Progress In Electromagnetics Research, PIER 44, 131–142, 2004

NUMERICAL ANALYSIS OF TWO DIMENSIONAL TAPERED DIELECTRIC WAVEGUIDE

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Abstract—A simple method is presented to obtain the scattering parameters of the two dimensional tapered dielectric waveguide, by discrete approximation to tapering, consisting of series of steps. The two dimensional step discontinuity of the junction of two different dielectric rectangular waveguides has been solved using integral equation arising from the field matching of the discrete modes and the continuous spectrum. Accurate numerical solution has been obtained using Ritz-Galerkin variational approach with appropriate sets of expanding functions. The results in the form of scattering parameters for varying tapered length have been depicted graphically. Computed results from generalized integral expressions are found to be in excellent agreement with results obtained in two-dimensional case. With this method it is possible to design the structure to enlarge the cross section of a mode in a slow and controlled manner.

1. INTRODUCTION

Discontinuities in dielectric waveguides play an important role in designing components in millimeter, submillimeter and optical circuitry. Quit often in these applications an open waveguide with one particular cross-section must be joined to a waveguide of another cross-section. The open waveguides to be connected differ sometimes not only in size but also in their cross-sectional form. Usually these waveguide connectors are required to launch as much as possible of the power that is incident in one waveguide into the other waveguide.

In such waveguide transitions power may be lost to reflection and radiation. The transition should be designed to keep radiation reflection loss at a minimum. Transitions between different dielectric wave guides in form of gradual waveguide tapers are particularly well suited for open waveguides because of their microscopic dimensions.

For the problems with small discontinuities Marcuse [1] approximated tapered dielectric waveguide by many infinitesimal step junctions, then assumed that the modes of the adjacent waveguides are approximately orthogonal. Miyanaga and Asakura [2] solved a linearly tapered grating coupler on basis of the first order perturbation theory of dividing the grating region in to short subsections.

For problems with large steps, Rozzi and In'tVeld [3, 4] solved an integral equation by the Ritz-Galerkin method for one-dimensional case. Asok De et al. [5] for two-dimensional case.

In this paper we have calculated scattering parameters for the two-dimensional tapered dielectric by many steps junctions. For each step junction scattering parameters [5] have been calculated and then cascaded to find out the overall value along taper length. The validity of the proposed method is examined by observing the convergence and by comparing the tapering effect on two different size waveguide.

2. GENERALIZED SCATTERING MATRIX TECHNIQUE

We consider first the abrupt transition between two dielectric waveguides of wave impedances Z_1 and Z_2 Fig. 1. The refractive index distribution for waveguide I ($z \leq 0$) and II ($z \geq 0$) is n_1 and outside the guide is n_2. Here $n_1^2 = \varepsilon_1$ and $n_2^2 = \varepsilon_2$ are dielectric constants. Fig. 1 shows the front view of the two dimensional dielectric waveguide with dimensions $2d$ and $2d2$ along x direction for waveguide I and II respectively. The top view also looks like front view with dimensions $2d1$ and $2d3$ along y direction for waveguide I and II respectively. The scattering matrix of this junction relates the forward and backward

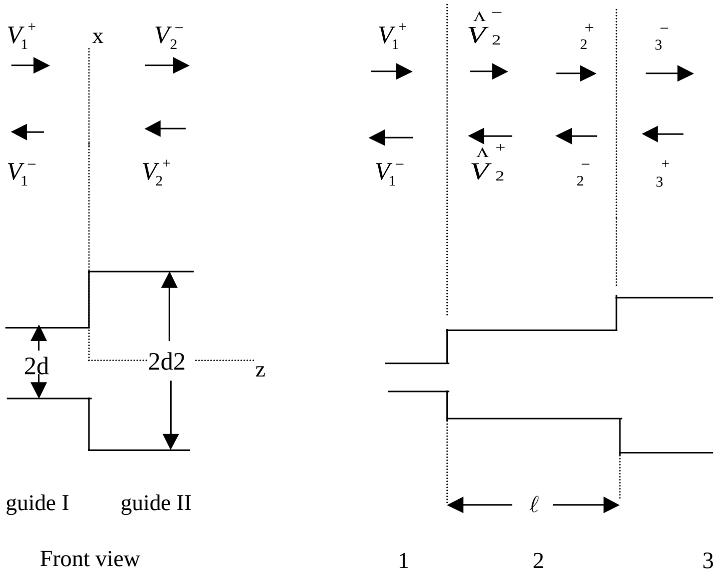

Figure 1. Front view of the two dimensional dielectric waveguide with abrupt junction and step approximation to the taper.

wave amplitudes $V_1^{\binom{+}{-}}$ in waveguide 1 and $V_2^{\binom{+}{-}}$ in guide 2 to each other.

$$\begin{bmatrix} V_1^- \\ V_2^- \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} V_2^+ \\ V_2^- \end{bmatrix} = [T_{21}] \begin{bmatrix} V_1^+ \\ V_1^- \end{bmatrix} \quad \text{where } T_{21} = \begin{bmatrix} \dfrac{S_{12}^2 - S_{11}S_{22}}{S_{12}} & \dfrac{S_{22}}{S_{12}} \\ -\dfrac{S_{11}}{S_{12}} & \dfrac{1}{S_{12}} \end{bmatrix} \tag{2}$$

The transition matrix $T_2' = \begin{bmatrix} e^{-j\phi_2} & 0 \\ 0 & e^{j\phi_2} \end{bmatrix}$ where $\phi_2 = \beta_2\ell$, β_2 is propagation constant and between the dielectric waveguide steps length ℓ relates the wave amplitudes at the output of the section to its input amplitudes by

$$\begin{bmatrix} V_2^+ \\ V_2^- \end{bmatrix} = [T_2'] \begin{bmatrix} \hat{V}_2^+ \\ \hat{V}_2^- \end{bmatrix} \tag{3}$$

The total transmission matrix $\hat{T}$ of the nonuniform dielectric waveguide of Fig. 1 including the two abrupt transitions on both sides is then given by

$$\hat{T} = T_{32}T_2'T_{21} \tag{4}$$

It provides us the relation

$$\begin{bmatrix} V_3^+ \\ V_3^- \end{bmatrix} = [\hat{T}] \begin{bmatrix} V_1^+ \\ V_1^- \end{bmatrix} \tag{5}$$

between the overall input and output wave amplitudes. A similar procedure can be applied to calculate the transmission characteristics of multisection dielectric waveguides, which consists of a series of uniform waveguides with abrupt transition at the junctions. A typical front view and top view of the taper Fig. 2, which involves gradual change in the guide cross-section. A discrete approximation to this shape is made consisting of a series of steps.

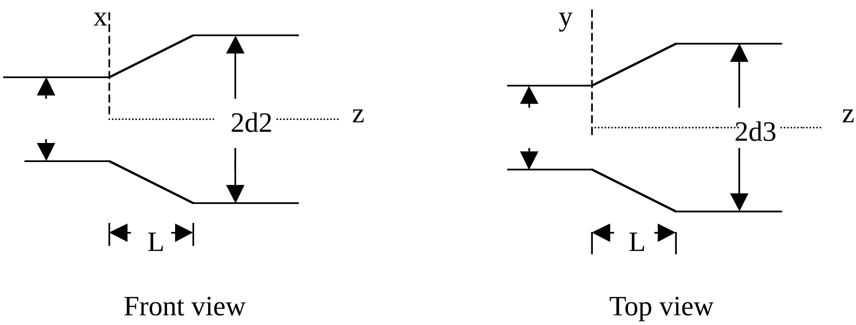

Figure 2. The front view and top view of the two dimensional dielectric waveguide with taper length L. The geometrical dimensions are $d/d2$, $d1/d3$ equals to 0.111, $\varepsilon_1 = 5$, $\varepsilon_2 = 1$.

3. ANALYSIS OF DISCONTINUITY

3.1. Scattering Matrix Formulation (TE Case)

Using a field matching technique, which requires field description on either side of the discontinuity in terms of modes, can solve the diffraction problem at an abrupt discontinuity. The complete field propagating in an open waveguide can be resolved into a finite set of surface wave modes and a continuum of radiative modes [5]. In the following we considered a two dimensional dielectric waveguide, excited by transverse electric (TE) waves with the transverse field components E_y and H_x.

For a TE mode excitation with variation along the x-y direction the field components are related as

$$H_x(x,y,z) = \frac{1}{j\omega\mu}\frac{\partial}{\partial z}E_y(x,y,z) = \frac{\beta}{\omega\mu}E_y(x,y,z) \tag{6}$$

where ω, μ, β are angular frequency, permeability and propagation constant respectively. $E_y(x,y,z)$ may be expressed as a modal expression

$$E_y = \left\{\sum_k a_k(x,y)\varphi_k(x,y)\int_0^\infty\int_0^\infty b(k_x,k_y)\phi(x,k_x:y,k_y)dk_x dk_y\right\} e^{j(\omega t-\beta z)} \tag{7}$$

$a_k(x,y)$ and $b(k_x,k_y)$ are unknown amplitudes of the surface $\varphi(x,y)$ and continuum $\phi(x,y)$ modes respectively. Let us consider a steady state and source free problem, with two different semi-infinite two dimensional waveguides forming a step discontinuity at $z = 0$ (Fig. 1). The incident field considered here will be composed of surface waves only. For now, let us assume that there are n_j surface modes which are capable of propagating in guide I (left), and n_r surface modes which can propagate in guide II (right), with the total number of propagating surface waves given by

$$n_i = n_j + n_r \tag{8}$$

Continuity of electric field $E_y(x,y,z)$ and the magnetic field $H_x(x,y,z)$ at $z = 0$ is expressed as

$$\begin{aligned}
E_y(x,y,0) &= \sum_{k=1}^{n_j}\left(V_k^i + V_k^r\right)\varphi_k(x,y) \\
&\quad + \int_0^\infty\int_0^\infty V^I(k_x,k_y)\phi^I(x;k_x,y;k_y)dk_x dk_y \\
&= \sum_{k=n_j+1}^{n_i}\left(V_k^i + V_k^r\right)\varphi_k(x,y) \\
&\quad + \int_0^\infty\int_0^\infty V^{II}(k_x,k_y)\phi^{II}(x;k_x,y;k_y)dk_x dk_y \qquad (9) \\
H_x(x,y,0) &= \sum_{k=1}^{n_j} -\frac{1}{z_k}\left(V_k^i - V_k^r\right)\varphi_k(x,y)
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^\infty\int_0^\infty \frac{V^I(k_x,k_y)\phi^I(x;k_x,y;k_y)}{Z^I(k_x,k_y)}dk_xdk_y \\
&= \sum_{k=n_j+1}^{n_i} -\frac{1}{z_k}\left(-V_k^i+V_k^r\right)\varphi_k(x,y) \\
&-\int_0^\infty\int_0^\infty \frac{V^{II}(k_x,k_y)\phi^{II}(x;k_x,y;k_y)}{Z^{II}(k_x,k_y)}dk_xdk_y \qquad (10)
\end{aligned}
$$

Since a scattering formulation is sought, the incident V_k^i and reflected V_k^r surface wave amplitudes are made explicit where as $V^I(k_x,k_y)$ and $V^{II}(k_x,k_y)$ represent the amplitudes of the (scattered) continuum fields in guide I and II, respectively. By using orthogonality of the modal amplitudes in Equation (10), one may express the unknown modal amplitudes in terms of the magnetic field $H_x(x,y,0)$ as

$$
V_k^r = V_k^i + s_k z_k \int_0^\infty\int_0^\infty \varphi_k(x,y)H_x(x,y,0)dxdy \qquad (11)
$$

$$
V^I(k_x,k_y) = \int_0^\infty\int_0^\infty \phi^I(x;k_x,y;k_y)z(k_x,k_y)H_x(x,y,0)dxdy \qquad (12)
$$

similarly for $V^{II}(k_x,k_y)$, where

$$
s_k = 1 \text{ for } k < n_j \text{ and } s_k = -1 \text{ for } k > n_j \qquad (13)
$$

Upon substituting the above equations in to Equation (9) and rearranging, one obtains

$$
\sum_{k=1}^{n_j} s_k V_k^i \varphi_k(x,y) = \int_0^\infty\int_0^\infty Z(x,y,x',y')H_x(x',y')dx'dy' \qquad (14)
$$

Where

$$
\begin{aligned}
Z(x,y,x',y') &= \frac{1}{2}\sum_{k=1}^{n_j} z_k\varphi_k(x,y)\varphi_k(x',y') \\
&+\frac{1}{2}\int_0^\infty\int_0^\infty \left[\begin{array}{c} Z^I(k_x,k_y)\phi^I(x;k_x,y;k_y)\phi^I(x';k_x,y';k_y)+ \\ Z^{II}(k_x,k_y)\phi^{II}(x;k_x,y;k_y)\phi^{II}(x';k_x,y';k_y)\end{array}\right]dk_xdk_y \qquad (15)
\end{aligned}
$$

The summation on the left hand side of the Equation (14) represents the total incident electric field impinging on either side of the discontinuity. The term $Z(x, y, x', y')$ is a Green's function which may be viewed as an "impedance" of the step discontinuity, then the scattered field due to magnetic field $H_x(x', y')$ over entire range 0 to infinity is given by the above formula. The linear relationship between scattered (reflected and transmitted) and incident modal amplitudes in Equation (11) suggests that we may obtain a scattering matrix formulation of the form

$$V^r = SV^i \tag{16}$$

Let first consider the case of single surface wave incident on the discontinuity. Since the amplitude of the incident surface mode is arbitrary, it is possible to set

$$V_j^i = 1 \qquad V_{k\neq j}^i = 0$$

and Equation (16) reduce to

$$V_k^r = S_{kj} \tag{17}$$

Let the corresponding scattered magnetic field be $h_j(x, y)$, then according to Equation (11) we have

$$s_j V_k^i \varphi_j(x, y) = \int_0^\infty\int_0^\infty Z(x, y, x', y') h_j(x', y') dx'dy' \tag{18}$$

In the above equation, $h_j(x, y)$ is the unknown function to be determined by discretization of Equation (18) by means of a Ritz-Galerkin procedure. In this procedure an orthonormal basis functions set (Cosine-Laguerre), [6] $f(x, y)$ is introduced in the interval $0 < x,\ y < \infty$ and the magnetic field is represented as

$$h_j(x, y) = \sum_{n=0}^{\infty} w_{nj} f_n(x, y) \tag{19}$$

By using the above Equation (19) in Equation (20) and testing latter Equation, with the weight function $f_k(x, y)$, we obtain

$$\begin{aligned} s_j V_j^i &\int_0^\infty\int_0^\infty f_k(x, y)\varphi_j(x, y) dxdy \\ &= \sum_{n=0}^{\infty} w_{nj} \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty f_k(x, y) Z(x, y, x', y') f_n(x, y) dxdydx'dy' \end{aligned} \tag{20}$$

$$Q_{kj} = \int_0^\infty \int_0^\infty f_k(x,y)\varphi_j(x,y),$$

$$Z_{kn} = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty f_k(x,y)Z(x,y,x',y')f_n(x,y)dxdydx'dy' \tag{21}$$

and of the vectors $Q_j = \begin{bmatrix} Q_{1j} \\ Q_{2j} \\ \cdot \\ \cdot \\ Q_{nj} \end{bmatrix}$ $w_j = \begin{bmatrix} w_{1j} \\ w_{2j} \\ \cdot \\ \cdot \\ w_{nj} \end{bmatrix}$ can be written in matrix form as

$$w_j = s_j Z^{-1} Q_j \tag{22}$$

The scattering matrix is obtained from Equations (11), (14), (22) as

$$\begin{aligned} S_{kj} &= \delta_{kj} + s_k z_k \int_0^\infty \int_0^\infty \varphi_k(x,y) \sum_{n=0}^{\infty} w_{nj} f_n(x,y)dxdy \\ &= \delta_{kj} + s_k s_j z_k Q_k^T Z^{-1} Q_j \end{aligned} \tag{23}$$

where δ_{kj} is the kronecker delta and Q_k^T denotes transposition of Q_k matrix.

The above equation specifies the scattering coefficient of the incident j^{th} surface mode to the k^{th} surface mode. In the Ritz-Galarkin approach, the infinite column matrices Q_k and Q_j and square matrix Z are replaced by their finite truncations $(0 < n < N)$.

By the careful choice of expanding functions, the oscillations in the solution will decrease rapidly with increasing order and convergence is quickly achieved. The step discontinuity is therefore represented by a generalised $(n_j + n_r)$ port scattering network.

However, the step discontinuity is a reciprocal junction and its scattering matrix ought to be symmetrical. This will only be so if all ports terminated by the same matching impedances and these are all normalised to unity. Hence, the normalised scattering matrix is introducing ideal transformers connected at ports. Such that

$$\overline{z} = 1 = n^2 z_k \tag{24}$$

Giving the transformation ratio (25)

$$n = \frac{1}{\sqrt{z_k}} \tag{25}$$

Table 1. Converged scattering parameters of tapered waveguide shown in Figure 2, for different values of L/λ.

L/λ	$\lvert S11 \rvert$	$\lvert S22 \rvert$
0	0.125	0.505
0.072	0.123	0.492
0.143	0.12	0.457
0.215	0.114	0.407
0.286	0.108	0.349
0.358	0.102	0.292
0.430	0.094	0.239
0.501	0.084	0.195
0.573	0.076	0.158
0.645	0.067	0.128

Where z_k is the characteristic impedance of the k^{th} surface mode.

$$\overline{S}_{kj} = \delta_{kj} + s_k s_j \sqrt{z_k z_j} Q_k^T Z^{-1} Q_j \tag{26}$$

which now displays the symmetry required by the reciprocity of the junction.

Owing to the orthogonality of modes, the scattering formulation between continuous modes and surface modes can be easily derived

$$S_{j,k'_x,k'_y} = s_k s'_k \sqrt{z_j z(k'_x, k'_y)} Q_j^T Z^{-1} Q(k'_x, k'_y) \tag{27}$$

4. NUMERICAL RESULTS

It is now interesting to observe the scattering behaviour of the taper. In step analysis all above formulas are valid for the waveguide II with dimensions $d2$ and $d3$ in x and y direction respectively. Table 1 shows the converged values of $|S11|$ and $|S22|$ for different L/λ ratios with $\lambda = 2\pi$. The numerically calculated scattering parameters are well matching with the result calculated by Asok De et al. [5] for taper length $L = 0$, indicating a two dimensional step with step ratio $d/d2 = d1/d3 = 0.111$ (Fig. 3). Fig. 4 depicted variation of the reflection coefficient for different tapered length. It is observed that as the tapered length increases the reflection coefficient decreases agreeing the well-known phenomenon of tapering in transmission line theory [7]. Fig. 5 shows the variation of $|S11|$ for different L/λ ratios with different λ values. If we observe the magnitude of the reflection coefficient is less as the λ value increases along the tapered length.

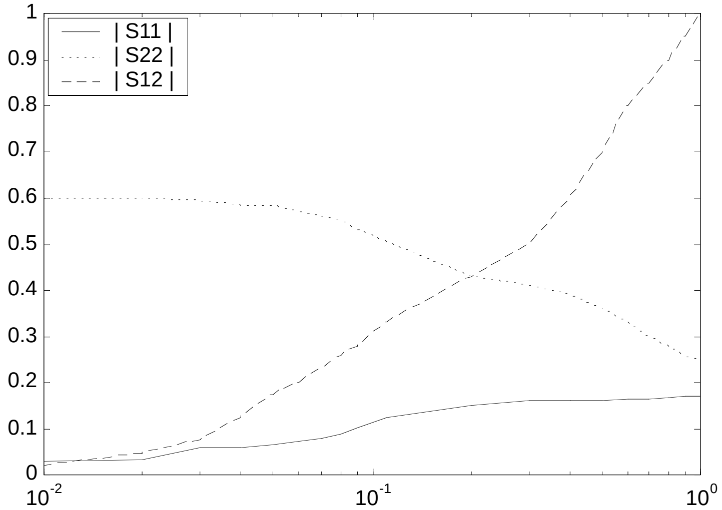

Figure 3. Scattering parameters $|S_{11}|, |S_{22}|, |S_{12}|$ for varying step ratio $d/d2$ with $d1/d3 = 0.000$. The geometric dimensions are $k_0 d2 = 1$, $k_0 d3 = 1$, $\varepsilon_1 = 5$, $\varepsilon_2 = 1$, where k_0 is free space wave number.

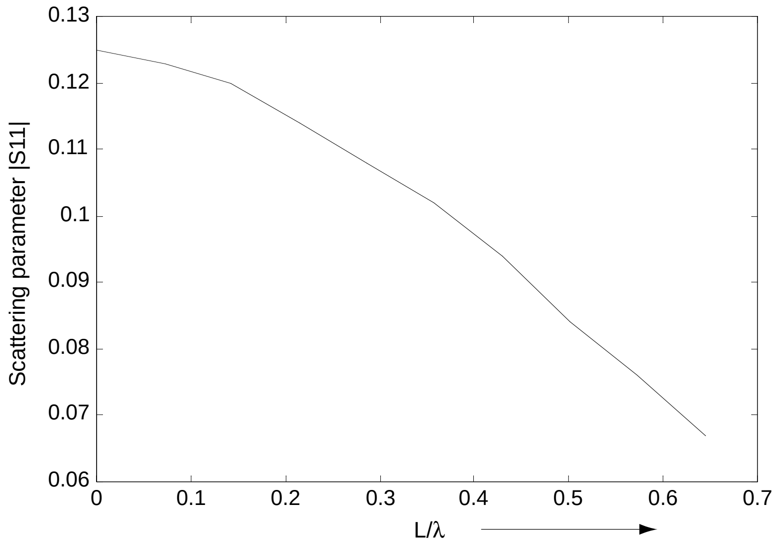

Figure 4. Reflection coefficient of a tapered two-dimensional dielectric waveguide versus L/λ with $\lambda = 2\pi$.

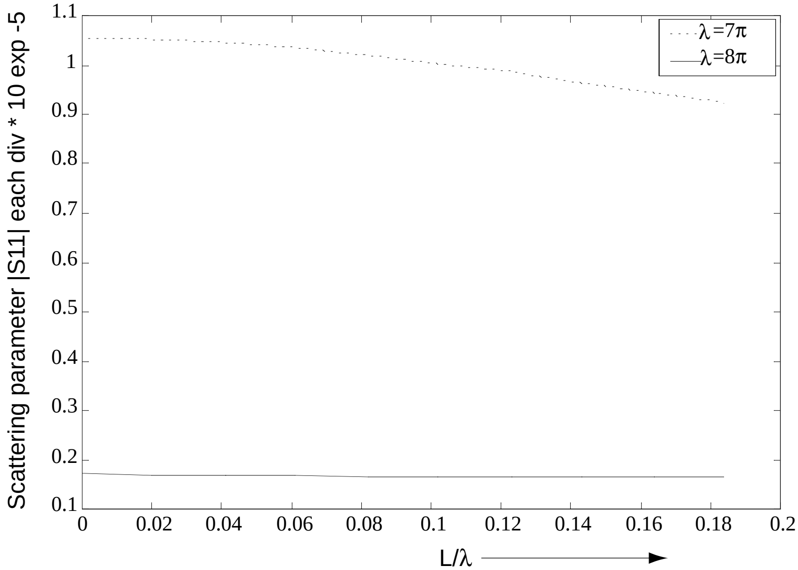

Figure 5. Reflection coefficient of a tapered two-dimensional dielectric waveguide versus L/λ.

5. CONCLUSION

In conclusion, a general analysis technique has been outlined for the tapered two- dimensional dielectric waveguide by approximating the series of steps. The key point is calculating the scattering parameters at each junction, which involves a continuous, as well as a discrete spectrum and then cascading. If we can avoid any mode conversion, we might then be able to design the structure to enlarge cross section of a particular mode in slow and controlled manner. The taper would then acts as a beam expander, for use as a matching section between two different dielectric guides.

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