

NON-RELATIVISTIC ELECTROMAGNETIC SCATTERING: “REVERSE ENGINEERING” USING THE LORENTZ FORCE FORMULAS

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Abstract—For almost a century, velocity dependent scattering problems are solved in the context of Einstein’s Special Relativity theory. Most interesting problems involve non-uniform motion, which is heuristically justified by assuming the validity of the “instantaneous velocity” approximation. The present study attempts to provide a consistent postulational foundation by introducing boundary conditions based on the Lorentz force formulas.

The methodology used here is dubbed “reverse engineering”: Being aware of the relativistic results, we show that they are replicated, (at least) to the first order in $\beta = v/c$ by the present method. Specific problems are discussed to demonstrate the power of the method, and pave the way to future research in this problem area.

Specifically, by realizing that at the boundary we deal with *signals*, which are derived from *waves*, only the latter being subject to the wave equations, it is feasible to apply boundary conditions and construct appropriately the scattered waves in space.

It is shown that the present approach is also consistent with the Minkowski constitutive relations which are exploited for solving problems where the medium moves parallel with respect to the boundaries.

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1. INTRODUCTION AND RATIONALE

It is almost a century since the publication of Einstein's original paper [1], which drastically changed our understanding of physics. For an authoritative review see Pauli [2]. Instead of the Newtonian absolute space and time trellis on which the physical world was until then described, we have now an infinitum of inertial reference systems related by the Lorentz transformation.

Against this new backdrop, many problems of physics have been investigated. Of particular interest to the research at hand are electromagnetic scattering problems in the presence of moving objects and moving media. In this respect it is interesting to note Abraham's article [3], because he has derived relativistic correct results even though Einstein's theory [1] was not yet published.

Over a period of many years, relativistic electromagnetic scattering problems have been investigated. A comprehensive review of various results, and an extended collection of references was given by Van Bladel [4], many results are also described and referenced by Kong [5]. These provide a link to the relevant literature. Many of these investigations are attempts to enrich our catalog of solved problems, and they indeed afford a deeper insight into the general problem area.

The aim of the present investigation is, more than to deal with specific problems, rather to provide a unifying critical point of view to the whole subject, in particular the approach based on the Lorentz force formulas, as explained below.

Strictly speaking, Einstein's Special Relativity theory [1] applies to constant velocities only. For example, proper time, which is a keystone concept of the theory, cannot be accounted for, for a particle moving with a varying velocity on an arbitrary trajectory. For a simple demonstration see Censor [6]. This is a well-known problem in the postulational foundation of Special Relativity. See for example Bohm [7, pp. 162–163]. Accordingly, the instantaneous velocity of the accelerated object is considered in a local unaccelerated inertial frame of reference moving with the object's instantaneous velocity. Within the short time interval for which this new frame is considered, until

it is replaced by another instantaneous velocity frame, the laws of Newtonian mechanics are claimed to be applicable in this frame.

At best, this makeshift heuristic approach is an empirically adequate approximation, but does not stem from the initial postulates of the Special Relativity theory. However, in the absence of better tools, the instantaneous velocity approximation is adopted for many analyses appearing in [4] and elsewhere. At the present time, we are equipped with those results and can propose other approaches and compare results with those derived exactly or approximately from special-relativistic arguments. This “reverse engineering” approach will be followed below. Accordingly, it is demonstrated that for instantaneous velocities, at least the first order relativistic velocity effects can be considered as valid, subject to the present approach.

Recently [8] the Doppler effect [9–11] and associated scattering problems have been discussed. Without invoking special-relativistic considerations, it has been shown that scattering of a plane wave incident on a moving plane scatterer gives rise to frequency, wavelength and direction of propagation of the reflected wave, commensurate with the relativistic equations derived in [1]. It must be stressed that these results were obtained neither by using the Lorentz nor the Galilean coordinate transformations, as explained below. Based on the Lorentz force formulas, these results are extended here to account for velocity dependent amplitude effects as well.

2. THE LORENTZ FORMULA APPROACH AND SPECIAL RELATIVITY

There exists a *conceptual difference* between the Lorentz force formula in its classical form and the relativistic electrodynamics approach. Let us start with the macroscopic Maxwell’s equations which are the fundamental “law of nature” concerning us here, see [6] for notation

$$\begin{aligned}\partial_{\mathbf{x}} \times \mathbf{E} &= -\partial_t \mathbf{B} - \mathbf{j}_m \\ \partial_{\mathbf{x}} \times \mathbf{H} &= \partial_t \mathbf{D} + \mathbf{j}_e \\ \partial_{\mathbf{x}} \cdot \mathbf{D} &= \rho_e \\ \partial_{\mathbf{x}} \cdot \mathbf{B} &= \rho_m\end{aligned}\tag{1}$$

In (1) the equations are given in the $\mathbf{X} = (\mathbf{x}, ict)$ spatiotemporal regime (in the Minkowski notation), e.g., $\mathbf{E} = \mathbf{E}(\mathbf{X})$. Indices *e*- (electric), or *m*- (magnetic) refer to electric and (virtual) magnetic sources, respectively. In the context of the Special Relativity theory this initial space is often referred to as the “laboratory frame

of reference,” from which moving objects are observed, in contradistinction to the “comoving frame” or “proper frame,” in which an object is observed by the comoving observer as being at rest. If the comoving frame is inertial, i.e., unaccelerated, then according to Einstein’s postulate [1], “the same” (i.e., having the same mathematical functional structure) Maxwell’s equations apply in it (see also [12])

$$\begin{aligned}
 \partial_{\mathbf{x}''} \times \mathbf{E}'' &= -\partial_{t''} \mathbf{B}'' - \mathbf{j}_m'' \\
 \partial_{\mathbf{x}''} \times \mathbf{H}'' &= \partial_{t''} \mathbf{D}'' + \mathbf{j}_e'' \\
 \partial_{\mathbf{x}''} \cdot \mathbf{D}'' &= \rho_e'' \\
 \partial_{\mathbf{x}''} \cdot \mathbf{B}'' &= \rho_m''
 \end{aligned} \tag{2}$$

where in (2) $\mathbf{E}'' = \mathbf{E}''(\mathbf{X}'')$, etc., with $\mathbf{X}'' = (\mathbf{x}'', ict'')$. The spaces are related by the Lorentz transformation

$$\begin{aligned}
 \mathbf{x}'' &= \tilde{\mathbf{U}} \cdot (\mathbf{x} - \mathbf{v}t) \\
 t'' &= \gamma(t - \mathbf{v} \cdot \mathbf{x}/c^2) \\
 \gamma &= (1 - \beta^2)^{-1/2} \\
 \hat{\mathbf{v}} &= \mathbf{v}/v, \quad \beta = v/c, \quad v = |\mathbf{v}| \\
 \tilde{\mathbf{U}} &= \tilde{\mathbf{I}} + (\gamma - 1)\hat{\mathbf{v}}\hat{\mathbf{v}},
 \end{aligned} \tag{3}$$

where in (3) the tilde denotes dyadics and $\mathbf{v}$ is the velocity observed from the laboratory frame. For brevity (3) can be denoted by $\mathbf{X}'' = \mathbf{X}''[\mathbf{X}]$, and it is easily shown that (3) when solved for the un-primed coordinates, yields $\mathbf{X} = \mathbf{X}[\mathbf{X}'']$, which has the same structure as (3) with interchanged double-primed and un-primed coordinates, and $\mathbf{v}$ replaced by $-\mathbf{v}$.

The fields in (1), (2) are related by the field transformation formulas (see [6] for notation)

$$\begin{aligned}
 \mathbf{E}'' &= \tilde{\mathbf{V}} \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \\
 \mathbf{B}'' &= \tilde{\mathbf{V}} \cdot (\mathbf{B} - \mathbf{v} \times \mathbf{E}/c^2) \\
 \mathbf{D}'' &= \tilde{\mathbf{V}} \cdot (\mathbf{D} + \mathbf{v} \times \mathbf{H}/c^2) \\
 \mathbf{H}'' &= \tilde{\mathbf{V}} \cdot (\mathbf{H} - \mathbf{v} \times \mathbf{D}) \\
 \mathbf{j}_{e,m}'' &= \tilde{\mathbf{U}} \cdot (\mathbf{j}_{e,m} - \mathbf{v}\rho_{e,m}) \\
 \rho_{e,m}'' &= \gamma(\rho_{e,m} - \mathbf{v} \cdot \mathbf{j}_{e,m}/c^2) \\
 \tilde{\mathbf{V}} &= \gamma\tilde{\mathbf{I}} + (1 - \gamma)\hat{\mathbf{v}}\hat{\mathbf{v}}
 \end{aligned} \tag{4}$$

where $\tilde{\mathbf{I}}$ is the idemfactor dyadic. What is crucial in (4) is that $\mathbf{E}'' = \mathbf{E}''(\mathbf{X}'')$ and $\mathbf{E} = \mathbf{E}(\mathbf{X})$, etc., depend on their native space coordinates, and the Lorentz transformation (3) $\mathbf{X}'' = \mathbf{X}''[\mathbf{X}]$ mediates between the coordinate systems. In summary—here we deal with two distinct but related spaces.

In contradistinction, the classical Lorentz force formula is given by

$$\begin{aligned}\mathbf{f}_e &= q_e \mathbf{E}' \\ \mathbf{E}' &= \mathbf{E} + \mathbf{v} \times \mathbf{B}\end{aligned}\tag{5}$$

where in (5) $\mathbf{E}'$ formally denotes a new *effective* $\mathbf{E}$ field, which includes the velocity effect $\mathbf{v} \times \mathbf{B}$, i.e., the velocity generated $\mathbf{E}$ field. All the fields in (5), e.g., $\mathbf{f}_e(\mathbf{X})$, $\mathbf{E}(\mathbf{X})$ etc., as well as the velocity field $\mathbf{v}(\mathbf{X})$ are measured in the laboratory frame of reference, in terms of the native spatiotemporal coordinates $\mathbf{X}$. Unlike the relativistic notions, nothing is assumed here regarding measurements in a comoving frame of reference, and no assumptions are made on the constancy of $\mathbf{v}$ —as far as we are concerned, it can be any nonuniform velocity field $\mathbf{v}(\mathbf{X})$.

To the first order in β , we have in (4) $\tilde{\mathbf{V}} = \tilde{\mathbf{I}}$, i.e., the idemfactor dyadic, hence to this approximation, the first equation (4) becomes

$$\mathbf{E}''(\mathbf{X}'') = \mathbf{E}'(\mathbf{X})\tag{6}$$

and similarly for other equations in (4), but the two fields depend on different spatiotemporal arguments. Multiplying (6) by q_e on both sides now tells us that (from the relativistic point of view) the Coulomb force in the comoving frame is identical in magnitude, to the first order in β , to the Lorentz force measured in the laboratory frame.

Therefore, in retrospect, (this is what is meant here by “reverse engineering”), one could deduce (5) from (4), incorporating the relativistic force four-vector properties (e.g., see [6] for a simple discussion). This means that if magnetic point sources were physically existent, one would be compelled, based on (4), to include in the theory an analog magnetic Lorentz force of the form

$$\begin{aligned}\mathbf{f}_m &= q_m \mathbf{H}' \\ \mathbf{H}' &= \mathbf{H} - \mathbf{v} \times \mathbf{D}\end{aligned}\tag{7}$$

It is noted that Sommerfeld [13] defines the fictitious magnetic Lorentz force formula by using the second equation in (4). The reason for that, or the advantage compared to (7) is not clear. Of course (7) must be understood as a *postulate* supplementing the existing theory comprised of (1, 5). This paradigm *does not include* the theory of Special Relativity, which we use here only as our “reverse engineering” benchmark tool.

3. BOUNDARY CONDITIONS FOR MOVING OBJECTS

An attempt will be carried out here to define adequate boundary conditions for moving boundaries, using the Lorentz force formulas (5), (7) and without invoking Special Relativity considerations. Only in retrospect will our findings be compared to the relativistically exact results.

The general boundary conditions are derived by invoking a limiting process near a boundary, such that (1) is valid on both sides of the boundary. The last two scalar equations in (1) then yield

$$\hat{\mathbf{n}} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = \rho_{mS} \quad (8)$$

$$\hat{\mathbf{n}} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_{eS} \quad (9)$$

i.e., in (8), (9) on the boundary between two regions (denoted “1” and “2”), the normal components of vector $\mathbf{B}$, $\mathbf{D}$, are discontinuous, indicated by the jump in the magnetic and electric surface charge densities ρ_{mS} , ρ_{eS} , respectively. The unit normal vector $\hat{\mathbf{n}}$ points into region “1”. The vector equations (1) yield

$$\hat{\mathbf{n}} \times (\mathbf{E}_1 - \mathbf{E}_2) = -\mathbf{j}_{mS} \quad (10)$$

$$\hat{\mathbf{n}} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{j}_{eS} \quad (11)$$

indicating the discontinuity of $\mathbf{E}$, $\mathbf{H}$, across the boundary with the magnetic and electric surface current densities, $\mathbf{j}_{mS}$, $\mathbf{j}_{eS}$, respectively.

When dealing with electrostatics and magnetostatics, in general all the four relations (8)–(11) are needed, however, in dynamical (time-dependent) systems, and in the absence of sources ρ_{eS} , ρ_{mS} , $\mathbf{j}_{eS}$, $\mathbf{j}_{mS}$, only two equations, e.g., (10), (11), are needed. This is well known but seldom emphasized in textbooks. Inasmuch as the Lorentz force formulas (5), (7) seem to suggest only two out of the four conditions for moving boundaries, this point is reviewed here succinctly and in great simplicity: For example, consider the first equation (1) in a current-less domain. Multiplying the equation by $\hat{\mathbf{n}}$, we obtain

$$\hat{\mathbf{n}} \cdot \partial_x \times (\mathbf{E}_1 - \mathbf{E}_2) = -\partial_t \hat{\mathbf{n}} \cdot (\mathbf{B}_1 - \mathbf{B}_2) \quad (12)$$

The right hand side of (12) then vanishes according to (8), and the left hand side involves only tangential field components, hence it agrees with (11), and therefore we have here a redundancy. Multiplying the second vector equation in (1) by $\hat{\mathbf{n}}$, a similar redundancy in (9) and (10) is revealed.

Boundary conditions for moving boundaries are discussed in the literature, e.g., see [4, 5]. What we wish to do here is to derive the essential expressions needed for scattering by moving objects, without

invoking relativistic considerations. To that end we stipulate boundary conditions based on (5), (7). Consider first a perfectly conducting boundary, on which the tangential $\mathbf{E}$ field according to (10) must vanish in the absence of imposed surface current density sources $\mathbf{j}_{ms}$. One way to justify this boundary condition is to consider the Coulomb force $\mathbf{f}_e = q_e \mathbf{E}$, i.e., (5) with $\mathbf{v} = 0$, acting on free charges at the surface. The field will “attempt” to separate the initially neutral charges, thus causing a current. In a perfectly conducting material this would induce infinite currents, hence the tangential component of the total field, $\mathbf{E}_{to}$, must vanish at the boundary. Consequently an excitation (*ex*-) field will be associated with a scattered (*sc*-) field such that at the boundary the total (*to*-) $\mathbf{E}$ field vanishes

$$\hat{\mathbf{n}} \times \mathbf{E}_{to} = \hat{\mathbf{n}} \times (\mathbf{E}_{ex} + \mathbf{E}_{sc}) = 0 \quad (13)$$

A perfect conductor produces the same effects as a material with $\varepsilon \rightarrow \infty$. In analogy to (13), we stipulate a magnetic Coulomb force $\mathbf{f}_m = q_m \mathbf{H}$, i.e., (7) with $\mathbf{v} = 0$. For such a “perfectly conducting” magnetic material possessing $\mu \rightarrow \infty$ we now have the analog of (13)

$$\hat{\mathbf{n}} \times \mathbf{H}_{to} = \hat{\mathbf{n}} \times (\mathbf{H}_{ex} + \mathbf{H}_{sc}) = 0 \quad (14)$$

For perfect conducting media the two conditions (13), (14) are mutually exclusive, otherwise the system would become over-determined. For a material with arbitrary finite μ , ε , even though we cannot assume electric and magnetic conduction currents, there are still polarization currents present, induced by the fields. At the boundary, the conditions now involve the internal (*in*-) fields in the form

$$\begin{aligned} \hat{\mathbf{n}} \times \mathbf{E}_{to} &= \hat{\mathbf{n}} \times (\mathbf{E}_{ex} + \mathbf{E}_{sc}) = \hat{\mathbf{n}} \times \mathbf{E}_{in} \\ \hat{\mathbf{n}} \times \mathbf{H}_{to} &= \hat{\mathbf{n}} \times (\mathbf{H}_{ex} + \mathbf{H}_{sc}) = \hat{\mathbf{n}} \times \mathbf{H}_{in} \end{aligned} \quad (15)$$

In view of the argument following (12), if there are no prescribed surface sources on the scatterer, one of the boundary conditions (13), (14), or the pair are necessary and sufficient for solving the scattering problem.

An explicit solution (15) requires knowledge of the constitutive relations involved, e.g., for simple media

$$\mathbf{D}_{ex,sc} = \varepsilon_0 \mathbf{E}_{ex,sc}, \mathbf{B}_{ex,sc} = \mu_0 \mathbf{H}_{ex,sc} \quad (16)$$

$$\mathbf{D}_{in} = \varepsilon \mathbf{E}_{in}, \mathbf{B}_{in} = \mu \mathbf{H}_{in} \quad (17)$$

Following (5), (7), for moving boundaries the Coulomb force formulas are to be replaced by the Lorentz force formulas. Accordingly, in (13)–(15), the fields $\mathbf{E}_{ex}$, $\mathbf{E}_{sc}$, should be replaced by the effective values $\mathbf{E}'_{ex}$,

$\mathbf{E}'_{sc}$, respectively. Similarly, replace $\mathbf{H}_{ex}$, $\mathbf{H}_{sc}$ by $\mathbf{H}'_{ex}$, $\mathbf{H}'_{sc}$, respectively

$$\begin{aligned}\hat{\mathbf{n}} \times \mathbf{E}'_{to} &= \hat{\mathbf{n}} \times (\mathbf{E}'_{ex} + \mathbf{E}'_{sc}) = \hat{\mathbf{n}} \times \mathbf{E}'_{in} \\ \hat{\mathbf{n}} \times \mathbf{H}'_{to} &= \hat{\mathbf{n}} \times (\mathbf{H}'_{ex} + \mathbf{H}'_{sc}) = \hat{\mathbf{n}} \times \mathbf{H}'_{in}\end{aligned}\quad (18)$$

with the same constitutive relations as in (17) in the form

$$\mathbf{D}'_{in} = \varepsilon \mathbf{E}'_{in}, \quad \mathbf{B}'_{in} = \mu \mathbf{H}'_{in} \quad (19)$$

and similarly effective primed fields in (16).

In view of (6), the basic limitation on (18), (19) is that they yields correct relativistic results only to the first order in the velocity factor β . Missing will be factors involving γ as in (4).

The results (18), (19) are different from the essentially non-relativistic kinematical boundary conditions cited by [4, 5], which were originally suggested by von Laue [14]. These replace (10), (11) in the form

$$\begin{aligned}\hat{\mathbf{n}} \times (\mathbf{E}_1 - \mathbf{E}_2) - (\hat{\mathbf{n}} \cdot \mathbf{v})(\mathbf{B}_1 - \mathbf{B}_2) &= -\mathbf{j}_{mS} \\ \hat{\mathbf{n}} \times (\mathbf{H}_1 - \mathbf{H}_2) + (\hat{\mathbf{n}} \cdot \mathbf{v})(\mathbf{D}_1 - \mathbf{D}_2) &= \mathbf{j}_{eS}\end{aligned}\quad (20)$$

The main difference seems to be in that (17), based on the Lorentz force formulas (5), (7), contains a term

$$\hat{\mathbf{n}} \times (\mathbf{v} \times \mathbf{B}) = (\hat{\mathbf{n}} \cdot \mathbf{B})\mathbf{v} - (\hat{\mathbf{n}} \cdot \mathbf{v})\mathbf{B} \quad (21)$$

and a similar term for $\mathbf{D}$. The term $(\hat{\mathbf{n}} \cdot \mathbf{B})\mathbf{v}$ in (21) and the corresponding $(\hat{\mathbf{n}} \cdot \mathbf{D})\mathbf{v}$ do not appear in (20). It is also noted that the extra terms in (20) containing $\hat{\mathbf{n}} \cdot \mathbf{v}$ vanish identically and do not feature in problems where the motion is parallel to the interface, as discussed below in relation with the Minkowski constitutive relations. In any event, it is not clear how a problem based on (20) could be used, without further assumptions, to solve arbitrary problems and account for Doppler effects involved.

Before leaving the subject of boundary conditions, we wish to draw attention to the fact that Van Bladel (see [4, p. 313]), solving the problem of scattering from an oscillating mirror, suggests the boundary condition (18) for the electric fields, but not for the magnetic fields. This point seems to be conflicting with the present results.

4. THE ROLE OF THE MINKOWSKI CONSTITUTIVE EQUATIONS

For the basic theory and historical remarks see Sommerfeld [13], referring to the original work by Minkowski [15], see also [6]. Minkowski's constitutive equations are pertinent to scattering problems where

the interface is moving parallel to itself. These constitutive equations relate the fields in the laboratory system in the presence of a moving medium.

It will be shown that the boundary conditions (18), (19), based on the Lorentz force formulas (5), (7), are consistent and equivalent to the Minkowski constitutive equations approach. In order to illustrate the principle, we adhere to the simplest situation of a homogeneous isotropic medium.

Based on (4), in the laboratory system, the fields inside a moving medium will be given by

$$\begin{aligned}\mathbf{D}_{in} + \mathbf{v} \times \mathbf{H}_{in}/c^2 &= \varepsilon(\mathbf{E}_{in} + \mathbf{v} \times \mathbf{B}_{in}) \\ \mathbf{B}_{in} - \mathbf{v} \times \mathbf{E}_{in}/c^2 &= \mu(\mathbf{H}_{in} - \mathbf{v} \times \mathbf{D}_{in})\end{aligned}\quad (22)$$

where in (22) $\mathbf{E}_{in} = \mathbf{E}_{in}(\mathbf{X})$ etc., are all measured in the laboratory frame of reference where the moving medium is observed, and expressed in terms of the native spatiotemporal coordinates $\mathbf{X}$. But note that the constitutive parameters μ, ε , in (22) are those measured in the self or proper (comoving) frame where the medium is at rest.

From (4)–(7) it is clear that to the first order in β , (22) amounts to (19). Scattering problems where the medium moves tangentially with respect to the interface can be solved by assuming a medium at rest with the boundary conditions given by (15), and with the constitutive properties (16), (19), (22).

Problems of this kind are known to exhibit depolarization effects, i.e., scattered field vector components in new directions. This phenomenon also happens for a cylinder moving along its axis [16], see also references to related problems of scattering by media moving parallel to the interface, e.g., planes, rotating spheres and cylinders, and cylinders moving along the axis, in [4, 5].

To show that (15) coupled with (22) are compatible with (18), (19), consider a boundary moving parallel to the interface of a medium possessing arbitrary μ, ε . With this geometry we get from (18), (21)

$$\begin{aligned}\hat{\mathbf{n}} \times \mathbf{E}'_{to} &= \hat{\mathbf{n}} \times \mathbf{E}_{to} + \hat{\mathbf{n}} \times (\mathbf{v} \times \mathbf{B}_{to}) = \hat{\mathbf{n}} \times \mathbf{E}_{to} + (\hat{\mathbf{n}} \cdot \mathbf{B}_{to})\mathbf{v} \\ \hat{\mathbf{n}} \times \mathbf{E}'_{in} &= \hat{\mathbf{n}} \times \mathbf{E}_{in} + \hat{\mathbf{n}} \times (\mathbf{v} \times \mathbf{B}_{in}) = \hat{\mathbf{n}} \times \mathbf{E}_{in} + (\hat{\mathbf{n}} \cdot \mathbf{B}_{in})\mathbf{v}\end{aligned}\quad (23)$$

But according to (8) $\hat{\mathbf{n}} \cdot \mathbf{B}_{to} = \hat{\mathbf{n}} \cdot \mathbf{B}_{in}$ for the geometry at hand. An analogous expression is obtained for the associated boundary conditions for the $\mathbf{H}$ fields

$$\begin{aligned}\hat{\mathbf{n}} \times \mathbf{H}'_{to} &= \hat{\mathbf{n}} \times \mathbf{H}_{to} - \hat{\mathbf{n}} \times (\mathbf{v} \times \mathbf{D}_{to}) = \hat{\mathbf{n}} \times \mathbf{H}_{to} - (\hat{\mathbf{n}} \cdot \mathbf{D}_{to})\mathbf{v} \\ \hat{\mathbf{n}} \times \mathbf{H}'_{in} &= \hat{\mathbf{n}} \times \mathbf{H}_{in} - \hat{\mathbf{n}} \times (\mathbf{v} \times \mathbf{D}_{in}) = \hat{\mathbf{n}} \times \mathbf{H}_{in} - (\hat{\mathbf{n}} \cdot \mathbf{D}_{in})\mathbf{v}\end{aligned}\quad (24)$$

and now according to (9) $\hat{\mathbf{n}} \cdot \mathbf{D}_{to} = \hat{\mathbf{n}} \cdot \mathbf{D}_{in}$. Thus it has been shown that the two approaches, with their pertinent constitutive relations, are consistent and compatible.

The question of the Minkowski constitutive relations goes beyond the plane half-space, circular cylinder and sphere, indeed rigid bodies moving or rotating such that the medium moves parallel to the interface. One could envisage a fluid medium contained by some arbitrary shaped boundary and at the boundary moving parallel to it. Although we deal now with non-uniform motion and velocity, we can still ask whether the Minkowski constitutive relations are valid, From the above analysis it follows that at least to the first order in β the Lorentz force formulas justify the relativistic Minkowski relations.

For media moving parallel to the interfaces, e.g., stratified plane, cylindrical, or spherical motion, there are cases when other methods can also be employed, e.g., see [17], using a Green function integral approach.

5. NORMAL SCATTERING AT A PLANE INTERFACE

The simplest example of scattering by moving objects is the normal reflection from a perfect conducting mirror moving in free space, according to $x = vt$, $\mathbf{v} = \hat{\mathbf{x}}v$ along the direction of propagation of the exciting wave. The plane harmonic exciting and scattered *waves* are given by

$$\begin{aligned} \mathbf{E}_{ex} &= \hat{\mathbf{z}}E_{ex}e^{i(k_{ex}x - \omega_{ex}t)}, & \mathbf{H}_{ex} &= -\hat{\mathbf{y}}H_{ex}e^{i(k_{ex}x - \omega_{ex}t)} \\ \mathbf{E}_{sc} &= -\hat{\mathbf{z}}E_{sc}e^{i(-k_{sc}x - \omega_{sc}t)}, & \mathbf{H}_{sc} &= \hat{\mathbf{y}}H_{sc}e^{i(-k_{sc}x - \omega_{sc}t)} \end{aligned} \quad (25)$$

$$k_{ex}/\omega_{ex} = k_{sc}/\omega_{sc} = (\mu_0\epsilon_0)^{1/2} = 1/v_{ph} = 1/c$$

$$E_{ex}/H_{ex} = E_{sc}/H_{sc} = (\mu_0/\epsilon_0)^{1/2} = \zeta$$

respectively, where in (25) v_{ph} is the phase velocity, in the present case the speed c of light in free space. As amply explained previously [8], we first substitute $x = vt$ in the incident wave in order to find the time dependent *signals* at the instantaneous position occupied by the moving surface at any given time. This is tantamount to saying that we define a local coordinate x_T

$$x_T = x - vt \quad (26)$$

parametrized by the time t , with the scatterer situated at $x_T = 0$. Note that we do not suggest a coordinate *transformation* in the Galilean or Lorentz transformation sense, and thus we do not seek *waves* in a new coordinate reference x_T .

Substituting from (26) into (25) yields a time dependent *signal* field rather than a *wave* satisfying the wave equation

$$\begin{aligned}\mathbf{E}_{ex} &= \hat{\mathbf{z}}E_{ex}e^{-i\omega_T t}, & \mathbf{H}_{ex} &= -\hat{\mathbf{y}}H_{ex}e^{-i\omega_T t} \\ \mathbf{E}_{sc} &= -\hat{\mathbf{z}}E_{sc}e^{-i\omega_T t}, & \mathbf{H}_{sc} &= \hat{\mathbf{y}}H_{sc}e^{-i\omega_T t} \\ \omega_T &= \omega_{ex}(1 - \beta) = \omega_{sc}(1 + \beta) \\ \beta &= v/v_{ph} = v/c\end{aligned}\tag{27}$$

This step is part of the evaluation of the boundary conditions and guarantees that the exciting and scattered signals have the same time dependence at the boundary. In order to have proper *waves* satisfying (1) and propagating at v_{ph} , it follows from (27) that

$$\omega_{sc}/\omega_{ex} = k_{sc}/k_{ex} = \xi, \quad \xi = (1 - \beta)/(1 + \beta)\tag{28}$$

Although ω_T is not the relativistically transformed frequency, we obtain for ω_{sc} the relativistically exact result obtained in [1], because of the ratios in (28), see also [8].

The amplitudes are calculated by the first line of (18), with $\mathbf{E}'_{in} = 0$

$$\begin{aligned}\mathbf{E}'_{ex} &= \mathbf{E}_{ex} + \mu_0 \mathbf{v} \times \mathbf{H}_{ex} = -\mathbf{E}'_{sc} = -(\mathbf{E}_{sc} + \mu_0 \mathbf{v} \times \mathbf{H}_{sc}) \\ |\mathbf{E}'_{ex}|/|\mathbf{H}'_{ex}| &= |\mathbf{E}'_{sc}|/|\mathbf{H}'_{sc}| = \zeta, \quad E_{sc}/E_{ex} = H_{sc}/H_{ex} = \xi\end{aligned}\tag{29}$$

where ζ, ξ are defined above, in (25), (28), respectively. Once again, (29) is the exact relativistic result obtained by Einstein [1].

For materials with arbitrary μ, ε , (18) is used. This amounts to implementing the classical Fresnel formulas for scattering of a plane wave at a plane interface (e.g., see [5]), except that here the effective fields $\mathbf{E}'_{ex}, \mathbf{E}'_{sc}$ are involved, and the result is relativistically exact only within the first order of the velocity factor β .

Note carefully that ω_T in (27) is *not* the frequency exciting the moving object as obtained from exact relativistic transformations. Actually a factor γ , (3), is missing. However, for the scattered frequency (28) this has no effect. But if we deal with dispersive media, the present frequency ω_T exciting the scatterer is correct only to the first order in β .

The case of normal incidence and a plane scatterer moving parallel to the interface yields vanishing cross terms, e.g., see (21). Consequently there will be no velocity effect present, and no motional effect of first order in β . Note, however, that relativistically we encounter the transverse Doppler effect involving second-order terms in β in the frequency. Once again, this shows that for dispersive media the results will be relativistically exact to within first order in β .

6. OBLIQUE SCATTERING AT A PLANE INTERFACE

It is always possible to resolve an arbitrarily polarized plane wave into two components, with one exciting field, $\mathbf{E}$, or $\mathbf{H}$, parallel to the reflecting half space. Let us therefore take the exciting wave as

$$\begin{aligned}\mathbf{E}_{ex} &= \hat{\mathbf{z}}E_{ex}e^{i\varphi_{ex}}, & \mathbf{H}_{ex} &= \hat{\mathbf{k}}_{ex} \times \hat{\mathbf{z}}H_{ex}e^{i\varphi_{ex}}, & \varphi_{ex} &= \mathbf{k}_{ex} \cdot \mathbf{r} - \omega_{ex}t \\ \mathbf{k}_{ex} \cdot \mathbf{r} &= k_{ex}r \cos(\theta - \alpha_{ex}) = k_{ex,x}x + k_{ex,y}y \\ \mathbf{E}_{sc} &= -\hat{\mathbf{z}}E_{sc}e^{i\varphi_{sc}}, & \mathbf{H}_{sc} &= -\hat{\mathbf{k}}_{sc} \times \hat{\mathbf{z}}H_{sc}e^{i\varphi_{sc}}, & \varphi_{sc} &= \mathbf{k}_{sc} \cdot \mathbf{r} - \omega_{sc}t \\ \mathbf{k}_{sc} \cdot \mathbf{x} &= k_{sc}r \cos(\theta - \alpha_{sc}) = -k_{sc,x}x + k_{sc,y}y\end{aligned}\tag{30}$$

The *signal* at the interface's instantaneous position $x_T = 0$, obtained from (26) by substituting $x = vt$ into the phases in (30), prescribes the same phase $\varphi_{ex} = \varphi_{sc}$ at the surface, hence

$$\begin{aligned}\omega_T &= \omega_{ex} - k_{ex,x}v = \omega_{sc} + k_{sc,x}v \\ &= \omega_{ex}(1 - \beta \cos \alpha_{ex}) = \omega_{sc}(1 + \beta \cos \alpha_{sc}) \\ k_{ex,z} &= k_{ex} \sin \alpha_{ex} = k_{sc,z} = k_{sc} \sin \alpha_{sc}\end{aligned}\tag{31}$$

Based on (30), (31), it has already been shown in [8] that the scattered frequency is given by

$$\omega_{sc}/\omega_{ex} = \gamma^2(1 - 2\beta \cos \alpha_{ex} + \beta^2)\tag{32}$$

being the exact relativistic result obtained in [1], including the second order velocity effects. Similarly, it is shown in [8] that in agreement with [1], the directions obtained from (31) are related by

$$\cos \alpha_{sc} = -[(1 + \beta^2) \cos \alpha_{ex} - 2\beta]/[1 - 2\beta \cos \alpha_{ex} + \beta^2]\tag{33}$$

and are therefore relativistically exact.

The aberration phenomenon described, e.g., by (33) is therefore not contingent on the Special Relativity theory. In fact, astronomers were aware of the aberration phenomenon (attributed to James Bradley (1693–1762) who discovered the aberration of light in 1725–26 (published 1729)) even before the advent of the theory of relativity.

If the medium is dispersive, then ω_T is involved in the results. Once again we note that ω_T , (31), differs from the exact relativistic transformed excitation frequency. The present results, based on the Lorentz force formulas (5), (7), lead therefore to results which are correct within the first order in the velocity factor β .

The calculation of the amplitudes requires (5), (7), (19), (30). Similarly to (31–33) we obtain

$$\begin{aligned}\mathbf{E}'_{ex} &= \mathbf{E}_{ex} + \mu_0 \mathbf{v} \times \mathbf{H}_{ex} = -\mathbf{E}'_{sc} = -(\mathbf{E}_{sc} + \mu_0 \mathbf{v} \times \mathbf{H}_{sc}) \\ \mathbf{E}'_{ex} &= \mathbf{E}_{ex}(1 - \beta \cos \alpha_{ex}), \quad \mathbf{E}'_{sc} = \mathbf{E}_{sc}(1 + \beta \cos \alpha_{sc})\end{aligned}\quad (34)$$

Consequently (34) yields the same amplitude ratios as in (32)

$$E_{sc}/E_{ex} = H_{sc}/H_{ex} = \gamma^2(1 - 2\beta \cos \alpha_{ex} + \beta^2) \quad (35)$$

and once again (35) is an exact relativistic result.

7. SCATTERING BY A LINEALLY MOVING CYLINDER

The relativistic problem of scattering by a cylinder, moving with constant velocity $x = vt$, $\mathbf{v} = \hat{\mathbf{x}}v$, perpendicularly to the cylindrical z axis, has been discussed before [4, 18]. Recently the non-relativistic treatment of the problem has been considered [8]. The method considered translation (as opposed to real motion involving effects as in (5), (7)), which introduced phase differences and accordingly frequency shifts, but did not take into account motional amplitude effects in the manner presently pursued. We will show now that a careful application of the Lorentz force formulas boundary conditions (18), (19) facilitates a relativistically correct first order in β analysis. This is a crucial observation, justifying the implementation of the method for general, motional modes, involving non-uniform velocity, as discussed later.

The excitation *wave* is taken as in the first line (25). Similarly to (26), (27), the *signal* at some special point on the scatterer is computed. This can be any point in the local coordinate system of the scatterer, since it only serves to provide a reference for the phase shift of the excitation wave relative to this point. The simplest choice is to take the cylinder's center $x_T = 0$ as the reference point, although it is physically inaccessible to the incident wave. As in (27), we have

$$\begin{aligned}\mathbf{E}_{ex} &= \hat{\mathbf{z}}E_{ex}e^{-i\omega_T t}, \quad \mathbf{H}_{ex} = -\hat{\mathbf{y}}H_{ex}e^{-i\omega_T t} \\ \omega_T/\omega_{ex} &= k_T/k_{ex} = 1 - \beta, \quad \omega_{ex}/k_{ex} = \omega_T/k_T = c\end{aligned}\quad (36)$$

Associated with the frequency ω_T (36), we also consider a propagation constant $k_T = \omega_T/c$ because the excitation wave propagates in the initial space, with speed c in our example. This is a crucial point, as mentioned below.

Define a local coordinate system

$$\begin{aligned}x_T &= x - vt = r \cos \theta - vt = r_T \cos \theta_T \\ y_T &= y = r \sin \theta = r_T \sin \theta_T\end{aligned}\quad (37)$$

whose origin $x_T = 0$, $y_T = 0$ is the center of the cross-sectional circle. Note that (37) could be construed as a Galilean coordinate transformation, and one is tempted to simply substitute (37) into the plane wave $\mathbf{E}_{ex}$ in (25), and consider it a *wave* in the comoving system of reference. Of course this would *not* yield the relativistically correct results to first order in β , as required. Instead, this would introduce obsolete Galilean concepts like the rule of combining propagation velocities, i.e., replacing c by $c - v$, which has no place in the present context. Here there is no attempt to introduce moving observers, or to perform measurements in a moving coordinate system: Conceptually everything is considered in the initial reference system $\mathbf{X}$. Time t in (37) serves only as a parameter tracking the position of the circle in question.

The phase difference between the origin $x_T = 0$, $y_T = 0$ and the circumference of the circle is given by $k_T R \cos \theta_T$, and the total phase factor of the excitation signal at the boundary, recast in a Fourier-Bessel integral, is therefore

$$e^{i\varphi_{ex}} = \sum_{m=-\infty}^{\infty} i^m J_m(k_T R) e^{im\theta_T - i\omega_T t}, \quad \varphi_{ex} = k_T R \cos \theta_T - i\omega_T t \quad (38)$$

Corresponding to (27) we define the effective excitation field

$$\begin{aligned} \mathbf{E}'_{ex} &= \hat{\mathbf{z}} E'_{ex} e^{i\varphi_{ex}}, \quad \mathbf{H}'_{ex} = -\hat{\mathbf{y}} H'_{ex} e^{i\varphi_{ex}} \\ E'_{ex}/E_{ex} &= H'_{ex}/H_{ex} = 1 - \beta \end{aligned} \quad (39)$$

From (5), (7), (34) the relation of the amplitudes is derived, as given in (39).

Consider first the perfectly conducting circular cylinder. Based on past experience in solving the classical problem of a cylinder at rest, and the associated relativistic problem of a lineally moving cylinder [8, 18], we choose for the effective scattered signal

$$\mathbf{E}'_{ex} = -\hat{\mathbf{z}} E'_{ex} \sum_{m=-\infty}^{\infty} i^m a_m H_m^{(1)}(k_T R) e^{im\theta_T - i\omega_T t} \quad (40)$$

where $H_m^{(1)}$ denotes the Hankel function of the first kind and order m . The combination of $H_m^{(1)}$ and the time factor $e^{-i\omega_T t}$ guarantees the correct radiation condition for the scattered *waves*, introduced below. Using the *signals* (36), (38)–(40) to satisfy (18) (with $\mathbf{E}'_{in} = 0$), and exploiting the orthogonality of the Fourier-Bessel series prescribes

$$a_m = J_m(k_T R)/H_m^{(1)}(k_T R) \quad (41)$$

which agrees with the problem of scattering by a cylinder at rest, and in the relativistic context—in the comoving frame of reference, it constitutes the correct relativistic result to first order in β , as computed before, see [4, 18].

But suggesting the coefficients (41), which indeed satisfy the boundary conditions for the *signals* at the boundary, constitutes only one step in creating a full solution to the problem. Now we have to find the corresponding scattered *wave*, which must be a solution of the wave equation and simultaneously reduce to the signal (40), (41) on the boundary. This is not trivial.

We start with an arbitrary plane wave as in the first line of (30), polarized along the cylindrical axis $\hat{\mathbf{z}}$ and propagating in the direction indicated by α

$$\begin{aligned}\mathbf{E}_\alpha &= \hat{\mathbf{z}}E_\alpha e^{i\varphi_\alpha}, & \mathbf{H}_\alpha &= \hat{\mathbf{k}}_\alpha \times \hat{\mathbf{z}}H_\alpha e^{i\varphi_\alpha}, \\ \varphi_\alpha &= \mathbf{k}_\alpha \cdot \mathbf{r} - \omega_\alpha t = k_{\alpha,x}x + k_{\alpha,y}y - \omega_\alpha t = k_\alpha r \cos(\theta - \alpha) - \omega_\alpha t \\ k_{\alpha,x} &= k_\alpha \cos \alpha, & k_{\alpha,y} &= k_\alpha \sin \alpha\end{aligned}\tag{42}$$

Computing the time signal for this wave at the local origin $x_T = 0$, $y_T = 0$ as in (36), we find

$$\begin{aligned}\mathbf{E}_\alpha &= \hat{\mathbf{z}}E_\alpha e^{-i\omega_\alpha T t}, & \mathbf{H}_\alpha &= \hat{\mathbf{k}}_\alpha \times \hat{\mathbf{z}}H_\alpha e^{-i\omega_\alpha T t} \\ \omega_{\alpha T}/\omega_\alpha &= k_{\alpha T}/k_\alpha = 1 - \beta \cos \alpha \\ \omega_\alpha/k_\alpha &= \omega_{\alpha T}/k_{\alpha T} = c\end{aligned}\tag{43}$$

In order to satisfy the boundary conditions, the frequencies in (36), (43) must be equal on the circle $r_T = R$, thus yielding

$$\omega_{\alpha T}/\omega_T = k_{\alpha T}/k_T = 1\tag{44}$$

and

$$\omega_\alpha/\omega_{ex} = k_\alpha/k_{ex} = (1 - \beta)/(1 - \beta \cos \alpha)\tag{45}$$

The result (45), because it is a ratio, once again eliminates factors γ involved in the relativistic transformations of frequencies, and turns out to be relativistically exact.

The phase shift on the circumference $r_T = R$ relative to the reference point $x_T = 0$, $y_T = 0$ is the phase incurred along the projection of the radius R on the direction of propagation. Instead of (38) we now have

$$e^{i\varphi_\alpha}, \varphi_\alpha = k_{\alpha T}R \cos(\theta_T - \alpha) - \omega_{\alpha T}t\tag{46}$$

Exploiting the Sommerfeld integral representations for the cylindrical functions, e.g., see [5, 19], a superposition of functions shown in (45) is constructed

$$\begin{aligned} \mathbf{E}'_{sc} &= -\hat{\mathbf{z}}E'_{ex} \int_{\theta_T-(\pi/2)+i\infty}^{\theta_T+(\pi/2)-i\infty} e^{ik_T R \cos(\theta_T-\alpha)-i\omega_T t} g(\alpha) d\alpha/\pi \\ g(\alpha) &= \sum_{m=-\infty}^{\infty} a_m e^{im\alpha} \end{aligned} \quad (47)$$

which is consistent with (40). Note that in (47) the integration is over a range of complex propagation angles α . This is a mathematical consequence of the Sommerfeld integral representation for the Hankel cylindrical functions.

According to (5) (see also (34) for similar considerations), in order to find the scattered field for which (47) provides $\mathbf{E}'_{sc}$, we need to supplement each signal in the integrand by a factor $(1 - \beta \cos \alpha)^{-1} \approx 1 + \beta \cos \alpha$, exact to the first order in β . Consequently, the scattered signal at the boundary is given by

$$\begin{aligned} \mathbf{E}_{sc} &= -\hat{\mathbf{z}}E'_{ex} \int_{\theta_T-(\pi/2)+i\infty}^{\theta_T+(\pi/2)-i\infty} e^{ik_T R \cos(\theta_T-\alpha)-i\omega_T t} G(\alpha) d\alpha/\pi \\ &= -\hat{\mathbf{z}}E'_{ex} \sum_{m=-\infty}^{\infty} i^m A_m H_m^{(1)}(k_T R) e^{im\theta_T - i\omega_T t} \end{aligned} \quad (48)$$

$$\begin{aligned} G(\alpha) &= (1 + \beta \cos \alpha) g(\alpha) = \sum_{m=-\infty}^{\infty} A_m e^{im\alpha} \\ A_m &= a_m + (\beta/2)(a_{m-1} + a_{m+1}) \end{aligned}$$

Recasting (48) in the form

$$\begin{aligned} \mathbf{E}_{sc} &= -\hat{\mathbf{z}}E'_{ex} \sum_{m=-\infty}^{\infty} i^m a_m e^{im\theta_T - i\omega_T t} \\ &\quad \cdot \left[H_m^{(1)} + (i\beta/2)(H_{m+1}^{(1)} e^{i\theta_T} - H_{m-1}^{(1)} e^{-i\theta_T}) \right] \end{aligned} \quad (49)$$

where all the Hankel functions have the argument $k_T R$ as in (48), emphasizes how an interaction of the cylindrical modes is created. Thus for example, a cylinder at rest with a dominant monopole term $a_0 \neq 0$, $a_m = 0$, $m \neq 0$, e.g., for an $\mathbf{E}$ field polarized along the axis of a thin perfectly conducting cylinder, scattering omni-directionally, the velocity effect introduces additional dipole terms. See [18] for some simulation graphical results.

For a cylinder of arbitrary constitutive parameters, the coefficients (41) are replaced by the appropriate coefficients involving the internal material parameters (19). The same procedure can be applied to cylinders of arbitrary cross section. All we need to know is the pertinent scattering amplitude function $g(\alpha) = \sum_{m=-\infty}^{\infty} a_m e^{im\alpha}$ (e.g., as an

experimental result), from which the coefficients a_m of the Fourier series can be derived. We do not elaborate on these aspects here.

The relativistic treatment of scattering by spheres and arbitrary three-dimensional linearly moving objects has been discussed before [18] in the relativistic context. The present approach based on the Lorentz force formulas will lead to the relativistically correct results, within the first order in β .

Finally, the scattered *wave* reducing to the *signal* (48), (49) at the boundary must be discussed. The way the subject was developed above, each of the signals appearing under the integral sign in (47) corresponds to a scattered plane wave given by (42), (43). Consequently, the scattered wave can be represented in the form

$$\mathbf{E}_{sc} = \hat{\mathbf{y}} E'_{ex} \int_{\theta_T - (\pi/2) + i\infty}^{\theta_T + (\pi/2) - i\infty} e^{ik_\alpha r \cos(\theta - \alpha) - i\omega_\alpha t} G(\alpha) d\alpha / \pi \quad (50)$$

where θ_T , θ are related by (37), and ω_α , ω_T , and the corresponding k_α , k_T are related by formulas given in (44), (45).

The actual integration of (50) is complicated, and probably will not be possible analytically, except for some limiting cases. However, we can say a few things about the qualitative nature of the scattering process: It is seen that (50) describes a continuous spectrum of ω_α and the corresponding k_α . Moreover, if the saddle point approximation is applied to (50), only a single instantaneous frequency appears. Thus as the distance from the scatterer increases, the spectrum increasingly narrows, and in the limit is described, similarly to the results (32), (33), (35), by scattering by a plane moving in an arbitrary direction, this direction changing with time.

8. SCATTERING BY AN ECCENTRICALLY ROTATING CYLINDER

The general methodology introduced above is adequate for dealing with arbitrary modes of motion. In order to focus on a concrete example, we choose the eccentrically rotating cylinder. Problems involving periodic motion of boundaries have been considered before in the relativistic context, e.g., scattering by an oscillating plane reflector [4]. Another

class of periodic motion, in the acoustical context, has also been discussed [20], treating periodically perturbed scattering boundaries.

Problems of this kind are technically interesting because periodic mechanical motion is found in many engineering applications, such as vibrating or rotating objects, be it aircrafts or marine vessels, or on smaller scales, equipment as found in the workshop. Acquiring spectral signatures of such objects could assist in remote sensing, for various applications.

The following is a computed example of the spectrum expected from an eccentrically rotating scatterer. We would expect the periodic motion to “modulate” the waves, and thus display a discrete spectrum which contains the incident frequency and additional sidebands corresponding to the motional frequencies and their harmonics. However, things are more complicated, because we have seen from the lineally moving object, analyzed above, that a continuous spectrum is created. Hence the purely discrete spectrum will be manifested only in limiting cases.

One such case involves eccentrically rotating, circular, thin (with respect to wavelength) cylinders. In this example we do not attribute rotation about some axis to the cylinder’s material itself, although problems belonging to this class can also be considered. Rather we refer to the motion of the cylindrical axis of the object, defined by a local coordinate system x_T, y_T . Similarly to (37) we define

$$x_T = x - Q \cos \theta, \quad y_T = y - Q \sin \theta, \quad \theta = \Omega t \quad (51)$$

where Q, Ω , are the radius and the frequency associated with the rotation, respectively. Accordingly (51) prescribes

$$\mathbf{v} = -\hat{\mathbf{x}}\Omega Q \sin \theta + \hat{\mathbf{y}}\Omega Q \cos \theta = \Omega Q \hat{\mathbf{e}} \quad (52)$$

for the motion of the axis.

The incident wave is chosen as in (25), hence at the position of the cylinder’s axis, $x_T = 0, y_T = 0$, the *signal* field is found by substituting from (51)

$$\begin{aligned} \mathbf{E}_{ex} &= \hat{\mathbf{z}}E_{ex}e^{i\varphi_{ex}}, \quad \mathbf{H}_{ex} = -\hat{\mathbf{y}}H_{ex}e^{i\varphi_{ex}} \\ \varphi_{ex} &= k_{ex}Q \cos \Omega t - i\omega_{ex}t, \quad e^{i\varphi_{ex}} = \sum_{m=-\infty}^{\infty} i^m A_m e^{-i\omega_m t} \\ \omega_m &= \omega_{ex} - m\Omega, \quad A_m = J_m(k_{ex}Q) \end{aligned} \quad (53)$$

We may call the effect demonstrated in (53) a “Doppler effect,” however it does not follow the usual Doppler frequency shift formula, expressed by the velocity, as in (27) for example.

The effective field signal is obtained according to (5), (52)

$$\mathbf{E}'_{ex} = \hat{\mathbf{z}}E'_{ex} = \hat{\mathbf{z}}E_{ex}e^{i\varphi_{ex}}(1 + \beta \sin \Omega t), \quad \beta = \Omega Q/c \quad (54)$$

and a similar expression for $\mathbf{H}'_{ex}$. Recasting (54) by redefining indices, and exploiting the formula $J_{m-1}(\rho) + J_{m+1}(\rho) = (2m/\rho)J_m(\rho)$ for cylindrical functions (e.g., see [19]) yields,

$$e^{i\varphi_{ex}}(1 + \beta \sin \Omega t) = \sum_{m=-\infty}^{\infty} i^m A'_m e^{-i\omega_m t} \quad (55)$$

$$A'_m = A_m(1 - \Omega m/\omega_{ex})$$

Consequently (55) shows that the spectrum content remains the same, although the amplitude of individual sidebands is changed.

For an arbitrary cylinder we need now to consider the additional phase shift of the wave at the boundary, relative to the local origin $x_T = 0$, $y_T = 0$ defined in (51). This calls for a procedure similar to that effected in (38), etc. Accordingly each sideband of frequency ω_m will lead to a continuous spectrum. Instead of analyzing this complicated problem, we consider here the case of the thin cylinder. For each frequency ω_m , the criterion for thinness is $\omega_m R/c \ll 1$. Obviously this can only be satisfied for a finite band of sidebands, but for many practical problems of electromagnetic propagation and feasible mechanical motion this seems to be a sound approximation for cylinders satisfying $\omega_{ex} R/c \ll 1$. When the criterion is met, then the cylinder's boundary is considered to be excited by (54), (55).

In order to construct the scattered field we start with a legitimate solution of the wave equation, this time we start with a cylindrical outgoing wave

$$\begin{aligned} \mathbf{E}_n &= \hat{\mathbf{z}}E_n i^n H_n^{(1)}(k_v r) e^{in\theta - i\omega_v t} \\ &= \hat{\mathbf{z}}E_n e^{-i\omega_v t} \int_{\theta - (\pi/2) + i\infty}^{\theta + (\pi/2) - i\infty} e^{ik_v r \cos(\theta - \alpha) + in\alpha} d\alpha / \pi \\ \mathbf{H}_n &= H_n e^{-i\omega_v t} \int_{\theta - (\pi/2) + i\infty}^{\theta + (\pi/2) - i\infty} e^{ik_v r \cos(\theta - \alpha) + in\alpha} \hat{\mathbf{k}}_v \times \hat{\mathbf{z}} d\alpha / \pi \\ E_n/H_n &= \zeta, \quad \omega_v/k_v = c \end{aligned} \quad (56)$$

where in (56) $\hat{\mathbf{H}}_n = \hat{\mathbf{k}}_v \times \hat{\mathbf{z}}$ defines the mutually perpendicular vectors associated with the plane waves in the superposition shown in (56). The signal corresponding to (56), at the cylinder's center is obtained by substituting $r = Q$, $\theta = \Omega t$, into (56). This can be done now or after computing the effective signal fields.

The effective signal values associated with (56) are defined by (5), (7), (52). For the $\mathbf{E}$ field, for example, this yields

$$\begin{aligned}
 \mathbf{E}'_n &= \hat{\mathbf{z}} E_n e^{-i\omega_v t} \int_{\theta-(\pi/2)+i\infty}^{\theta+(\pi/2)-i\infty} e^{ik_v r \cos(\theta-\alpha)} h(\alpha) d\alpha / \pi \\
 h(\alpha) &= e^{in\alpha} (1 + \beta \sin(\Omega t - \alpha)) \\
 &= e^{in\alpha} + (\beta/2i) (e^{i(n-1)\alpha+i\Omega t} - e^{i(n+1)\alpha-i\Omega t}) \\
 \mathbf{E}'_n &= \hat{\mathbf{z}} E_n e^{-i\omega_v t} i^n e^{in\theta} G_n \\
 G_n &= H_n^{(1)}(k_v r) - (\beta/2) (H_{n-1}^{(1)}(k_v r) e^{-i(\theta+\Omega t)} + H_{n+1}^{(1)}(k_v r) e^{i(\theta+\Omega t)})
 \end{aligned} \tag{57}$$

Note carefully that (57), being a *signal* at some location, does not satisfy the *wave* equation any more.

For our scattered wave we choose an infinite sum of the waves appearing in (56)

$$\mathbf{E}_{sc} = \sum_{n=-\infty}^{\infty} \mathbf{E}_n = \hat{\mathbf{z}} \sum_{n=-\infty}^{\infty} E_n i^n H_n^{(1)}(k_v r) e^{in\theta - i\omega_v t} \tag{58}$$

and the associated $\mathbf{H}_{sc}$ field. Accordingly the effective field expressed in (57) becomes

$$\begin{aligned}
 \mathbf{E}'_{sc} &= \sum_{n=-\infty}^{\infty} \mathbf{E}'_n = \hat{\mathbf{z}} \sum_{n=-\infty}^{\infty} E_n e^{-i\omega'_v t} i^n G_n, \quad \omega'_v = \omega_v - n\Omega t \\
 G_n &= H_n^{(1)}(k_v Q) - (\beta/2) (H_{n-1}^{(1)}(k_v Q) e^{-i2\Omega t} + H_{n+1}^{(1)}(k_v Q) e^{i2\Omega t})
 \end{aligned} \tag{59}$$

The two field signals (54), (55) and (59), must satisfy boundary conditions as in (18), (19) at the surface of the cylinder. Before material considerations are taken into account, the spectral components must be identified, because the orthogonality of the spectral exponentials prescribes that the excitation and scattered frequencies at the scatterer's surface must coincide. It follows from (53) that each frequency ω_m must be paired with each frequency ω'_v , $\omega'_v \pm 2\Omega$ prescribed in (59). Only then the geometrical and material considerations can be effected. Inasmuch as the present study is not interested in solutions *per se*, but rather with their feasibility, the particulars are not further pursued here.

However, by assuming a thin, perfectly conducting cylinder, the salient characteristics of the solution become apparent: In such a case we expect the spectrum to be discrete, displaying in the scattered

wave the incident frequency and sidebands separated by the mechanical frequency Ω . This only applies to thin cylinders (or to plane interfaces and their ray approximation in the other extreme), as we have seen that the finite cylinder, even moving with a constant velocity, creates a continuous spectrum scattered field. Moreover, we are now confident that the solution is physically correct to the first order in the velocity, and agrees (to within first order in the velocity) with the quasi-relativistic approach of assuming instantaneous velocities.

9. DISCUSSION AND CONCLUDING REMARKS

After almost a century since the appearance of Einstein's monumental paper [1], the subject area of scattering in velocity dependent systems reached the stage where it can be re-examined and integrated into the Maxwell electromagnetic theory in a fundamental manner. Until now, the research in this area was mainly advanced by solving special problems. Almost invariably, the tool to solve velocity dependent problems relied on Einstein's theory of Special Relativity, even though for non-uniform motion the theory is not valid. It must be remembered that this "instantaneous velocity" approach is heuristic.

The present approach uses special-relativistic results only as a benchmark. What is attempted here is to directly tackle this class of problems by invoking the Lorentz force formulas and derive from them the boundary conditions (5), (7).

We have examined scattering by moving plane interfaces, moving cylinders, and periodically moving objects.

The various problems discussed above show that the derived results agree with the strictly relativistic and the "instantaneous velocity" approach to at least within the first order in $\beta = v/c$. In the case of plane interfaces and nondispersive media, exact relativistic results are derived for the scattered waves. However, for arbitrary shapes, where structural (geometric) dispersion is present, and/or when macroscopic material dispersion is displayed, the excitation frequency differs, with second order velocity factors γ (e.g., see (3)) involved.

With the new impetus imparted to the area of velocity dependent scattering, and with the availability of strong numerical packages for solving electromagnetic problems, one may hope that problems involving moving and rotating scatterers will be considered.

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