left(2 - \omega_0 \dfrac{s_{1R}}{k_0}\right)}{2k_0} \tag{18}$$

where the dispersion of μ is ignored. Examine now the dependence of ν_{pul}. This can be shown (see Appendix B) that

$$\frac{d\nu_{pul}}{d\beta}_{|\beta=0} = 0 \tag{19.1}$$

$$\frac{d\nu_{pul}}{d\beta}_{|\beta>0} < 0 \tag{19.2}$$

$$\frac{d\nu_{pul}}{d\beta}_{|\beta<0} > 0 \tag{19.3}$$

Thus, $\beta = 0$ provides maximum of the pulse velocity defined by (4) and (5). *This proves the statement $\nu_{pul} < c$ and thus justifies the present definition of the pulse velocity, as a velocity associated with a physical object propagation.*

Generally, dispersion leads to stretching pulses in space. A factor that can neutralize this tendency is a nonlinearity of the dielectric properties. It is known that in a medium with the cubic type nonlinearity, interplay of dispersion and nonlinear interaction can lead to the stable pulse propagation (see, e.g., Newell and Moloney, [26]). This is significantly, that the forming of a stable pulse is a result of nonlinear interaction without spectral transformation, i.e., without harmonic generation. Nonlinear interaction, which is associated with harmonic generation, does not prevent the pulse shape corruption. Furthermore, such an interaction leads to permanent energy redistribution among different harmonics. Thus, for a certain harmonic, this process reveals itself as an effective medium activity (or losses), as was discussed by Gurwich and Censor [22]. They showed also that propagation of a pulse on a certain harmonic frequency cannot be described by the group velocity $\nu_g = \partial\omega/\partial k$. Proceeding from these parallels, we intend to include nonlinearity as an activity (losses) of an effective medium. It is possible to define effective dielectric parameters in such a medium in a manner, which allows for concealing the nonlinear field dependence in these parameters (see also discussion in Gurwich and Censor [23]). The constitutive relation in a linear medium can be written as

$$\mathbf{D}(t) = \underset{\sim}{\varepsilon}(t)\otimes\mathbf{E}(t) \tag{20.1}$$

or for the vector form spectral components

$$\mathbf{F}_D(\omega) = \underset{\sim}{\varepsilon}(\omega)\cdot\mathbf{F}_E(\omega) \tag{20.2}$$

Symbol $\otimes$ denotes here the convolution integral in the vector form (say, for tensor $\underset{\sim}{a}$ and vector $\mathbf{b}$, $\underset{\sim}{a}(t)\otimes\mathbf{b}(t) = \int \underset{\sim}{a}(\tau)\cdot\mathbf{b}(t-\tau)d\tau$), and symbol $\underset{\sim}{\varepsilon}$ is used for the tensor of the dielectric permittivity both in the time and frequency domain. The constitutive relation for a general type nonlinear medium can be written by using the Volterra series formalism as (see, e.g., [23])

$$\mathbf{D} = \sum \mathbf{D}^{(n)} \tag{21}$$

where $\mathbf{D}^{(n)}$ describes the nth order nonlinear contribution

$$\mathbf{D}^{(n)}(t) = \underset{\sim}{\varepsilon}^{(n)}(t)\otimes\mathbf{E}(t)\otimes\ldots\otimes\mathbf{E}(t) \tag{22.1}$$

prescribing the appearance of new spectral components on the harmonic frequencies (for the fundamental mode having ω_0 as the central frequency, the harmonic modes have the central frequencies $2\omega_0$, $3\omega_0$, $\ldots, n\omega_0$). The nonlinear permittivity $\underset{\sim}{\varepsilon}^{(n)}$ determines the intensity of in the nth order nonlinear interaction (n-photon process in the quantum mechanics terms). For the spectral component $\mathbf{D}^{(n)}$ with the frequency $\omega = p\omega_0 + \Omega$ we have ([22, 27])

$$\mathbf{F}_D^{(n)}(\omega) = \int\ldots\int \delta(\omega-\omega_1-\ldots-\omega_n)\,\underset{\sim}{\varepsilon}^{(n)}(\omega_1,\ldots,\omega_n) \cdot\mathbf{F}_E(\omega_1)\cdot\ldots\cdot\mathbf{F}_E(\omega_n)d\omega_1\ldots d\omega_n \tag{22.2}$$

where the presence of the Dirac function $\delta(\omega - \omega_1 - \ldots - \omega_n)$ provides the energy conservation. Denoting $\Omega_m = \omega_m - m\omega_0$, we can rewrite this relation as

$$\begin{aligned}
&\mathbf{F}_D{}^{(n)}(m\omega_0 + \Omega_m) \\
&\quad = \int \cdots \int \sum_{m_1 \pm m_2 \ldots \pm m_n = m} \underset{\sim}{\varepsilon}^{(n)}(\omega_1, \ldots, \omega_n) \cdot \mathbf{F}_E(m_1\omega_0 + \Omega_1) \\
&\qquad \cdot \ldots \cdot \mathbf{F}_E(m_n\omega_0 - \Omega_1 - \ldots \Omega_{n-1}) d\Omega_1 \ldots d\Omega_{n-1} \qquad (22.3)
\end{aligned}$$

where symbol $\pm$ indicates that the proper frequencies can be involved in the linear combination with signs $+$ or $-$, as well (see [23, 25]). Introducing now the dielectric permittivity of the effective medium as

$$\begin{aligned}
\underset{\sim}{\varepsilon}_{eff}(\omega) = &\int \cdots \int \sum_{m_1 \pm m_2 \ldots \pm m_n = m} \underset{\sim}{\varepsilon}^{(n)}(\{\omega\}) \cdot \mathbf{F}_E(m_1\omega_0 + \Omega_1) \\
&\cdot \ldots \cdot \mathbf{F}_E(m_{n-1}\omega_0 + \Omega_{n-1}) \frac{F_E(m_n\omega_0 - \Omega_1 - \ldots - \Omega_{n-1})}{F_E(\omega)} \\
&\cdot d\Omega_1 \ldots d\Omega_{n-1} \qquad (23)
\end{aligned}$$

where for the brevity's sake symbol $\{\omega\}$ substitutes the previous $(\omega_1, \ldots, \omega_n)$ in (22.3), we can write (22.3) formally in the linear form similar to (20.2). Let the given coordinate provide $\underset{\sim}{\mu}$ and $\underset{\sim}{\varepsilon}$ tensors to be diagonal, thus the scalar form equations can be used again (see [23] for details). Then, relation (7) still holds if we treat ε there as ε_{eff}. Proceeding from another Maxwell's equation

$$\nabla \times \mathbf{H} = -\frac{\partial(\varepsilon_{eff}\mathbf{E})}{\partial t} \qquad (24)$$

we obtain in turn

$$F_E(\Omega, y) = \sqrt{\frac{\mu}{\varepsilon_{eff}}} F_H(\Omega, y) - i\frac{1}{\varepsilon_{eff}\omega_0(1 + \Omega/\omega_0)} \frac{\partial}{\partial y} F_H(\Omega, y) \qquad (7.1)$$

Compatibility of (7) and (7.1) requires $\frac{\partial}{\partial y}\sqrt{\mu}F_H(\Omega, y) = \frac{\partial}{\partial y}\sqrt{\varepsilon_{eff}}F_E$. Now we can introduce the combined dielectric permittivity of the effective medium ε_{effC} as

$$\sqrt{\varepsilon_{effC}} = \sqrt{\varepsilon_{eff}} + i\frac{1}{\sqrt{\mu}\omega} \frac{\frac{\partial}{\partial y} F_E(\Omega, y)}{F_E(\Omega, y)} \qquad (25)$$

and reduce (7) and (7.1) to the single relation

$$\sqrt{\mu}F_H(\Omega,y)=\sqrt{\varepsilon_{effC}}F_E(\Omega,y) \tag{26}$$

In order to be consistent with the previous consideration, we take the spectral function figured in (4) as, $F=\sqrt{\varepsilon_{effC}}F_E$ or $F=\sqrt{\mu}F_H$.

Now, introduce the amplitude of a certain (say, pth) harmonic mode as $\rho(y)=|F(p\omega_0,y)|$, which varies slowly because of nonlinear interaction. The spectral function of this mode (F_p) can be represented as $F_p(\Omega_p,y)=\rho(y)F_{pN}(\Omega_p,y)$, where new F_{pN} function is normalized $|F_{pN}(0,y)|=1$. This redefinition can be written in the form $F_p(\Omega_p,y)=e^{\ln[\rho(y)]}F_{pN}(\Omega_p,y)$. Then one can introduce the combine β-parameter

$$\tilde{\beta}(y)=\beta\frac{\ln\rho(y)}{y} \tag{27}$$

(further, index $_p$ of the harmonic mode is omitted). The parameter $\tilde{s}=\partial K/\partial\omega_{|\Omega=0}$ and also $\tilde{R}_1(\Omega,\Omega_1)$ and $\tilde{J}(\Omega)$, are determined by the dispersion equation containing amplitudes of interacting harmonics. Such an equation can be written for the steady state where all the amplitudes are presumed as remaining constant; another possibility is in using the dispersion equation in the linear approximation (see Gurwich and Censor [23]). Thus we can write in the general form

$$\begin{aligned}\int y|f(t,y)|^2dy&=\frac{1}{2}e^{\frac{2\tilde{\beta}}{\tilde{s}_{1R}}t}\\ \times\iint e^{\frac{\tilde{s}_{1J}\Omega_\Sigma}{\tilde{s}_{1R}}t}&\left\{\begin{aligned}&\frac{1}{\tilde{s}_{1R}^2}\int\xi_1 e^{\frac{2\tilde{\beta}-\tilde{s}_{1J}\Omega_\Sigma}{\tilde{s}_{1R}}\xi_1}e^{i\xi_1\Omega_d}\Phi_{\Omega_\Sigma,\Omega_d}(\xi_1,t)d\xi_1\\ &-\frac{t}{\tilde{s}_{1R}^2}\int e^{\frac{2\tilde{\beta}-\tilde{s}_{1J}\Omega_\Sigma}{\tilde{s}_{1R}}\xi_1}e^{i\xi_1\Omega_d}\Phi_{\Omega_\Sigma,\Omega_d}(\xi_1,t)d\xi_1\end{aligned}\right\}d\Omega_\Sigma d\Omega_d\end{aligned} \tag{28}$$

The Sommerfeld reasoning concerning the front velocity (Sommerfeld and Brillouin [6]) can be applied to the present case as well. Indeed, they proceed from the point that the medium reaction, determining phase (group, and other types) velocity, starts only with the initial field oscillations reaching the medium. The latter thus propagate in the "rest medium" and have the same velocity as in the vacuum, i.e., c. In the presence of the nonlinear interaction, the representation of pulse as a superposition of two non-terminated waves (as in Fig. 1) cannot be used directly. However, the following arguments show that the Sommerfeld reasoning still holds. Indeed, the initial weak oscillations, or forerunners, as Sommerfeld and Brillouin [6] called them, are not involved actually in the nonlinear interaction because of their weak intensity. Therefore, the nonlinear effects do not contribute to the front velocity. Formally, relations determining the pulse velocity in nonlinear media are also of a similar type to that was used by Sommerfeld and Brillouin [6]. Thus, their approach of the complex contour integrals is valid. To conclude $\nu_{pul} < c$, we turn again to considerations resulting in relations (19.1), (19.2) and (19.3).

3. GAUSSIAN PULSE IN A QUADRATIC TYPE NONLINEAR MEDIUM

As an example, consider the pulse with the Gaussian spectrum propagating in an absorbing medium with the quadratic type nonlinearity (i.e., only $\mathbf{D}^{(2)}$ contributes in (21), and thus only $\underset{\sim}{\varepsilon}^{(2)}$ nonlinear permittivity is involved). Its propagation associated with the generation of harmonic modes has been considered by Gurwich and Censor [22]. The normalized spectrum of a certain harmonic mode can be described as

$$F_N(\Omega, y) = e^{-a(y)\Omega^2 - b(y)\Omega} \tag{29}$$

with $b(0) = 0$, where a and b, even if being real initially, gradually obtain imaginary parts. The index referring to the mode number is omitted here since only one mode is under explicit consideration and all others are impacted in the "effective medium". As has been shown by Gurwich and Censor [22], equations for the amplitudes of the proper modes (ρ) and for their a and b parameters can be separated in the first approximation for extended pulses. In this approximation, one can write (see [22, 28])

$$\rho(y) \approx \sqrt{\overline{a}}\,\Im(y) \tag{30}$$

where $\overline{a}$ is the a-parameter, averaged in a certain manner on the space interval $[0, y]$, and $\Im(y)$ is the proper solution (amplitude) for plane periodic waves, which turns to be a slowly varying periodic function (see, e.g., [23, 24, 28, 29]). Its space period is determined by the components of the nonlinear dielectric tensor $\underset{\sim}{\varepsilon}^{(2)}$ (denoted here as $T^{y}_{\varepsilon_{NL}}$). The amplitude of these periodic variations is determined in turn by the distortion of the phase matching condition between interacted modes.

This condition requires $\mathbf{k}_n = n\mathbf{k}_0$, where $\mathbf{k}_0$ is the wave vector of the fundamental mode. Denote this distortion measure as $\Delta\kappa$ (in the case where only two modes, the first and the second, interact efficiently $\Delta\kappa = k_2 - 2k_0$). Expanding $\Im(y)$ in the Fourier series, we can write

$$\tilde{\beta}(y) = \beta + \frac{\ln\left(\rho_0 + R(\Delta\kappa)\sum c_j \cos\left[jT^y_{\varepsilon_{NL}} y + \varphi_j\right]\right)}{y} \tag{31}$$

where $R(\Delta\kappa) \to \rho_0$ with $\Delta\kappa \to 0$, and $R(\Delta\kappa) \to 0$ for large $\Delta\kappa$. Thus, the combined β-parameter is also the periodic function. Using the substitution $y = (t-\xi_1)/\tilde{s}_{1R}$, it is concluded that the pulse velocity also receives the periodical variations in time. Gurwich and Censor [12] have shown that the asymptotic behavior of b can be estimated as

$$b(y) \propto b_\infty + C_b y e^{i\Delta\kappa y} \tag{32}$$

where C_b is a certain constant. As can be seen from (31), (30), and relation (A5) in Appendix A, the y-dependent part of the b-parameter appears in similar form as $\tilde{s}_{1J}y$, i.e., $C_b\cos(\Delta\kappa y)$ is to be added in the proper expression for $\tilde{s}_{1J}$. This also contributes to the periodical time variations of the pulse velocity. These variations reflect the periodic changes in the absorbing-active character of the "effective medium", which is compounded by the material medium and the fields in it, and is "glued" by nonlinearity.

It is again emphasized, that the velocity introduced here provides adequate description of the pulse motion as a whole entity. The usual description using the group velocity conception fails where the propagation is accompanied by pulse shape corruption. This is because its local character, i.e., the group velocity conception is based on the local properties of the pulse either in space or spectral coordinates (see, e.g., [30]). The present definition proceeds from the integral pulse description, therefore it holds in a general case. *The only objection, which can be made in subject of this velocity definition, is in calculating it for a particular case.* The answer is in relation (A5), Appendix A. Expanding function $sh\left(\frac{\overline{s}_{1J}W}{\overline{s}_{1R}}(t-\xi)\right)/(t-\xi)$ in (A5) in the power series

$$\frac{sh\left(\dfrac{\overline{s}_{1J}W}{\overline{s}_{1R}}(t-\xi)\right)}{t-\xi} = \sum_{n=0,\ldots}\left(\frac{\overline{s}_{1J}W}{\overline{s}_{1R}}\right)^{2n+1}\frac{(t-\xi)^{2n}}{(2n+1)!} \tag{33}$$

one can represent (A5) as

$$\nu_{pul} = \frac{1}{\overline{s}_{1R}} - \frac{d}{dt}\frac{1}{\overline{s}_{1R}} \frac{\sum\limits_{n=0,\dots} A_n \left(\frac{\overline{s}_{1J}}{\overline{s}_{1R}}\right)^{2n+1} \frac{t^{2n}}{(2n+1)!}}{\sum\limits_{n=0,\dots} B_n \left(\frac{\overline{s}_{1J}}{\overline{s}_{1R}}\right)^{2n+1} \frac{t^{2n}}{(2n+1)!}} \tag{34}$$

where A_n and B_n are the coefficients determined by (A5) and (33). For rather weak absorption, and where the *time instance is not too large*, relation (34) can be reduced to the form

$$\nu_{pul} = \frac{1}{\overline{s}_{1R}} - \sum_{n=1,\dots} C_n \frac{\overline{s}_{1J}^{2n+1}}{\overline{s}_{1R}^{2n+2}} \frac{t^{2n-1}}{(2n-1)!(2n+1)} \tag{34.1}$$

where new coefficient C_n are now determined by A_n and B_n. These expansion coefficients depend generally on a W-parameter describing pulse spectrum, a β-parameter, and $\overline{\overline{\overline{\Phi}}}_\xi$ also containing averaged dispersion properties.

With the presence of nonlinearity, relations (34) and (34.1) still hold, but A_n, B_n, C_n and $\overline{s}_{1J}$ are periodically time dependent.

Relations (34) and (34.1) keep their form for an arbitrary pulse shape, and not only for the Gaussian one. The pulse shape determines the expansion coefficients A_n, B_n, C_n. It is not surprising that the pulse velocity turns out to be dependent on time. The velocity in the present definition depends on the pulse shape, which is deformed during the propagation. One can see that for pulses with a wide spectrum, the second term in the right hand side of (A5) behaves as $1/W$. By extracting this factor from the coefficients C_n, relation (34.1) can be rewritten as

$$\nu_{pul} = \frac{1}{\overline{s}_{1R}} - \frac{1}{\overline{s}_{1R}} \frac{\overline{s}_{1J}}{W} Z\left(t, W, \frac{\overline{s}_{1J}}{\overline{s}_{1R}}\right) \tag{34.2}$$

where function Z is easily derived from (34.1). *Thus, additionally we can state that for very narrow pulses (wide spectrum), the real group velocity (i.e., its mean-spectral value* $1/\overline{s}_{1R}$*) holds its role.* This is a clear result: The group velocity cannot adequately describe pulse motion if propagation is associated with the pulse corruption; however, for narrow pulses the shape corruption is not so significant. The pulse stretching due to strong dispersion can disturb this consideration, and that is why relation (34.3) appears under the condition formulated above: for the not too large time instance.

It is worth to discuss here a point of possible misunderstanding: if the present approach can be applied for a case where the pulse is splitted in two parts propagating in opposite directions. Really, using the f-function of the whole pulse one can get $\nu_{pul} = 0$. However (again parallels with non-rigid body mechanics: a case of several bodies), one can use as well the f-function of each part separately for describing this part motion.

4. PARALLELS WITH THE QUANTUM MECHANICS

Now study the subject of applying the present definition of the pulse velocity to the de Broglie wave-packets.

Consider the Schrödinger (one-dimensional) equation for an electron in a potential field

$$i\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2}\psi - U(y)\psi = 0 \tag{35}$$

where $U(y)$ is the electron potential energy. Let $\psi_E(y,t)$ be a solution corresponding to a state with a certain energy E. This wave function can be normalized as

$$\int \psi_E \psi^*_{E_1} dy = \delta(E - E_1) \tag{36}$$

Forming the wave-packet by considering a state with a certain energy spectrum $\rho(E)$ we can build the proper wave function

$$\psi(y,t) = A \int \psi_E(y,t)\rho(E) dE \tag{37}$$

with the proper normalization coefficient A, which is normalized as

$$\int \psi\psi^* dy = 1 \tag{38}$$

Hence the pulse velocity is now defined by the relation

$$\nu_{pul} = \frac{d}{dt}\int y|\psi(t,y)|^2 dy \tag{5.1}$$

or its derivative $d\nu_{pul}/dt$ by the relation

$$\frac{d}{dt}\nu_{pul} = \frac{d^2}{dt^2}\int y|\psi(t,y)|^2 dy \tag{39}$$

It is well known (see, e.g., Landau and Lifshitz [31]), that in the operator form, the Newton motion law holds in the quantum mechanics

$$\hat{\dot{\mathbf{v}}} = -\frac{1}{m_e}\nabla U \tag{40}$$

where m_e is the electron mass and U is its potential energy. Proceeding from the interpretation of the time derivative in the quantum mechanics [31]: $\overline{\dot{f}} \equiv \dot{\overline{f}}$, and passing to the scalar form equation, we can conclude also

$$\frac{\partial}{\partial t}\nu_{pul} = \frac{1}{m_e}\overline{F} \tag{40.1}$$

where $\mathbf{F} = -\nabla U$, and the overbar denotes the mean value on the wave-packet length interval. In other words, ν_{pul} describes the de Broglie wave-packet motion (in terms of average characteristics), and this description corresponds to the Newton motion law.

5. SUMMARY

We have introduced here the novel definition of the velocity describing wave-packet propagation. It is shown that the newly defined velocity describes an electromagnetic pulse propagation in an absorbing media without the problems associated with other definitions characterizing pulse propagation, i.e., it is always real and does not exceed the light velocity in the vacuum. This formalism can also be expanded for pulse propagation in nonlinear media. The present definition can be justified as possessing the physical sense, which is analogous to velocity of non-rigid objects in the classical mechanics on the one hand, and the de Broglie wave-packet motion in the quantum mechanics on the other hand. In the latter example, this illustrates the parallels between an electromagnetic pulse (describing the motion of a photon ensemble) and the electron motion (describing in the terms of the de Broglie wave-packet). It is also shown that the group velocity conception holds so far as the electromagnetic pulse can be treated as a stable-form uncorrupted object. All this supports considering of the present definition as a substantial conception.

APPENDIX A

Consider relation (13). Assuming that F, R_1 and J are continuous functions in the whole frequency range, we can use the mean value theorem and then fulfill the integration with respect to Ω-variables. Applying this theorem for a case where the integrand contains complex valued functions should be commented. Let $f(x)$ and $g(x)$ be complex value functions of the real variable x. Let $f(x)$ also be continuous and differentiable and thus can be represented by the truncated Taylor series on the integration interval

$$f(x) = f_0 + f_0 x + \ldots + f_0^{(n)} x^n \tag{A1}$$

Introduce the following designation: $G_x = \int g dx$, thus G_a means G_x for $x = a$, G_0 means G_x for $x = 0$, and $G = \int_0^a g dx = G_a - G_0$; further $G_{xx} = \int G_x dx$, $G_{xxx} = \int G_{xx} dx$, and so on. Then, for the integral $\int_0^a f g dx$ we have

$$\begin{aligned}\int_0^a f g dx &= f_0 \int_0^a g dx + f_0' \int_0^a x g dx + \ldots + f_0^{(n)} \int_0^a x^n g dx \\ &= f_0 G + f_0' \int_0^a x g dx + \ldots + f_0^{(n)} \int_0^a x^n g dx\end{aligned} \tag{A2}$$

where the lower limit of the integration is chosen in order to reduce the following relations. In turn

$$\int_0^a x g dx = a G_a \left(1 - \frac{G_{aa} - G_{00}}{a G_a}\right) = a G \left(\frac{G_a}{G} - \frac{G_{aa} - G_{00}}{a G}\right) = a G \Delta_1 \tag{A3.1}$$

where Δ_1 denotes the expression in the parenthesis. Similarly

$$\begin{aligned}\int_0^a x^2 g dx &= a^2 G_a \left(1 - 2\frac{G_{aa}}{a G_a} + 2\frac{G_{aaa} - G_{000}}{a^2 G_a}\right) \\ &= a^2 G \left(\frac{G_a}{G} - 2\frac{G_{aa}}{a G} + 2\frac{G_{aaa} - G_{000}}{a^2 G}\right) = a^2 G \Delta_2\end{aligned} \tag{A3.2}$$

and so on. Finally we obtain

$$\begin{aligned}\int_0^a f g dx &= f_0 G + f_0' a G \Delta_1 + \ldots + f_0^{(n)} a^n G \Delta_n \\ &= G \left(f_0 + f_0' a \Delta_1 + \ldots + f_0^{(n)} a^n \Delta_n\right)\end{aligned} \tag{A4}$$

The whole procedure is correct if $G \neq 0$ and this is presumed in our case. We take the expression in the parenthesis as the mean value of $f(\overline{f})$. If function g depends on a certain parameter y, then $\overline{f}$ also depends on it, and this is indicated further by the proper subindex $(\overline{f}_y)$. Note, that $\overline{f}$ can be a complex value even in a case where $f(x)$ is a real function.

Performing the integration in (13) with respect to Ω_Σ leads to function $shW(t-\xi)/W(t-\xi)$, where W depends on the range of Ω_Σ and the ξ-coordinate is defined in the text. Let the pulse have a finite space extension for any time instance t, i.e., in any time its extension X_t can be estimated as $X_t < X_{max}$. (Actually, this parameter determines the interval of integration with respect to y in (4), and thus also with respect to ξ in the following relations. Subindex $_t$ indicates changing of the pulse extension together with its shape by propagation through the medium). Then we can chose $t \gg \xi$ on the whole interval of the integration and investigate the asymptotic expression for the pulse velocity. For the case of a narrow pulse frequency band where the contributions of $R1(\Omega, \Omega_1)$ and $J(\Omega)$ can be neglected (or at least substituted by certain average values), the integration with respect of frequency variables yields

$$\nu_{pul} = \frac{1}{\overline{s}_{1R}} - \frac{1}{\overline{s}_{1R}} \frac{d}{dt} \left\{ \frac{\int \overline{\overline{\Phi}}_\xi e^{\frac{2\beta - \overline{s}_{1J} W}{\overline{s}_{1R}} \xi} \frac{sh\left(\frac{\overline{s}_{1J} W}{\overline{s}_{1R}}(t-\xi)\right)}{t-\xi} \sin \xi W d\xi}{\int \overline{\overline{\Phi}}_\xi e^{\frac{2\beta - \overline{s}_{1J} W}{\overline{s}_{1R}} \xi} \frac{sh\left(\frac{\overline{s}_{1J} W}{\overline{s}_{1R}}(t-\xi)\right)}{t-\xi} \frac{\sin \xi W}{\xi} d\xi} \right\} \tag{A5}$$

where $\overline{s}_{1R}$ and $\overline{s}_{1J}$ are constants obtained by averaging $R_1(\Omega, \Omega_1)$ and $J(\Omega)$ in the proper expressions (12), and $\overline{\overline{\Phi}}_\xi$ is the result of double averaging of $\Phi_{\Omega_\Sigma, \Omega_d} = F\left(\frac{\Omega_\Sigma + \Omega_d}{2}\right) F^*\left(\frac{\Omega_\Sigma - \Omega_d}{2}\right)$ for the integration with respect to Ω_Σ and Ω_d. It is reasonable to treat $\overline{\overline{\Phi}}_\xi$ as a slowly varying function of ξ compared to other factors under the integrals. Factor $e^{-\frac{2\beta}{\overline{s}_{1R}} t}$, being involved initially both in the numerator and denominator of (A5) and also frequency independent, vanishes in the final relation. For time moments $t \gg \xi$ in the whole ξ-interval, one can fulfil the integration, applying the mean value theorem for the function $f(\xi) = sh\left(\frac{\overline{s}_{1J} W}{\overline{s}_{1R}}(t-\xi)\right)/(t-\xi)$. Show first, that one can find such $\overline{\xi}$, which satisfies the equation $f(\overline{\xi}) = \overline{f}$, where $\overline{f}$ is also real. Indeed, for large t, we can write

$$f(\overline{\xi}) \approx \frac{e^{\left|\frac{\overline{s}_{1J}W}{\overline{s}_{1R}}\right|(t-\xi)}}{t-\xi} = \frac{e^{a(t-\xi)}}{t-\xi} \tag{A6}$$

where positive coefficient a is introduced for brevity's sake and this relation is valid in the whole integration interval. Function $f(\xi)$ is already monotonic in the integration interval and thus the mean value theorem can be used in its standard form. After using the mean value theorem in the numerator and denominator and substituting $\overline{f} = \frac{e^{a(t-\overline{\xi})}}{t-\overline{\xi}}$, the integration yields

$$\nu_{pul} \approx \frac{1}{\overline{s}_{1R}}\left[1 - \frac{\int \overline{\overline{\Phi}}_\xi e^{\frac{2\beta-\overline{s}_{1J}W}{\overline{s}_{1R}}\xi} \sin\xi W d\xi}{\int \overline{\overline{\Phi}}_\xi e^{\frac{2\beta-\overline{s}_{1J}W}{\overline{s}_{1R}}\xi} \frac{\sin\xi W}{\xi}\Omega_d d\xi} \frac{d}{dt}\left\{\frac{\left(t-\overline{\overline{\xi}}\right)e^{\left|\frac{\overline{s}_{1J}W}{\overline{s}_{1R}}\right|(t-\overline{\xi})}}{\left(t-\overline{\xi}\right)e^{\left|\frac{\overline{s}_{1J}W}{\overline{s}_{1R}}\right|(t-\overline{\overline{\xi}})}}\right\}\right]$$

$$= \frac{1}{\overline{s}_{1R}}\left[1 - e^{\left|\frac{\overline{s}_{1J}W}{\overline{s}_{1R}}\right|\left(\overline{\overline{\xi}}-\overline{\xi}\right)} \frac{\int \overline{\overline{\Phi}}_\xi e^{\frac{2\beta-\overline{s}_{1J}W}{\overline{s}_{1R}}\xi} \sin\xi W d\xi}{\int \overline{\overline{\Phi}}_\xi e^{\frac{2\beta-\overline{s}_{1J}W}{\overline{s}_{1R}}\xi} \frac{\sin\xi W}{\xi}\Omega_d d\xi} \frac{d}{dt}\left\{\frac{\left(t-\overline{\overline{\xi}}\right)}{\left(t-\overline{\xi}\right)}\right\}\right] \tag{A7}$$

where $\overline{\xi}$ and $\overline{\overline{\xi}}$ are the proper mean values of ξ in the numerator and denominator of (A5) respectively. Relation (A7) can be written in the form

$$\nu_{pul} = \frac{1}{\overline{s}_{1R}}\left[1 - \frac{D}{\left(t-\overline{\xi}\right)^2}\left(\overline{\overline{\xi}}-\overline{\xi}\right)\right] \approx \frac{1}{\overline{s}_{1R}}\left[1 - \frac{D}{t^2}\left(\overline{\overline{\xi}}-\overline{\xi}\right)\right] \tag{A8}$$

where

$$D = \frac{\int \overline{\overline{\Phi}}_\xi e^{\frac{2\beta-\overline{s}_{1J}W}{\overline{s}_{1R}}\xi} \sin\xi W d\xi}{\int \overline{\overline{\Phi}}_\xi e^{\frac{2\beta-\overline{s}_{1J}W}{\overline{s}_{1R}}\xi} \frac{\sin\xi W}{\xi}\Omega_d d\xi} \tag{A9}$$

and is determined by the pulse shape.

APPENDIX B

Consider relation (4), (5), and (13). The pulse velocity can be expressed as

$$\nu_{pul} = \frac{d}{dt}\frac{\iiint e^{-\vartheta t}\left[G_1 - tG_2\right] d\xi_1 d\Omega_d d\Omega_\Sigma}{\iiint e^{-\vartheta t} G_3 d\xi_1 d\Omega_d d\Omega_\Sigma} \tag{B1}$$

where $\vartheta = \frac{2\beta}{s_{1R}} - \frac{s_{1J}\Omega_\Sigma}{s_{1R}}$ and functions G_1, G_2 and G_3 are determined according to relations (13) and the proper one for $\int |f(t,y)|^2 dy$:

$$G_1 = \frac{\xi_1}{s_{1R}^2} e^{\vartheta\xi_1} e^{i\xi_1\Omega_d} F\left(\frac{\Omega_\Sigma + \Omega_d}{2}\right) F^*\left(\frac{\Omega_\Sigma - \Omega_d}{2}\right) \tag{B2.1}$$

$$G_2 = \frac{1}{s_{1R}^2} e^{\vartheta\xi_1} e^{i\xi_1\Omega_d} F\left(\frac{\Omega_\Sigma + \Omega_d}{2}\right) F^*\left(\frac{\Omega_\Sigma - \Omega_d}{2}\right) \tag{B2.2}$$

$$G_3 = \frac{1}{s_{1R}} e^{\vartheta\xi_1} e^{i\xi_1\Omega_d} F\left(\frac{\Omega_\Sigma + \Omega_d}{2}\right) F^*\left(\frac{\Omega_\Sigma - \Omega_d}{2}\right) \tag{B2.3}$$

It can be seen immediately that these functions satisfy the relations: $G_1 = \xi_1 G_2$, $G_3 = s_{1R} G_2$. Examine now the dependence of ν_{pul} on ϑ. Assume that ratio $\varepsilon_{J1}/\varepsilon_j^0$ determined according to (17) remains defined (and finite) if $\varepsilon_J^0 \to 0$. Then one can rewrite function ϑ as $\vartheta = \beta\vartheta_1$, where $\vartheta_1 > 0$ (for the whole frequency interval if $\beta > 0$, and at least for a certain frequency interval if $\beta < 0$). Now we can differentiate (B1) with respect to β, that yields

$$\begin{aligned}\frac{d}{d\beta}\nu_{pul} = &-\frac{d}{dt}\frac{\iiint \vartheta_1 t e^{-\vartheta t}\left[G_1 - tG_2\right] d\xi_1 d\Omega_d d\Omega_\Sigma}{\iiint e^{-\vartheta t} G_3 d\xi_1 d\Omega_d d\Omega_\Sigma} \\ &+ \frac{d}{dt}\frac{\iiint e^{-\vartheta t}\left[G_1 - tG_2\right] d\xi_1 d\Omega_d d\Omega_\Sigma \iiint \vartheta_1 t e^{-\vartheta t} G_3 d\xi_1 d\Omega_d d\Omega_\Sigma}{\left[\iiint e^{-\vartheta t} G_3 d\xi_1 d\Omega_d d\Omega_\Sigma\right]^2}\end{aligned} \tag{B3}$$

Now let $\beta = 0(\varepsilon_J^0 = 0)$. Then differentiating with respect to t leads to

$$\frac{d\nu_{pul}}{d\beta}_{|\beta=0} = -\frac{\iiint \vartheta_1 G_1 d\xi_1 d\Omega_d d\Omega_\Sigma}{\iiint G_3 d\xi_1 d\Omega_d d\Omega_\Sigma} + t\frac{\iiint \vartheta_1 G_2 d\xi_1 d\Omega_d d\Omega_\Sigma}{\iiint G_3 d\xi_1 d\Omega_d d\Omega_\Sigma}$$
$$+ \frac{\iiint G_1 d\xi_1 d\Omega_d d\Omega_\Sigma \iiint \vartheta_1 G_3 d\xi_1 d\Omega_d d\Omega_\Sigma}{\left[\iiint G_3 d\xi_1 d\Omega_d d\Omega_\Sigma\right]^2}$$
$$- t\frac{\iiint G_2 d\xi_1 d\Omega_d d\Omega_\Sigma \iiint \vartheta_1 G_3 d\xi_1 d\Omega_d d\Omega_\Sigma}{\left[\iiint G_3 d\xi_1 d\Omega_d d\Omega_\Sigma\right]^2} \qquad \text{(B4)}$$

Substitute ϑ_1 in the integrals in (B4) by its mean value, denoting the latter as $\overline{\vartheta}_{11}$, $\overline{\vartheta}_{12}$, or $\overline{\vartheta}_{13}$, according to the index of G-function involved in the proper integral. Relation (B4) can now be rewritten as

$$\frac{d\nu_{pul}}{d\beta}_{|\beta=0}$$
$$= \left(\overline{\vartheta}_{13} - \overline{\vartheta}_{11}\right)\frac{\iiint G_1 d\xi_1 d\Omega_d d\Omega_\Sigma}{\iiint G_3 d\xi_1 d\Omega_d d\Omega_\Sigma} - t\left(\overline{\vartheta}_{13} - \overline{\vartheta}_{12}\right)\frac{\iiint G_2 d\xi_1 d\Omega_d d\Omega_\Sigma}{\iiint G_3 d\xi_1 d\Omega_d d\Omega_\Sigma} \qquad \text{(B5)}$$

Suppose now that the pulse spectrum allows for ignoring the frequency dependence of s_{1R}, thus $\overline{\vartheta}_{13} = \overline{\vartheta}_{12}$. As can be seen from (B2), for $\beta = 0$ functions G_1 and G_3 have the similar Ω-dependence and hence also $\overline{\vartheta}_{13} = \overline{\vartheta}_{11}$. This results in

$$\frac{d\nu_{pul}}{d\beta}_{|\beta=0} = 0 \qquad \text{(B6)}$$

At the same time, it can be seen that a positive value of β increases the average value $\overline{\vartheta}_{11}$ and a negative β decreases it. Indeed, consider integrals

$$\iiint \vartheta_1 e^{-\vartheta t} G_1 d\xi_1 d\Omega_d d\Omega_\Sigma \quad \text{and} \quad \iiint \vartheta_1 e^{-\vartheta t} G_3 d\xi_1 d\Omega_d d\Omega_\Sigma$$

involved in (B1). Their explicit forms are

$$\iint \left\{ \frac{1}{s_{1R}^2} \int \xi_1 e^{\beta\vartheta_1\xi_1} e^{i\xi_1\Omega_d} d\xi_1 \right\} \vartheta_1 e^{-\beta\vartheta_1 t} \Phi_{\Omega_\Sigma,\Omega_d} d\Omega_\Sigma d\Omega_d \tag{B7}$$

and

$$\iint \left\{ \frac{1}{s_{1R}} \int e^{\beta\vartheta_1\xi_1} e^{i\xi_1\Omega_d} d\xi_1 \right\} \vartheta_1 e^{-\beta\vartheta_1 t} \Phi_{\Omega_\Sigma,\Omega_d} d\Omega_\Sigma d\Omega_d \tag{B8}$$

Consider integral (B7). Assuming the pulse possesses a finite space extension, we can perform inner integration in a proper finite interval, which can be taken symmetrical $[\Lambda, -\Lambda]$. Then integrating yields

$$\begin{aligned} &\iint \frac{2}{s_{1R}^2} \frac{\Lambda}{\beta\vartheta_1 + i\Omega_d} ch\Lambda(\beta\vartheta_1 + i\Omega_d)\vartheta_1 e^{-\beta\vartheta_1 t} \Phi_{\Omega_\Sigma\Omega_d} d\Omega_\Sigma d\Omega_d \\ -&\iint \frac{2}{s_{1R}^2} \frac{1}{(\beta\vartheta_1 + i\Omega_d)^2} sh\Lambda(\beta\vartheta_1 + i\Omega_d)\vartheta_1 e^{-\beta\vartheta_1 t} \Phi_{\Omega_\Sigma\Omega_d} d\Omega_\Sigma d\Omega_d \end{aligned} \tag{B9}$$

The similar treatment of (B8) leads to

$$\iint \frac{2}{s_{1R}} \frac{1}{\beta\vartheta_1 + i\Omega_d} sh\Lambda(\beta\vartheta_1 + i\Omega_d)\vartheta_1 e^{-\beta\vartheta_1 t} \Phi_{\Omega_\Sigma\Omega_d} d\Omega_\Sigma d\Omega_d \tag{B10}$$

According to (B9) and (B10), the value $\overline{\vartheta}_{11}$ is determined as

$$\overline{\vartheta}_{11} = \frac{J_3}{J_{3a}} \tag{B11}$$

where J_3 is integral (B9),

$$\begin{aligned} J_{3a} = &\iint \frac{2}{s_{1R}^2} \frac{\Lambda}{\beta\vartheta_1 + i\Omega_d} ch\Lambda(\beta\vartheta_1 + i\Omega_d) e^{-\beta\vartheta_1 t} \Phi_{\Omega_\Sigma\Omega_d} d\Omega_\Sigma d\Omega_d \\ &- \iint \frac{2}{s_{1R}^2} \frac{1}{(\beta\vartheta_1 + i\Omega_d)^2} sh\Lambda(\beta\vartheta_1 + i\Omega_d) e^{-\beta\vartheta_1 t} \Phi_{\Omega_\Sigma\Omega_d} d\Omega_\Sigma d\Omega_d \end{aligned}$$

and the value $\overline{\vartheta}_{13}$ is determined as

$$\overline{\vartheta}_{13} = \frac{\displaystyle\iint \frac{\Lambda}{s_{1R}} \frac{\vartheta_1}{\beta\vartheta_1 + i\Omega_d} sh\Lambda(\beta\vartheta_1 + i\Omega_d) e^{-\beta\vartheta_1 t} \Phi_{\Omega_\Sigma\Omega_d} d\Omega_\Sigma d\Omega_d}{\displaystyle\iint \frac{1}{s_{1R}} \frac{1}{\beta\vartheta_1 + i\Omega_d} sh\Lambda(\beta\vartheta_1 + i\Omega_d) e^{-\beta\vartheta_1 t} \Phi_{\Omega_\Sigma\Omega_d} d\Omega_\Sigma d\Omega_d} \tag{B12}$$

Let the pulse be long enough for accepting the approximation $sh\Lambda(\beta\vartheta_1 + i\Omega_d) \cong \Lambda(\beta\vartheta_1 + i\Omega_d) \cong e^{\Lambda(|\beta\vartheta_1|+i\Omega_d)}$ (note, that the actual pulse length is determined by the spectral function F contained in Φ; however one can extend the integration interval in order to make the approximation valid). Here sign $\pm$ indicates two possibilities: $+$ for positive b (lossy medium) and $-$ for negative b (active medium). Long pulse duration in turn prescribes a rather narrow pulse spectrum. Thus s_{1R} and s_{1J} can be treated as constants, and integration with respect to Ω_d does not effect the ratio of values $\overline{\vartheta}_{11}$ and $\overline{\vartheta}_{13}$. Taking this into account one can write

$$\overline{\vartheta}_{11} = \frac{J_3}{J_{3a}} = \frac{\displaystyle\int \frac{\vartheta_1}{\beta\vartheta_1 + i\Omega_d} e^{\Lambda|\beta\vartheta_1|} e^{-\beta\vartheta_1 t} \left(\Lambda - \frac{1}{\beta\vartheta_1 + i\Omega_d}\right) \Phi_{\Omega_\Sigma,\Omega_d} d\Omega_\Sigma}{\displaystyle\int \frac{1}{\beta\vartheta_1 + i\Omega_d} e^{\Lambda|\beta\vartheta_1|} e^{-\beta\vartheta_1 t} \left(\Lambda - \frac{1}{\beta\vartheta_1 + i\Omega_d}\right) \Phi_{\Omega_\Sigma,\Omega_d} d\Omega_\Sigma} \tag{B13}$$

$$\overline{\vartheta}_{13} = \frac{\displaystyle\int \frac{\vartheta_1}{\beta\vartheta_1 + i\Omega_d} e^{\Lambda|\beta\vartheta_1|} e^{-\beta\vartheta_1 t} \Phi_{\Omega_\Sigma,\Omega_d} d\Omega_\Sigma}{\displaystyle\int \frac{1}{\beta\vartheta_1 + i\Omega_d} e^{\Lambda|\beta\vartheta_1|} e^{-\beta\vartheta_1 t} \Phi_{\Omega_\Sigma,\Omega_d} d\Omega_\Sigma} \tag{B14}$$

These relations show that, if $|\beta|$ is not too small, its positive value augments the weight of large values of ϑ_1 in (B13), and thus enlarges $\overline{\vartheta}_{11}$ compared with $\overline{\vartheta}_{13}$, and vice versa for negative β. (For rather small $|\beta|$, the analysis of the dispersion equation and relations (4), (5), (12), (13) prescribe the pulse velocity to be close to the real part of the group velocity $Re\{\nu_g\}$, always being smaller than c, see, e.g., Sommerfeld [6]). In a more general case, one can get similar results by applying the mean value theorem to s_{1R} and s_{1J}.

Now let pulse be very short, such as $sh\Lambda(\beta\vartheta_1 + i\Omega_b) \cong \Lambda(\beta\vartheta_1 + i\Omega_b) - \Lambda^3(\beta\vartheta_1 + i\Omega_b)^3/6$ and $ch\Lambda(\beta\vartheta_1 + i\Omega_b) \cong 1 - \Lambda^2(\beta\vartheta_1 + i\Omega_b)^2/2$. Again inserting this into (B11), (B12) yields

$$\overline{\vartheta}_{11} = \frac{\displaystyle\iint \frac{\vartheta_1}{s_{1R}^2} \Lambda^3 (\beta\vartheta_1 + i\Omega_d)(\beta\vartheta_1 + i\Omega_d - 1/3) e^{-\beta\vartheta_1 t} \Phi_{\Omega_\Sigma,\Omega_d} d\Omega_\Sigma d\Omega_d}{\displaystyle\iint \frac{1}{s_{1R}^2} \Lambda^3 (\beta\vartheta_1 + i\Omega_d)(\beta\vartheta_1 + i\Omega_d - 1/3) e^{-\beta\vartheta_1 t} \Phi_{\Omega_\Sigma,\Omega_d} d\Omega_\Sigma d\Omega_d} \tag{B15}$$

$$\overline{\vartheta}_{13} = \frac{\displaystyle\iint \frac{\vartheta_1}{s_{1R}} e^{-\beta\vartheta_1 t} \Phi_{\Omega_\Sigma,\Omega_d} d\Omega_\Sigma d\Omega_d}{\displaystyle\iint \frac{1}{s_{1R}} e^{-\beta\vartheta_1 t} \Phi_{\Omega_\Sigma,\Omega_d} d\Omega_\Sigma d\Omega_d} \tag{B16}$$

We have again to conclude that the positive value of β (if $|\beta|$ is not too small) augments the weight of large values of ϑ_1 in (B15), and thus enlarges $\overline{\vartheta}_{11}$ compared with $\overline{\vartheta}_{13}$, and vice versa for negative β. Therefore

$$\frac{d\nu_{pul}}{d\beta}_{|\beta>0} < 0 \tag{B17}$$

$$\frac{d\nu_{pul}}{d\beta}_{|\beta<0} > 0 \tag{B18}$$

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