

THEORETICAL ASPECTS OF WAVE PROPAGATION IN RANDOM MEDIA BASED ON QUANTY AND STATISTICAL FIELD THEORY

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Abstract—In this work, we summarize the existing theoretical methods based on statistical and quanta theory and give some non-standard mathematical approaches based on such theories to explain the principal scalar and vector electrodynamic problems for future applications to acoustic, radio and optical wave propagation in homogeneous, isotropic, anisotropic and inhomogeneous media. We show how the statistical description of wave equations can be evaluated based on quantum field theory with presentation of Feynman's diagrams by a limited-to-zero finite set of expanded Green functions according to perturbation theory for single, double, triple, etc, scattering phenomenon. It is shown that at very short wavelengths, the Green's function is damped over a few wavelengths if the refractive index fluctuations in the medium are strong; at long wavelengths the effective phase velocity of electromagnetic waves may be increased. It is shown, that the coupling between different wave modes and the energy transfer between different wave modes, may be important, even for weak random fluctuations of parameters of the medium, but it takes a very long time.

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1. INTRODUCTION

The entire subject of wave propagation through random media, such as terrestrial, atmospheric and ionospheric (e.g., plasma), has been investigated theoretically by many authors [1–35]. As follows from numerous theoretical works, the problem of wave propagation through a random medium could be understood by use of non-standard mathematical tools that are based on relations between the statistical description of the wave field (electromagnetic and scalar) and the treatments in quantum field theory. Because the problems of random equations are not tractable with standard mathematical tools, we must use some special methods such as Feynman's diagram method [1–3], the method of renormalization [9, 24], etc.

The main goal of this work is to summarize the existing theoretical methods based on statistical and quantity theory and to give some non-standard mathematical approaches based on such theories to explain the principal problems of wave propagation for future applications to problems of radio, acoustic and optical communications. An example of one of the successful application of this approach to radio propagation in terrestrial environment, rural, mixed and urban, can be found in [31–35], where the combination of statistical description of the terrain, as an array of randomly distributed discrete obstructions, and the statistical description of wave equations have been evaluated based on quantum field theory with presentation of Feynman's diagrams by a limited-to-zero finite set of expanded Green functions according

to perturbation theory for single, double, triple, etc., scattering phenomenon.

Below, in Section 2, we introduce briefly main random equations and random functions that describe stochastic processes in the random medium. Here, we give a direct definition of the Fourier transforms of a stationary random function. We explain also the mathematical aspects of random equations. In Section 3, we introduce the perturbation method according to [9, 15, 18–20, 24] for description of wave multi-scattering processes by use of the Feynman's diagram procedure [1–3]. Here, we *extend this method to non-Gaussian functions* by means of a cluster expansion of the random refractive index [4–6]. In Section 4, we introduce a one-dimensional random propagation equation having an exact solution, and containing already most features of the general theory. Section 5 continues to describe a formal perturbation method and its approximations for scalar wave equation with random refractive index, with or without time dependence by use the same Feynman's diagram approach [1–3] to the formal perturbation method of all orders. As in Section 3, here we continue to discuss several approximate procedures related to the perturbation method, such as Born's approximation [9, 18–20, 24], Keller's expansion [16], Bourret's closure assumption [15] and Kraichnan's random coupling model [23]. In Section 6, we use a random Taylor expansion [9, 24] of the random refractive index, that is, we find under what conditions it is possible to replace the random index by a mere random variable or a linear function with random coefficients. Here we show that at very short wavelengths compared to the range of index correlations, the mean propagator (or Green's function) is damped over a few wavelengths if the refractive index fluctuations are strong. Section 7 describes the exact solution of the scalar wave equation through functional integration. In Section 8, we introduce the vector electromagnetic wave equation with random index and harmonic time dependence. It is shown here that at long wavelengths the effective phase velocity of electromagnetic waves is *increased* because of the coupling between transverse and longitudinal waves. Section 9 describes wave propagation in a statistically inhomogeneous random medium. It is shown here that the Bourret's approximation [15] for the mean propagator and the subsequent damping remain valid if the random medium is statistically inhomogeneous with a scale of variation which is very long compared to the wavelength. In Section 10, we describe more general propagation equations of linear waves in anisotropic random medium (namely, in plasma), and especially the coupling between different wave modes and the subsequent energy transfer. It is shown here the energy transfer between different wave

modes, in e.g., a turbulent plasma, may be important, even for weak random fluctuations of parameters, but it takes then a very long time.

Despite the fact that the material in the text presented below is based mostly on theoretical aspects of wave propagation in random media developed and described by Bourret [15], Furutsu [7, 9, 24] and Tatarskii [18–20], it is based on some non-standard mathematical receipts recommended by ourselves to explain the same propagation subject through the prism of the unified fully coupled stochastic approach based on statistical and quantum field theory to describe the corresponding physical phenomena.

2. MAIN WAVE EQUATIONS AND RANDOM FUNCTIONS

2.1. Wave Equations

We investigate a random medium as a medium whose parameters, such as pressure, density, temperature, or location of the discrete obstructions are random functions of position and time. A random medium can also be thought of as a collection of inhomogeneous media, each of which may be either continuous (turbulent medium) or discrete (medium with random inclusions or obstructions). Below we will briefly introduce the main equations that describe stochastic processes in the random medium. The propagation phenomena of linear waves in random medium is described by different linear differential equations with random coefficients:

A. Scalar wave equation can be presented in the following form:

$$\Delta\Psi(r, t) - \frac{n^2(r, t)}{c^2} \frac{\partial^2\Psi(r, t)}{\partial t^2} = 0 \quad (1)$$

where $\Psi(r, t)$ is the wave field amplitude in space and time domain, $n(r, t)$ is the refractive index which is a random function of space (r) and time (t), c is the wave velocity in free space.

B. Reduced scalar wave equation with any source $g(r)$ can be presented as follows:

$$\Delta\Psi(r) - k^2 n^2(r) \Psi(r) = g(r) \quad (2)$$

Here (2) is deduced from (1) assuming a harmonic time dependence $\sim \exp\{i \cdot c \cdot k \cdot t\}$ and a time independent refractive index n , where $k = \frac{2\pi}{\lambda}$ is the wave number and λ is the wavelength. The source term $g(r)$ is assumed to be given and non random. For mathematical convenience the wave number k is assumed to have a small positive imaginary part which ensures that we are looking for the outgoing wave. Moreover,

this imaginary part does not correspond to a damping process and may be canceled after the correct solution has been found.

C. Electromagnetic wave equation can be presented by the following form:

$$\Delta \mathbf{E}(r, t) - \nabla(\nabla \cdot \mathbf{E}(r, t)) - \frac{n^2(r, t)}{c^2} \frac{\partial^2 \mathbf{E}(r, t)}{\partial t^2} = 0, \quad (3)$$

Here $\mathbf{E}(r, t)$ is the vector presentation of electromagnetic field in space and time domain.

We must note that theoretical investigations here will be limited to the two first equations (1) and (2), and a one-dimensional model equation will be introduced in Section 4. Nevertheless, the results of the theoretical investigations will be applied to the electromagnetic wave equation in Section 8, and to more general equation in Section 9.

We shall always treat the refractive index as a time independent random function, which is equivalent to the assumption that the characteristic time of index fluctuations is much longer than the period of the wave. The medium in such conditions will be taken statistically homogeneous. This assumption excludes a medium where the turbulence is concentrated in a small volume of space. This restriction will be partially dropped in Section 10, where the scale of the inhomogeneity will be taken very large compared to the wavelength.

To conclude this subsection, let us show that the scalar wave equation (1) and the reduced scalar wave equation (2) may be treated simultaneously. Equation (1) corresponds to an initial value problem that well-known as the Cauchy problem, that is, we must be given $\Psi(r, 0)$ and $\frac{\partial \Psi(r, 0)}{\partial t}$ in order to find $\Psi(r, t)$. Equation (2) corresponds to a radiation problem. Let us introduce the Laplace transform of wave function $\Psi(r, t)$

$$\Psi(r, q) = \int_0^{\infty} \Psi(r, t) \cdot \exp\{i \cdot q \cdot t\} \cdot dt, \quad \text{Im}(q) > 0. \quad (4)$$

It satisfies the following equation, which is the Laplace transform of (1):

$$\Delta \Psi(r, q) + \frac{q^2}{c^2} \cdot n^2(r) \cdot \Psi(r, q) = \frac{n^2(r)}{c^2} \left[i \cdot q \cdot \Psi(r, t = 0) - \frac{\partial \Psi(r, t = 0)}{\partial t} \right] \quad (5)$$

Equations (5) and (2) can be identified if one takes

$$q = c \cdot k, \quad \Psi(r, t = 0) = 0, \quad \frac{\partial \Psi(r, t = 0)}{\partial t} = -\frac{c^2}{n^2(r)} g(r) \quad (6)$$

We shall always choose these initial conditions for equation (1) and treat (1) and (2) simultaneously, interchanging q and $c \cdot k$, whenever necessary.

2.2. Random Functions and Their Moments

A detailed treatment of stochastic processes may be found in [16]. In connection with random equations, see also [17, 18]. Here we give the main definitions in three dimensional (3D) space. By introducing any finite number of points r_1, r_2, \dots, r_n , we assume that the the mean value of random functions $\mu(r_1) \cdot \mu(r_2) \cdots \mu(r_n)$ always exists. It will denote with an angular bracket, that is,

$$\langle \mu(r_1) \cdot \mu(r_2) \cdots \mu(r_n) \rangle = \int_{\Omega} \mu(r_1) \cdot \mu(r_2) \cdots \mu(r_n) \cdot P(d\omega) \quad (7)$$

This expression is usually called a moment of order n . A random function (r. f.) will often be characterized by the infinite set of all its moments. The r. f. $\mu(r)$ is *centered* if

$$\langle \mu(r) \rangle = 0 \quad (8)$$

It is *stationary* if the joint distribution of any finite number of random variables $\mu(r_1) \cdot \mu(r_2) \cdots \mu(r_n)$ is invariant with respect to any simultaneous translation of its arguments. For space dependent r. f., it would perhaps be better to call it a *stationary homogeneous* r. f. (homogeneous turbulencies). If the random function is also real valued, the second order moment

$$\Gamma(r_1, r_2) = \langle \mu(r_1) \cdot \mu(r_2) \rangle \quad (9)$$

is called the *covariance function*.

If now the random function $\mu(r)$ is stationary, the covariance function $\Gamma(r)$ is only a function of $(r_1 - r_2)$, i.e.,

$$\Gamma(r_1, r_2) = \Gamma(r_1 - r_2) \quad (10)$$

The function $\Gamma(r)$ has a Fourier transform which is a positive measure $\Gamma(k)$, called the spectral measure of the stationary r. f., or spectral density function if it reduces to an ordinary function of wave number k . It is sometimes necessary to assume the existence of mean square derivatives of the random function $\mu(r)$ up to a given order; this subject will be discussed in Section 6.

Gaussian random function. A random function $\mu(r)$ is called Gaussian if the joint distribution of any finite number of random

variables $\mu(r_1) \cdot \mu(r_2) \cdots \mu(r_n)$ is Gaussian. This function is of very great theoretical interest and has many practical applications, especially because of following property:

Any scalar linear functional of a Gaussian random function is a Gaussian random variable.

Furthermore, we note the important property of the moments of a centered Gaussian random function

$$\langle \mu(r_1) \cdot \mu(r_2) \cdots \mu(r_{2n+1}) \rangle = 0, \tag{11a}$$

$$\langle \mu(r_1) \cdot \mu(r_2) \cdots \mu(r_{2n}) \rangle = \underbrace{\sum \langle \mu(r_i) \cdot \mu(r_j) \rangle \langle \mu(r_k) \cdot \mu(r_m) \rangle \cdots}_{p \text{ factors}} \tag{11b}$$

Here the summation extends over all $\frac{(2n)!}{2^n \cdot n!}$ partitions of $r_1 \dots r_{2n}$ into pairs. For example, for $n = 2$ one can easily obtain from (11)

$$\begin{aligned} \langle \mu(r_1) \cdot \mu(r_2) \cdot \mu(r_3) \cdot \mu(r_4) \rangle &= \langle \mu(r_1) \cdot \mu(r_2) \rangle \cdot \langle \mu(r_3) \cdot \mu(r_4) \rangle + \\ &\langle \mu(r_1) \cdot \mu(r_3) \rangle \cdot \langle \mu(r_2) \cdot \mu(r_4) \rangle + \langle \mu(r_1) \cdot \mu(r_4) \rangle \cdot \langle \mu(r_2) \cdot \mu(r_3) \rangle \end{aligned} \tag{12}$$

Fourier transforms of stationary random functions. Let us consider random valued measures as Fourier transforms of stationary random functions. A stationary random function on real line $\mu(\kappa, \omega)$, with continuous covariance function has a spectral representation

$$\mu(\kappa, \omega) = \int_{-\infty}^{\infty} dZ(k, \omega) \exp\{ik\kappa\} \tag{13}$$

Let us explain briefly the meaning of this formula. $Z(k, \omega)$ is a random function with orthogonal increments; this means that whenever the parameter values satisfy

$$\begin{aligned} k_1 < k_2 \leq k_3 < k_4 \\ \langle [Z(k_2) - Z(k_1)][Z(k_4) - Z(k_3)] \rangle &= 0 \end{aligned} \tag{14}$$

The integral in (13) is to be understood as a Stieltjes integral. With this definition the Fourier transforms of a stationary r. f. does not appear as another random function but as some *derivative* of a r. f. with orthogonal increments. Integral presentation (13) can be generalized for the case of three dimensional (3D) r.f.; we do not enter in this subject transmitting the reader to [1-3].

The cluster expansion of the centered r. f. and its F. T. If the r. f. $\mu(r)$ is centered, its covariance is else its two-point correlation function, but this is not true for higher moments. But, as was

shown in [1-6], the n -point correlation functions are not simultaneously correlated. We introduce therefore the correlation functions $h(r_1, r_2), h(r_1, r_2, r_3), \dots, h(r_1, r_2, \dots, r_p)$ through the following cluster expansions:

$$\begin{aligned} \langle \mu(r_1)\mu(r_2) \rangle &= h(r_1, r_2) \\ \langle \mu(r_1)\mu(r_2)\mu(r_3) \rangle &= h(r_1, r_2, r_3) \\ \langle \mu(r_1)\mu(r_2)\mu(r_3)\mu(r_4) \rangle &= h(r_1, r_2)h(r_3, r_4) + h(r_1, r_3)h(r_2, r_4) \\ &\quad + h(r_1, r_4)h(r_2, r_3) + h(r_1, r_2, r_3, r_4) \end{aligned} \quad (15)$$

$$\langle \mu(r_1) \cdot \mu(r_2) \cdot \dots \cdot \mu(r_p) \rangle = \sum h(r_{i_1}, \dots, r_{i_k})h(r_{j_1}, \dots, r_{j_m})h(r_{l_1}, \dots, r_{l_n}) \dots$$

where the summation is extended over all parameters of the set $1, 2, \dots, p$ into clusters of at least two points. From (11) it follows that for a centered Gaussian random function (g. r. f.) all correlation functions except the second order one vanish.

A graphic representation in terms of Mayer diagrams [1-3] may be helpful. The correlation function $h(r_1, r_2, \dots, r_p)$ is represented by a set of p points connected by p lines:

$$h(r_1, r_2) = \begin{array}{c} \bullet \\ | \\ \bullet \end{array}^1_2, \quad h(r_1, r_2, r_3) = \begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array}, \quad \dots \quad (16)$$

the cluster expansion is then written graphically. For example

$$\langle \mu(r_1)\mu(r_2)\mu(r_3)\mu(r_4) \rangle = \begin{array}{c} \bullet \quad \bullet \\ | \quad | \\ \bullet \quad \bullet \end{array}^1 \quad \begin{array}{c} \bullet \quad \bullet \\ \text{---} \quad \text{---} \\ \bullet \quad \bullet \end{array}^1 \quad \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array}^1 \quad \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array}^1 \quad (17)$$

This definition of the correlation functions ensures that they vanish if the points r_1, r_2, \dots, r_p are not inside a common sphere of radius ℓ (the proof follows by induction). We shall also need the F. T. of the correlation functions

$$h(k_1, k_2, \dots, k_p) = \frac{1}{(2\pi)^{3p}} \int h(r_1, r_2, \dots, r_p) \cdot \exp\{-i(k_1 r_1 + \dots + k_p r_p)\} d^3 r_1 \dots d^3 r_p \quad (18)$$

If the random function $\mu(r)$ is stationary, this is not a function, but a measure concentrated in the hyperplane $k_1 + k_2 + \dots + k_p = 0$. We shall therefore write

$$h(k_1, k_2, \dots, k_p) = g(k_1, k_2, \dots, k_p) \cdot \delta(k_1 + k_2 + \dots + k_p) \quad (19)$$

and call the ordinary functions $g(k_1, k_2, \dots, k_p)$, somewhat improperly the F.T. of the correlation functions, or simply the correlation functions in k -space. Using these functions, we can write the cluster expansion of the moments in k -space as

$$\begin{aligned}\langle \mu(k_1)\mu(k_2) \rangle &= g(k_1, k_2) \cdot \delta(k_1 + k_2) \\ \langle \mu(k_1)\mu(k_2)\mu(k_3) \rangle &= g(k_1, k_2, k_3) \cdot \delta(k_1 + k_2 + k_3) \\ \langle \mu(k_1)\mu(k_2)\mu(k_3)\mu(k_4) \rangle &= g(k_1, k_2) \cdot g(k_3, k_4) \cdot \delta(k_1 + k_2) \cdot \delta(k_3 + k_4) + \dots\end{aligned}\quad (20)$$

2.3. Random Equations

A random equation such as (2)

$$\Delta \Psi(r) - k^2 n^2(r) \Psi(r) = g(r) \quad (21)$$

describes linear waves and does not constitute a linear problem because the mean solutions not satisfy the mean equation. This is because

$$\langle n^2(r) \Psi(r) \rangle \neq \langle n^2(r) \rangle \langle \Psi(r) \rangle$$

In other words, the wave function and the refractive index are not statistically independent. If we try to evaluate $\langle n^2(r) \Psi(r) \rangle$, we must multiply (21) by $n^2(r)$ and average afterwards; this will rise to a form $\sim \langle n^2(r_1) n^2(r) \Psi(r) \rangle$, and so on.

Keller [16] has obtained an equation for a functional generating the whole set of moments. This equation may perhaps suggest now approximation procedures, but it does not solve the problem. The fact that even the lowest order moment of the wave function $\langle \Psi(r) \rangle$ depends upon all the infinite set of moments of the refractive index seems to make the problem hopelessly difficult. It happens however, that in certain limiting cases one may obtain solutions which do not depend upon all the moments of the refractive index.

Therefore, the perturbation method described in Section 3, gives the Bourret equation which depends only on the mean value and the covariance of the refractive index. It is only valid for wavelengths which are longer compared to the range of index correlations. Conversely, for the random Taylor expansion (see Section 6) we need only the probability distribution of the index and some of its derivatives at one fixed point. It is valid for wavelengths which are very short compared to the range of index correlations.

Another case of great interest is when $n^2(r)$ is a Gaussian random function (g.r.f.). It is then possible to get an exact solution of (21) through functional integration, which gives all the moments of the wave functions in terms of mean value and covariance of $n^2(r)$ (see

Section 7). Unfortunately this method cannot be generalized to other equations such as the electromagnetic wave equation (3). Finally, it must be stressed that no rigorous mathematical treatment (existence, unity) of (21) has been given up to now. This is mainly because one is not able to solve linear partial differential equations with nonconstant coefficient in the large.

3. THE PERTURBATION METHOD FOR MULTIPLE SCATTERING

The multiple scattering perturbation method is a general method for studying propagation equations with linear coefficients. It has been first introduced by Bourret [15] and Furutsu [7, 9, 24], and studied later by Tatarskii et al. [18–20]. It is not a rigorous method and cannot be made rigorous because it makes constant use of divergent series (see Section 5). Divergent series however, have been used successfully both in quantum field theory and in non-equilibrium statistical mechanics. It is our aim to show that the perturbation method and especially Bourret's approximation can be used to find uniform approximation of the mean wave function, when the wavelength is long compared to the range of index correlations. In this section, this will be proved rigorously for a one dimensional model which has exact solution. It will also be justified for the scalar wave equation, using the method of extraction of the most divergent secular terms (terms which increase as some power t or R) and n dimensional analysis of all perturbation terms. Such a method has been used previously by Ishimaru [26, 27] and Balescu [28] in non-equilibrium statistical mechanics. It is not rigorous but can be considered as physically satisfactory. In order to avoid unnecessary complication of algebra, we shall only consider the scalar wave equation

$$\Delta\Psi(r, t) - \frac{n^2(r)}{c^2} \frac{\partial^2\Psi(r, t)}{\partial t^2} = 0 \quad (22)$$

together with the initial conditions

$$\Psi(r, 0) = 0, \quad \frac{\partial\Psi}{\partial t} = -\frac{c^2}{n^2(r)} j(r) \quad (23)$$

In Section 2 this equation was shown to be equivalent to the random problem (2). The solution of (2.2) is obtained by changing z into ck_0 in Laplace transform (L. T.) of $\Psi(r, t)$. We shall make the assumption that the refractive index is a stationary random function of position, and is time independent. The assumption of strict stationary (i.e.,

not only for the two first moments) is essential. We separate now the constant mean value of $n^2(r)$ and its random part.

$$\begin{aligned} n^2(r) &= \langle n^2(r) \rangle [1 + \varepsilon \cdot \mu(r)] \\ \langle \mu(r) \rangle &= 0 \end{aligned} \tag{24}$$

Here ε is a dimensionless small positive parameter characterizing the relative strength of index fluctuations. Equation (22) is now written

$$\Delta \Psi(r, t) - \frac{1}{c^2} [1 + \varepsilon \cdot \mu(r)] \frac{\partial^2 \Psi(r, t)}{\partial t^2} = 0 \tag{25}$$

where $\langle n^2(r) \rangle$ has been incorporated into $\frac{1}{c^2}$. Laplace transform (L. T.) with respect to time and Fourier transform (F. T.) with respect to space will be two powerful tools in our future investigations. The L. T. and F. T. of $\Psi(r, t)$ are defined as

$$\Psi(r, z) = \int_0^\infty \exp\{izt\} \Psi(r, t) dt, \quad \text{Im}(z) > 0 \tag{26}$$

and

$$\Psi(k, \cdot) = \int_0^\infty \exp\{-ik \cdot r\} \Psi(r, \cdot) d^3r \tag{27}$$

and similar definition for $j(k)$ the F. T. of the source function. It should be observed that the same $\Psi(v, u)$ is used for the wave as a function of (r, t) , (r, z) , (k, t) , and (k, z) .

The stationary r. f. is written in terms of its F. T. $\mu(r)$ which is a *random valued measure* (r. v. m.)

$$\mu(r) = \int \exp\{ik \cdot r\} \mu(k) d^3k \tag{28}$$

The L. T. of (22), taking into account the initial conditions (23) is

$$\Delta \Psi(r, z) + \frac{z^2}{c^2} [1 + \varepsilon \mu(r)] \Psi(r, z) = j(r) \tag{29}$$

and the F. T. of this equation is

$$\left[-k^2 + \frac{z^2}{c^2} \right] \Psi(k, z) + \frac{\varepsilon z^2}{c^2} \int \mu(k - k') \Psi(k', z) d^3k' = j(k) \tag{30}$$

Equation (29) and (30) are both of the type

$$(L_0 + \varepsilon L_1)\Psi = j \tag{31}$$

where L_0 is a non-random operator whose inverse $G^{(0)} = I_0^{-1}$, called the unperturbed propagator (or Green's function), is known, and L_1 is a random operator. In r -space

$$L_0 = \Delta + \frac{z^2}{c^2}, \quad G^{(0)}(r, r'; z) = \frac{\exp\{iz|r - r'|\}}{-4\pi|r - r'|}, \quad L_1 = \frac{z^2}{c^2}\mu(r) \tag{32}$$

acting as an integral convolution operator. In k -space

$$L_0 = -k^2 + \frac{z^2}{c^2}, \quad G^{(0)}(k; z) = \frac{c^2}{z^2 - c^2k^2}, \quad L_1 = \frac{z^2}{c^2}\mu(k - k') \tag{33}$$

acting as an integral convolution operator. In r -space L_1 is diagonal operator and L_0 is not; in k -space it is the converse. The solution of (31) is now formally expanded in powers of ε yielding

$$\Psi = (L_0 + \varepsilon L_1)^{-1}j = L_0^{-1}j - \varepsilon L_0^{-1}L_1L_0^{-1}j + \varepsilon^2 L_0^{-1}L_1L_0^{-1}L_1L_0^{-1}j + \dots \tag{34}$$

where $(L_0 + \varepsilon L_1)^{-1} = G$ is called the *perturbed propagator* (or Green's function).

Let us represent the perturbation series for G with the aid of diagrams which will be called *bare* diagrams to discriminate between them and other *dressed* diagrams to be introduced afterwards. We make the following conventions:

- a) The unperturbed propagator $G^{(0)}(r, r')$ is represented by a solid line $r \ r'$.
- b) The random operator $-\varepsilon L_1$ is represented by a dot \bullet .
- c) Operators act to the right.

If so, we may write

$$G = \text{---} + \text{---}\bullet\text{---} + \text{---}\bullet\bullet\text{---} + \text{---}\bullet\bullet\bullet\text{---} + \dots \tag{35}$$

Let us write down explicitly a few terms of the perturbation series in r - and k -space.

$$G(r, r'; z) = G^{(0)}(r, r'; z) - \varepsilon \frac{z^2}{c^2} \int G^{(0)}(r, r_1; z)\mu(r_1)G^{(0)}(r_1, r'; z)d^3r_1 + \varepsilon^2 \frac{z^4}{c^4} \iint G^{(0)}(r, r_2; z)\mu(r_2)G^{(0)}(r_2, r_1; z)\mu(r_1)G^{(0)}(r_1, r'; z)d^3r_1d^3r_2 \tag{36}$$

$$\begin{aligned}
 G(k, k'; z) &= G^{(0)}(k; z)\delta(k - k') - \varepsilon \frac{z^2}{c^2} G^{(0)}(k; z)\mu(k - k')G^{(0)}(k'; z) \\
 &+ \varepsilon^2 \frac{z^4}{c^4} \int G^{(0)}(k; z)\mu(k - k_1)G^{(0)}(k_1; z)\mu(k_1 - k')G^{(0)}(k'; z)d^3k_1 \quad (37)
 \end{aligned}$$

where $\delta(k - k')$ is Dirac's measure. In order to help the interpretation of bare diagrams, it is sometimes useful to introduce subscripts under certain elements:

$$\begin{aligned}
 G(r, r'; z) &= \overline{r \ r'} + \overline{r \ \bullet \ r'} + \overline{r \ \bullet \ \bullet \ r'} + \dots \\
 G(k, k'; z) &= \overline{k \ k'} + \overline{k \ \bullet \ k'} + \overline{k \ \bullet \ \bullet \ k'} + \dots
 \end{aligned} \quad (38)$$

The subscripts are always written under the operators which are diagonal (the dot in r -space, the solid line in k -space). If so, the dashed curve will connect the concrete points for which r. v. m. $\mu(r_1)$ and $\mu(r_2)$ (or $\mu(k - k_1)$ and $\mu(k_1 - k')$) are inside the integrals, i.e.,

$$\mu(r_1)\mu(r_2) \sim \bullet \text{---} \text{---} \bullet \quad \text{or} \quad \mu(k - k_1)\mu(k_1 - k') \sim \bullet \text{---} \text{---} \bullet$$

We give now the physical interpretation of the perturbation expansion. The r -space diagrams correspond to multiple scattering of the wave at points r_1, r_2, \dots, r_N . The k -space diagrams correspond to multiple interactions between Fourier components of the wave and of the random inhomogeneities: at each vortex of a diagram a Fourier component k_p of the wave function interacts with a Fourier component ($k_{p+1} - k_p$) of the random inhomogeneities, giving, as a result, a Fourier component $k_{p+1} \equiv k_{p+1} - k_p + k_p$ of the wave function. Both viewpoints are useful; the first one, particularly for single or double scattering, and the second one for multiple scattering, because of the wave vector conservation conditions.

In future description, we also need the expansion of the perturbed double propagator $G \otimes G^*$, i.e., the tensor product of the perturbed propagator and its complex conjugate. In r -space

$$G \otimes G^* = G(r, r'; z)G^*(r_1, r'_1; z') \quad (39)$$

In k -space

$$G \otimes G^* = G(k, k'; z)G^*(k_1, k'_1; z') \quad (40)$$

This expansion can also be written in terms of diagrams:

$$\begin{array}{cccccc}
 \text{---} & \text{---} \bullet \text{---} & \text{---} & \text{---} \bullet \text{---} & \text{---} \bullet \bullet \text{---} & \\
 G \otimes G^* = & + & + & + & + & + \dots \\
 \text{---} & \text{---} & \text{---} \bullet \text{---} & \text{---} \bullet \text{---} & \text{---} &
 \end{array} \tag{41}$$

If we make the convention that operators of the lower line are the complex conjugate of the usual ones, for example

$$\begin{array}{l}
 \overline{\text{---} \bullet \text{---}} \\
 \text{---} \bullet \text{---} \\
 \overline{\text{---} \bullet \text{---}} \\
 \text{---} \bullet \text{---}
 \end{array}
 = \varepsilon^2 \frac{z^2 z'^2}{c^4} G^{(0)}(k; z) \mu(k - k') G^{(0)}(k'; z) \cdot G^{(0)*}(k_1; z') \mu^*(k_1 - k'_1) G^{(0)*}(k'_1; z') \tag{42}$$

we can now to present the mean perturbed propagator.

The mean perturbed propagator. Let us produce it in r -space first. We take the mean value of (35)

$$\langle G(r, r'; z) \rangle = \left\langle \overline{\text{---} \text{---}} \right\rangle + \left\langle \overline{\text{---} \bullet \text{---}} \right\rangle + \left\langle \overline{\text{---} \bullet \bullet \text{---}} \right\rangle + \dots \tag{43}$$

$$\left\langle \overline{\text{---} \text{---}} \right\rangle = \text{---} \text{ because it is non random measure;}$$

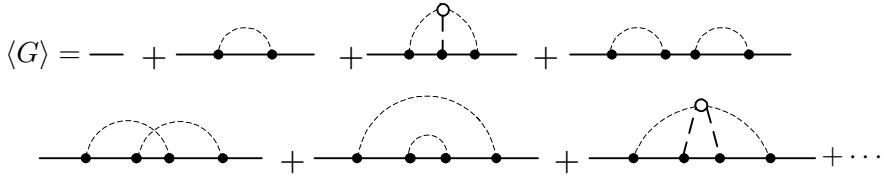
$$\left\langle \overline{\text{---} \bullet \text{---}} \right\rangle = 0 \text{ because of } \langle \mu(r) \rangle = 0.$$

Higher order diagrams contain moments of $\mu(r)$ such as $\langle \mu(r_1) \mu(r_2) \dots \mu(r_p) \rangle$, which must be calculated before integration over r_1, r_2, \dots, r_p is performed. Such a moment can now be written as a sum of products of correlation functions corresponding to all possible partitions of r_1, r_2, \dots, r_p into clusters of at least two points (see Section 2).

To every partition we associate a dressed diagrams constructed from the bare perturbation diagrams according to the following rules:

- a) If r_i and r_j belong to a two point cluster, we draw a curved dotted line from r_i to r_j (see above, $\overline{\text{---} \bullet \text{---}}$).
- b) If r_i, r_j, r_k, \dots belong to a p -point cluster ($p > 2$), we draw a set of dotted lines (curved or not, namely, $\overline{\text{---} \bullet \text{---} \bullet \text{---}}$) starting from the points r_i, r_j, r_k, \dots and ending at a common circle.

To each bare diagram, we associate as many dressed diagrams as there are different partitions. Their number is increasing very fast with n . Let us for example write down all diagrams up to fourth order of perturbation



Such diagrams are called Feynman diagrams [1–3]. For random equations, they have been introduced by Bourret [15], in the Gaussian case, where only two point clusters are needed. If we want to write down explicitly the contribution of any diagram, we first write the multiple integral with $\mu(r_1) \cdot \mu(r_2) \cdots \mu(r_p)$ for the corresponding bare diagram, replace then $\langle \mu(r_1) \cdot \mu(r_2) \cdots \mu(r_p) \rangle$ by the product of the correlation functions corresponding to the clusters which appear in the diagram, and finally perform the integration over r_1, r_2, \dots, r_p . For example:

$$\begin{aligned}
 \text{---} \cdot \text{---} \cdot \text{---} \cdot \text{---} \text{---} \quad r \quad r_4 \quad r_3 \quad r_2 \quad r_1 \quad r' &= \varepsilon^4 \frac{z^8}{c^8} \int G^{(0)}(r, r_4) \mu(r_4) G^{(0)}(r_4, r_3) \mu(r_3) G^{(0)}(r_3, r_2) \\
 &\times \mu(r_2) G^{(0)}(r_2, r_1) \mu(r_1) G^{(0)}(r_1, r') d^3 r_1 d^3 r_2 d^3 r_3 d^3 r_4 \quad (44a)
 \end{aligned}$$

from which according to presented above follows

$$\begin{aligned}
 \text{---} \cdot \text{---} \cdot \text{---} \cdot \text{---} \text{---} &= \varepsilon^4 \frac{z^8}{c^8} \int G^{(0)}(r, r_4) G^{(0)}(r_4, r_3) G^{(0)}(r_3, r_2) G^{(0)}(r_2, r_1) \\
 &\times G^{(0)}(r_1, r') h(r_1, r_2) h(r_3, r_4) d^3 r_1 d^3 r_2 d^3 r_3 d^3 r_4 \quad (44b)
 \end{aligned}$$

The same diagrams can be constructed in k -space, because the cluster expansion is valid in both spaces. We recall that the F.T. of a correlation function being singular, it is preferable to write them as a product of original function and a δ -measure:

$$h(k_1, k_2, \dots, k_p) = g(k_1, k_2, \dots, k_p) \delta(k_1 + k_2 + \dots + k_p) \quad (45)$$

If we want to calculate a diagram in k -space, we first write the multiple integral with $\mu(k - k_p) \mu(k_p - k_{p-1}) \cdots$ for the corresponding bare diagram, replace then $\langle \mu(k - k_p) \cdots \mu(k_1 - k') \rangle$ by the product of k -space correlation functions and δ -measures corresponding to the clusters which appear in the diagram, and finally perform in integration over k_1, k_2, \dots, k_p . For example

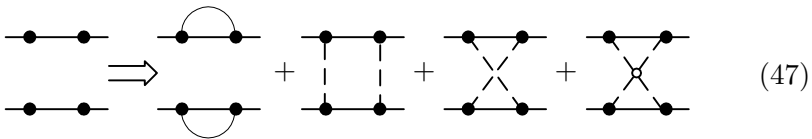
$$\begin{aligned}
 \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} &= \varepsilon^4 \frac{z^8}{c^8} \int G^{(0)}(k) \mu(k-k_3) G^{(0)}(k_3) \mu(k_3-k_2) G^{(0)}(k_2) \\
 &\quad \times \mu(k_2-k_1) G^{(0)}(k_1) \mu(k_1-k') G^{(0)}(k') d^3 k_1 d^3 k_2 d^3 k_3 \quad (46a)
 \end{aligned}$$

or

$$\begin{aligned}
 \text{---} \overset{\curvearrowright}{\bullet} \text{---} \overset{\curvearrowright}{\bullet} \text{---} &= \varepsilon^4 \frac{z^8}{c^8} \int G^{(0)}(k) G^{(0)}(k_3) G^{(0)}(k_2) G^{(0)}(k_1) G^{(0)}(k') \\
 &\quad \times g(k-k_3, k_3-k_2) g(k_2-k_1, k_1-k') \delta(k-k_2) \delta(k_2-k') d^3 k_1 d^3 k_2 d^3 k_3 \quad (46b)
 \end{aligned}$$

The vector differences $k_1 - k'$, $k_2 - k_1, \dots$ etc., which appear as arguments of correlation functions, are called *transition vectors*. The wave vector conservation condition $k_1 + k_2 + \dots + k_p = 0$ states that the sum of the transition vectors of a given cluster is zero. An immediate consequence of this is that any diagram has the same wave vector at both ends. In other words, *the mean perturbed propagator is a diagonal operator in k -space*. It will be noted as $\langle G(k; z) \rangle$.

The mean double propagator. If we assume the random function to be real, the extension of the diagram technique to the mean double propagator is straightforward in r -space. To any bare double diagram, we associate as dressed double diagrams as there are partitions of the whole set of upper and lower points into clusters, for example:



The explicit calculation of a diagram is performed exactly as for the mean propagator, remembering that operators in the lower lines are complex conjugate of the usual ones.

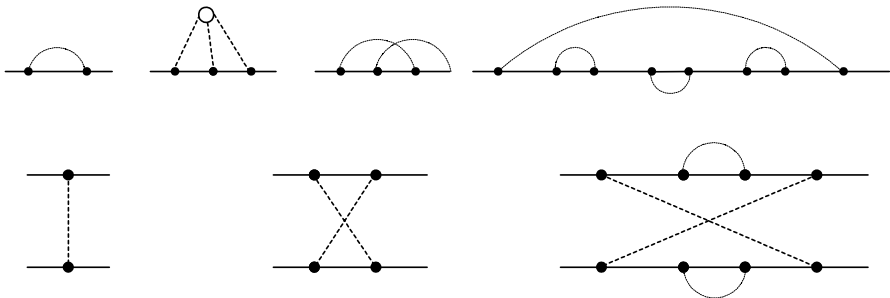
In k -space the situation is somewhat different, because the F. T. of a real function is not real, but enjoys the property $f^*(k) = f(-k)$. When we calculate a diagram in k -space, we must therefore replace transition vectors appearing in the lower line by their opposite in the correlation functions. For example,

$$\begin{aligned}
 \overline{\frac{k}{k_1} \frac{k'}{k'_1}} &= \varepsilon^2 \frac{z^2 z'^2}{c^4} G^{(0)}(k'; z) G^{(0)}(k; z) G^{*(0)}(k'_1; z') G^{*(0)}(k_1; z') \\
 &\quad \times g(k - k', k'_1 - k_1) \delta(k - k' + k'_1 - k_1) \quad (48)
 \end{aligned}$$

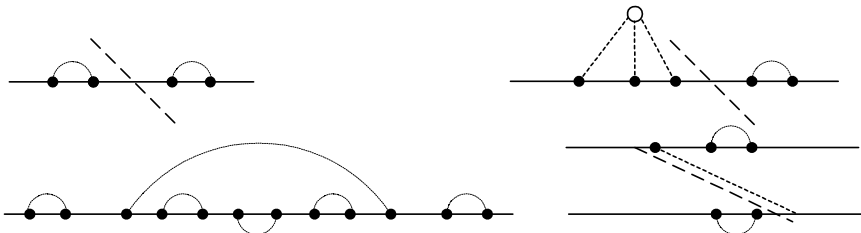
A consequence of the wave vector conservation condition is that the difference of the wave vectors appearing to the right and to the left of any double diagram is the same. If we take them equal, we obtain the following result:

*The mean spectral energy propagator $\langle |G(k, k'; z)|^2 \rangle$
satisfies a separate equation*

Mass operator and Dyson equation. Feynman diagram (single or double) is said to be unconnected if it can be cut into two or more diagrams, without cutting any dotted lines. The following diagrams are connected:

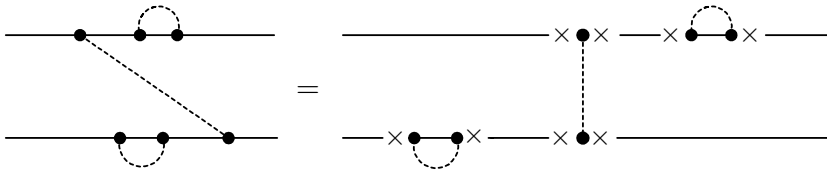


The following are unconnected (we give a possible cut)



The strong lines at the end of a diagram are called its *terminals*. We can write any unconnected diagram as a product of connected diagrams without terminals and strong lines. The decomposition is unique for the single propagator, for example:

$$\text{---} \overset{\text{arc}}{\curvearrowright} \text{---} \overset{\text{arc}}{\curvearrowright} \text{---} = \text{---} \times \overset{\text{arc}}{\curvearrowright} \times \text{---} \times \overset{\text{arc}}{\curvearrowright} \times \text{---}$$



We define now the mass operator M as the sum of all connected diagrams without terminals contributing to the single propagator:

$$M = \text{[diagram 1]} + \text{[diagram 2]} + \text{[diagram 3]} + \text{[diagram 4]} + \dots \tag{49}$$

The diagrams in the equation represent different types of self-energy corrections to a propagator: a simple loop, a loop with a vertex correction (a circle), two overlapping loops, and a loop with a vertex correction and a self-energy loop on the vertex.

In k -space M is diagonal operator $M(k; z)$; in r -space, it is an integral convolution operator $M(r, r'; z)$. It is useful to introduce two new symbols for the mean perturbed propagator and the mass operator

$$\langle G \rangle = \text{[double line symbol]}, \quad M = \text{[hatched semi-circle symbol]} \tag{50}$$

Using the decomposition of unconnected diagrams into products of connected ones, the following expansion is easily derived:

$$\text{[double line symbol]} = \text{[single line symbol]} + \text{[diagram 1]} + \text{[diagram 2]} + \text{[diagram 3]} + \dots \tag{51}$$

The expansion shows the double line symbol as a sum of a single line symbol and three diagrams where the hatched semi-circle symbol is inserted into the propagator line at different positions.

This is formally equivalent to an equation called the Dyson equation in quantum field theory:

$$\text{[double line symbol]} = \text{[single line symbol]} + \text{[diagram with hatched semi-circle]} \tag{52}$$

If $M(k; z)$ is known, this is an ordinary equation for $\langle G(k; z) \rangle$ in k -space; solving it we get

$$\langle G(k; z) \rangle = \frac{G^{(0)}(k; z)}{1 - G^{(0)}(k; z)M(k; z)} \tag{53}$$

In order to find the double propagator counter part of the Dyson equation, we define the operator \boxtimes as the sum of all connected double diagrams without terminals. Using this operator the following expansion is derived for the mean double propagator:

$$\langle G \otimes G^* \rangle = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \quad (54)$$

Equation (54) is formally equivalent to an equation called sometimes the Bethe-Salpeter equation:

$$\langle G \otimes G^* \rangle = \text{Diagram 1} + \text{Diagram 2} \langle G \otimes G^* \rangle \quad (55)$$

If the mean perturbed propagator and the operator \boxtimes are known, this is an integral equation for the mean double propagator.

Before, we try to use all this perturbation formalism to get approximate solutions for the mean propagator and double propagator, we shall study in the next section a one-dimensional (1D) random equation which has exact solution. It will help us to find out what kind of approximation are acceptable and what kind are not.

4. AN EXACT SOLUTION OF 1D-EQUATION

In this section, we study the one-dimensional equation:

$$\frac{\partial \Psi(x, t)}{\partial x} + \frac{1}{c} [1 + \varepsilon \mu(x)] \frac{\partial \Psi(x, t)}{\partial t} = 0 \quad (56)$$

where $\mu(x)$ is a real, centered and stationary Gaussian random function (g. r. f.) with covariance function

$$\Gamma(x, x') = \langle \mu(x) \mu(x') \rangle \quad (57)$$

and the associated radiation problem

$$\frac{\partial \Psi(x)}{\partial x} - ik_0 [1 + \varepsilon \mu(x)] \Psi(x) = \delta(x) \quad (58)$$

where $\delta(x)$ is Dirac's distribution at the origin. The wave number $k_0 = 2\pi/\lambda = 2\pi f/c$ is taken positive. Here λ is the wavelength, f is the radiated frequency and c is the velocity of light. Equations (56) and (58) can be treated simultaneously if we take the initial conditions

$$\Psi(x, 0) = \frac{\delta(x)}{[1 + \varepsilon \mu(x)]} \quad (59)$$

The L. T. of (56) and (58) are then identified by introducing z and ck_0 .

The underlying physical problem is the following: The monochromatic source of frequency $\omega = 2\pi f = ck_0$ is radiating into a semi-infinite one-dimensional medium whose refractive index

is $n(x) = 1 + \varepsilon\mu(x)$. Only propagation toward ($x > 0$) is considered; reflections are assumed to be negligible. Integration of (58) gives

$$\Psi(x) = Y(x) \exp[ik_0x] \exp \left\{ ik_0 \int_0^x \varepsilon\mu(y)dy \right\} \quad (60)$$

where $Y(x)$ being Heavieside's step function.

To calculate now the mean value of (60), the only random term is the second potential. For fixed x , $\int_0^x \varepsilon\mu(y)dy$ being a linear functional of the g. r. f., $\mu(x)$, is a centered Gaussian random variable (g. r. v.) φ . If so, $\langle e^{ik_0\varphi} \rangle$ is the characteristic function of this random variable. As φ is Gaussian

$$\langle e^{ik_0\varphi} \rangle = e^{-\frac{1}{2}k_0^2\langle\varphi^2\rangle} \quad (61)$$

we evaluate now

$$\langle\varphi^2\rangle = \left\langle \varepsilon^2 \left| \int_0^x \mu(y)dy \right|^2 \right\rangle = \varepsilon^2 \int_0^x dy \int_0^y \Gamma(y-y')dy' \quad (62)$$

and finally obtain

$$\langle\Psi(x)\rangle = Y(x) \exp[ik_0x] \exp \left\{ -\frac{1}{2}k_0^2\varepsilon^2 \int_0^x dy \int_0^y \Gamma(y-y')dy' \right\} \quad (63)$$

The mean wave function is thus expressed in terms of the covariance function of the refractive index. Higher order moments such as $\langle\Psi(x)\Psi(x')\rangle$ are easily obtained, using the characteristic function of a multivariate Gaussian distribution [18–20]. We introduce now the covariance function

$$\Gamma(x-x') = \exp \left\{ -\left| \frac{x-x'}{\ell} \right| \right\} \quad (64)$$

where ℓ is the range of index correlation. The mean wave function can now be calculated strictly

$$\langle\Psi(x)\rangle = Y(x) \exp[ik_0x] \exp \left\{ -\varepsilon^2 k_0^2 \ell^2 \left(\frac{x}{\ell} + e^{-\frac{x}{\ell}} - 1 \right) \right\} \quad (65)$$

The dimensionless parameter which determines the behavior of the solution is $\varepsilon k_0 \ell$. There are two interesting limiting approximations:

a) $\varepsilon k_0 \ell \ll 1$. It is a long wave length approximation ($\lambda \gg \ell$), and corresponds to weak interactions in quantum field theory. A uniform approximation for $\langle \Psi(x) \rangle$ is then

$$\langle \Psi(x) \rangle = Y(x) \exp[ik_0 x] \exp \left\{ -\varepsilon^2 k_0^2 \ell x \right\} \quad (66)$$

As follows from (66), the initial excitation is damped with an extinction length

$$x_{ex} = \left(\varepsilon^2 k_0^2 \ell \right)^{-1} \quad (67)$$

Let us compare x_{ex} and the wavelength $\lambda \sim (k_0)^{-1}$

$$\frac{x_{ex}}{\lambda} \sim \frac{1}{\varepsilon^2 k_0 \ell} = \frac{1}{\varepsilon} \cdot \frac{1}{\varepsilon k_0 \ell} \gg 1 \quad (68)$$

The damping is thus very slow; it is due to phase mixing and is not related to any dissipative mechanism. The mean wave function $\langle \Psi(x) \rangle$ can also be written as

$$\langle \Psi(x) \rangle = Y(x) \exp \left\{ i \left(k_0 - i\varepsilon^2 k_0^2 \ell \right) x \right\} \quad (69)$$

The effect of randomness on the mean wave function, as follows from (69), is simply a renormalisation of the wave number. The renormalised wave number is now equals $k = k_0 - i\varepsilon^2 k_0^2 \ell$ which has a small imaginary part (because $\varepsilon k_0 \ell \ll 1$). In the next section, we shall obtain this wave approximation as a sum of an infinite series extracted from the perturbation expansion of the mean propagator.

b) $\varepsilon k_0 \ell \gg 1$. It is a short wave length approximation ($\lambda \ll \ell$) corresponding to strong interactions in quantum field theory. A uniform approximation for $\langle \Psi(x) \rangle$ is then

$$\langle \Psi(x) \rangle = Y(x) \exp[ik_0 x] \exp \left\{ -\frac{1}{2} \varepsilon^2 k_0^2 x^2 \right\} \quad (70)$$

The initial excitation is damped again, with an extinction length $x_{ex} = (\varepsilon k_0)^{-1} \sim \lambda/\varepsilon$; the damping is more rapid than in the preceding case. This approximation is equivalent to a renormalization of the wave number, because x^2 appears in the second exponent in (70).

Should be noted that this approximate solution is the exact solution of the initial equation (58) if μ is not a random function, but a mere random variable. We can conclude that when the range of index correlations is much longer than the wavelength, the random index behaves like a random variable. This observation will be generalized

in Section 6. In the sequel, we shall also need F. T. of the exact mean function (65)

$$\begin{aligned} \langle \Psi(k) \rangle &= \int_{-\infty}^{\infty} \exp(-ik_0x) \langle \Psi(k) \rangle dx \\ &= \int_{-\infty}^{\infty} \exp[i(k_0 - k)] \exp(\varepsilon^2 k_0^2 \ell^2 x) \exp(-\varepsilon^2 k^2 \ell x) \exp\left\{-\varepsilon^2 k_0^2 \ell^2 e^{-\frac{x}{\ell}}\right\} dx \end{aligned} \quad (71)$$

Expand the last exponential term in a uniformly convergent series and integrate (71), we finally obtain

$$\langle \Psi(x) \rangle = \exp(\varepsilon^2 k_0^2 \ell^2) \sum_{n=0}^{\infty} \frac{(-\varepsilon^2 k_0^2 \ell^2)^n}{n!} \frac{1}{ik - ik_0 + \varepsilon^2 k_0^2 \ell + n/\ell} \quad (72)$$

As follows from (72), $\langle \Psi(x) \rangle$ has the poles $k_n = k_0 + i\varepsilon^2 k_0^2 \ell + in/\ell$, which correspond to more and more damped partial waves in r -space. If $\varepsilon k_0 \ell \ll 1$, we can approximate $\langle \Psi(x) \rangle$ by the first partial wave ($n = 0$) which gives again (69), apart a factor $\exp(\varepsilon^2 k_0^2 \ell^2) \neq 1$.

5. THE PERTURBATION METHOD AND ITS APPROXIMATIONS

In this section, we shall try to justify certain approximation procedures using the formal perturbation series of Section 3. We recall, once more, that the random function $\mu(r)$ is strictly stationary with respect to space translations. Our investigations will be only concerned with the mean propagator $\langle G \rangle$.

5.1. Low Order Approximations

In precedent section the mean wave function of the 1D-model was shown to be damped through destructive phase mixing, the damping length located with this phenomenon being very long compared to the wave length. The more general situations such as the scalar wave equation, we expect the no thing to happen. It is therefore necessary to get approximate solutions for the mean propagator which are valid at distances long compared to the wavelength (radiation problem) and times long compared to the period (initial wave problem). We shall show that low order perturbation approximations do satisfy this condition, even for very small strength of random fluctuations ε .

Born approximation for the mean propagator is the lowest order non vanishing approximation of the corresponding perturbation series, that is,

$$\langle G \rangle_{\text{Born}} = \frac{1}{k} + \frac{\text{---} \overset{\text{---}}{\text{---}} \text{---}}{k \quad k' \quad k} \quad (73)$$

i.e., explicitly in k -space

$$\langle G \rangle = \frac{c^2}{z^2 - c^2k^2} + \frac{\epsilon^2 z^4 c^2}{(z^2 - c^2k^2)^2} \int \frac{\Gamma(k - k')}{z^2 - c^2k'^2} d^3k' \quad (74)$$

If so, $G(k - k') = g(k - k', k' - k)$ is the F. T. of the covariance function. The squared perturbed propagator appearing in the second term on the random homogeneous space of (74) is consequence of the wave vector conservation conditions. This term has two double poles $z = \pm cK = \pm c|k|$. It is well known that such double poles will be two contributions to the inverse Laplace transform $\langle G(k, t) \rangle$, proportional to $t \cdot \exp[icKt]$ and $t \cdot \exp[-icKt]$. In other words the first perturbation term has a secular behavior, i.e., it increases without any limit as $t \rightarrow +\infty$. As this prevents any damping of the mean propagator, we conclude:

*The Born approximation is only a short time
(or short distance) approximation*

It will be useful in the sequel to have a better knowledge of the time dependence of the Born approximation. Let us, for example, take the covariance function as

$$\Gamma(r) = \exp \left[-\frac{R}{\ell} \right], \quad R \equiv |r| \quad (75)$$

Here ℓ , as above, is the range of refraction index correlations. To evaluate the convolution integral

$$\text{---} \overset{\text{---}}{\text{---}} \text{---} = \int \frac{c^2 \Gamma(k - k')}{z^2 - c^2k'^2} d^3k' \quad (76)$$

we note that it is the F. T. of

$$\exp \left[-\frac{R}{\ell} \right] \frac{\exp[izR/c]}{-4\pi R} = \frac{\exp[(i/c) \cdot [z + (ic/\ell)R]}{-4\pi R} \quad (77)$$

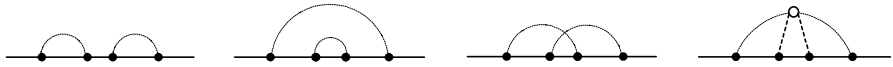
and changing z into $z + (ic/\ell)$, we finally find

$$\begin{array}{c} \text{---} \bullet \text{---} \bullet \text{---} \\ \text{---} \text{---} \end{array} \overset{\text{dashed arc}}{\curvearrowright} = \frac{c^2}{[z + (ic/\ell)]^2 - c^2k^2} \tag{78}$$

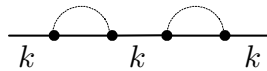
and

$$\begin{array}{c} \text{---} \bullet \text{---} \bullet \text{---} \\ \text{---} \text{---} \end{array} \overset{\text{dashed arc}}{\curvearrowright} = \frac{\varepsilon^2 c^2 z^4}{(z^2 - c^2k^2)^2 \{ [z + (ic/\ell)]^2 - c^2k^2 \}} \tag{79}$$

Besides the double poles $z = \pm cK$, two other poles have appeared in (79): $z = \pm cK - ic/\ell$. The corresponding contributions to L. T. $\langle G(k, t) \rangle$ are proportional to $\exp\{\pm icKt\} \cdot \exp\{ic(t/\ell)\}$. They are thus damped with a damping time $t_d = \ell/c$. This damping time is the time the wave takes to travel a distance ℓ equal to the size of the scattering blobs. We call it as in quantum field theory the interaction time t_{int} . We turn back now to the perturbation series for the mean propagator, and show that, as we take more and more perturbation terms we get more and more divergent secular terms. The fourth order diagrams are (see Section 3):



Because of the wave vector conservation condition the same vector occurs in the middle and at the terminals of the diagram



The factor $G^{(0)}(k; z) = \frac{c^2}{z^2 - c^2k^2}$ occurs thus twice in this diagram, producing secular terms proportional to $t^2 \cdot \exp[icKt]$ and $t^2 \cdot \exp[-icKt]$. More generally, any unconnected diagram which is the product of p connected diagrams produces secular terms proportional to $t^p \cdot \exp[icKt]$ and $t^p \cdot \exp[-icKt]$. We call them *the leading terms* of the diagram. Besides of leading terms, there are other secular terms with lower power of t , and also damped terms, with a damping time which is found to be always of the order of $t_{\text{int}} = \ell/c$. If the damping time of the mean propagator is much longer than t_{int} , the asymptotic time dependence of the mean propagator will be governed essentially by the leading terms.

Let us show that secular terms arise in the radiation problem too. We are now looking for $\langle G(r) \rangle$, whose F. T. may be obtained by changing z into ck_0 in $\langle G(k; z) \rangle$. Taking again the covariance function

$\exp(-R/\ell)$, we obtain

$$\langle G(k) \rangle = \frac{1}{k_0^2 - k^2} + \frac{\varepsilon^2 k_0^2}{(k_0^2 - k^2)^2} \frac{1}{[(k_0 + i/\ell)^2 - k^2]} + \dots \quad (80)$$

This is the function of $K = |k|$ only. Its inverse F. T. $\langle G(R) \rangle$ can be obtained by a single integration

$$\langle G(R) \rangle = \frac{1}{(2\pi)^2} \int_0^\infty \frac{e^{iKR} - e^{-iKR}}{iKR} K^2 \langle G(k) \rangle dK \quad (81)$$

In $\langle G(k) \rangle$ the poles $K = k_0$ appear again with increasing multiplicity, as we take more and more perturbation terms. As a consequence of this, we can write

$$\langle G(R) \rangle = e^{ik_0 R} \cdot [1 + AR + BR^2 + \dots] \quad (82)$$

where A, B, \dots do not depend on R . As we take the limit $\text{Im}(k_0) \rightarrow 0$, we obtain secular terms in R .

We conclude that no *finite* order approximation of the perturbation series can be used because it would diverge as $t \rightarrow \infty$ or $R \rightarrow \infty$. It is trivial observation however that an *infinite* sum of secular terms may be non-secular; for example

$$\exp(-t^2) = \sum_{n=0}^\infty \frac{(-t^2)^n}{n!} \quad (83)$$

but any finite sum has a secular behavior. If we want to do something with the formal perturbation series, we must thus use at least an infinite subseries. This result is independent of ε , the strength of refractive index fluctuations, because secular terms do only disappear for $\varepsilon = 0$.

5.2. Convergence of the Perturbation Expansion

A fundamental question arises as above: *does the perturbation series converge?* It is rather difficult to give a general answer to this question because we do not say what kind of convergence we expect (or do not expect). Let us first indicate that there is a proof in [1-6] that the perturbation series for $G(r)$ (radiation problem) is mean square convergent for Gaussian $\mu(r)$.

Let us now consider the 1D-model. For $\varepsilon k_0 \ell \ll 1$ ($\ell \ll \lambda$), we can write the mean propagator

$$\langle G(x, x') \rangle = Y(x - x') \exp \{ik_0(x - x')\} \exp \left\{ -\varepsilon^2 k_0^2 \ell (x - x') \right\} \quad (84)$$

The perturbation expansion of the mean propagator in power of ε is thus

$$\langle G(x, x') \rangle = Y(x - x') \exp \{ ik_0(x - x') \} \sum_{n=0}^{\infty} \frac{[-\varepsilon^2 k_0^2 \ell(x - x')]^n}{n!} \quad (85)$$

If x and x' are fixed, this is an analytic function of ε ; but this is of no interest because $\langle G(x, x') \rangle$ does not act as a multiplication, but as an integral convolution operator. The convolution product of a bounded source function with $Y(x) \exp \{ ik_0 x \} \times \exp \{ -\varepsilon^2 k_0^2 \ell x \}$ is convergent, but the convolution product with $Y(x) \exp \{ ik_0 x \} x^n$ is generally not, because the secular behavior of this term.

As last example, let us consider the convergence of the F. T. $\langle G(k) \rangle$ for the 1D-model. In Section 4, we proved that (72) presented now as

$$\langle G(x, x') \rangle = \exp \left(\varepsilon^2 k_0^2 \ell^2 \right) \sum_{n=0}^{\infty} \frac{(-\varepsilon^2 k_0^2 \ell^2)^n}{n!} \frac{1}{[ik - ik_0 + \varepsilon^2 k_0^2 \ell + n/\ell]}$$

are series on random homogeneous space (r. h. s.) is convergent, but this is not the perturbation series, because ε appears in the denominator; $\langle G(k) \rangle$ acts as a multiplication. We ask if this is an analytic function of ε in some neighbourhood of $\varepsilon = 0$ and all values of k .

For $n \geq 1$, $[ik - ik_0 + \varepsilon^2 k_0^2 \ell + n/\ell]$ is an analytic function of ε for $|\varepsilon| < \frac{n}{k_0 \ell^2}$ and all values of k . But for $n = 0$, $\varepsilon k_0 \ell \ll 1$ ($\ell \ll \lambda$) is not an analytic function of ε in any neighbourhood of $\varepsilon = 0$ because $k - k_0$ may vanish. We conclude that the perturbation expansion does not converge. However, it may easily be shown that for $\varepsilon k_0 \ell < 1$ ($l < \lambda$) one can write

$$\langle G(x, x') \rangle = [ik - ik_0 + \varepsilon^2 k_0^2 \ell + n/\ell]^{-1} \left\{ 1 + \varepsilon^2 k_0^2 \ell^2 R(k, \varepsilon) \right\} \quad (86)$$

where $R(k, \varepsilon)$ is a bounded function of k and ε . This means that we can approximate $\langle G(k) \rangle$ uniformly by the first term of the series (72) for $k_0 \ell \ll 1$. This term, though not analytic, can be *formally* expanded in powers of ε :

$$\frac{1}{[ik - ik_0 + \varepsilon^2 k_0^2 \ell]} = \frac{1}{(ik - ik_0)} \sum_{p=0}^{\infty} \left(-\frac{\varepsilon^2 k_0^2 \ell}{ik - ik_0} \right)^p \quad (87)$$

In the sequel we shall do exactly the converse: given a formal divergent series we shall extract from it another divergent subseries whose formal sum easy to calculate.

5.3. Bourret's Bilocal and Kraichnan's Random Coupling Models

We describe now two attempts to overcome the difficulty of divergent perturbation series arising from random equations. Both methods [15, 25] use the fact that there are certain infinite subseries of the formal perturbation series for the mean propagator, the exact solution of which is possible. As has been pointed by Kraichnan [25] in a fundamental paper on the dynamics of non-linear stochastic systems, the use of subseries of a perturbation series is very dangerous, because they may give unphysical results such as negative energy densities. It may also be happen that a certain subseries containing a given class of diagrams gives a good approximation, and that if one tries to improve it by enlarging this class, the resulting sum becomes unacceptable. It will therefore be necessary to proceed very carefully. we give now the first method.

Bourret's bilocal approximation [13]. We take all diagrams whose connected parts have only two vertices; the resulting series is called the Bourret series $\langle G \rangle_B$:

$$\langle G \rangle_B = \text{---} + \text{---} \overset{\text{---}}{\text{---}} \text{---} + \text{---} \overset{\text{---}}{\text{---}} \overset{\text{---}}{\text{---}} \text{---} + \text{---} \overset{\text{---}}{\text{---}} \overset{\text{---}}{\text{---}} \overset{\text{---}}{\text{---}} \text{---} + \dots \tag{88}$$

We have taken into account here all multiple scattering which are a succession of double scatterings. This series has been introduced by Bourret [15] for Gaussian random function (g. r. f.), and also studied by Tatarskii et al. [18–20]. Recalling that the mass operator is the sum of all connected diagrams without terminals, we find that the Bourret series corresponds to the lowest order approximation of the mass operator. The corresponding Dyson equation, which is immediately derived from (88) is

$$\langle G \rangle_B = \text{---} + \text{---} \overset{\text{---}}{\text{---}} \langle G \rangle_B \tag{89}$$

This equation has also been derived by Keller [16] but his derivation must be considered incorrect, because he treats the second term of the full perturbation series (—●—●—) as a small perturbation. As we know, this is only time of the mean propagator. Before we discuss Bourret's approximation for the single propagator; let us give the corresponding double propagator approximation

$$\langle G \otimes G^* \rangle_B = \text{====} + \text{====} + \text{====} + \dots \tag{90}$$

Bourret's equation (89) for the single propagator is easily solved in k -space; we shall not derive the corresponding solution for the radiation

problem, because this may be found in [15, 16, 18–20]. It is also instructive to study the behavior of $\langle G(k, t) \rangle_B$ for a given k , i.e., the time dependence of an initial excitation proportional to $\exp\{ikr\}$. The main result is that the natural frequencies $\omega = \pm cK$ are renormalized. For an $\exp(-R/\ell)$ covariance function, the renormalized frequencies are

$$\omega_r = \pm cK \mp \frac{\varepsilon^2 c K^3 \ell^2}{2} - i\varepsilon^2 c K^4 \ell^3 \quad (91)$$

The damping time corresponding to the imaginary part of ω_r is $t_d = \frac{1}{\varepsilon^2 c K^4 \ell^3}$. We turn not to the discussion of the Bourret's approximation; we shall show what it is an uniform approximation which is only valid when the wavelength is much longer than the range of refractive index correlations: $K\ell \ll 1$.

Bourret's derivation is based upon the following bilocal closure assumption:

$$\langle \mu(r_1)\mu(r_2)\Psi(r_2) \rangle = \langle \mu(r_1)\mu(r_2) \rangle \langle \Psi(r_2) \rangle \quad (92)$$

Such closure assumptions have been studied by Kraichnan [23]; he has shown that they are generally not uniformly valid for $t \rightarrow \infty$. However the examples considered by him are rather strong perturbations (such as random oscillator, or the non-linear stochastic Navier Stokes equation). As we shall see the case $K\ell \ll 1$ corresponds to weak perturbation (i.e., weak interactions in quantum field theory and weakly coupled gases in statistical mechanics). The damping time associated with the solution of Bourret's equation is $t_d = \frac{1}{\varepsilon^2 c K^4 \ell^3}$; for $K\ell \ll 1$

$$t_d \gg t_{\text{int}} = \frac{\ell}{c} \quad (93)$$

The asymptotic behavior of any diagram is thus governed by its leading term (see Section 5.1). Bourret diagrams having p connected parts gives a leading term proportional to $\varepsilon^{2p} \cdot t^p \cdot \exp(\pm icKt)$; for $t \sim t_d$ this becomes ε -independent. Any other diagram will give rise to uncompensated powers of ε , and may thus be constructed as *small*. This is rather poor justification of the Bourret approximation, because we did not make a dimensional analysis of the diagram with respect to the other parameters c , K . Let us give the main lines of a more rigorous justification. The leading term of

$$\underbrace{\text{diagram with } P \text{ times}} = \left[\frac{c^2}{z^2 - c^2 k^2} \right]^{p+1} (\text{diagram})^p \quad (94)$$

is easily found to be

$$i\left(\frac{c}{2K}\right)^{p+1} t^p \left[\exp(-icKt) \left(\lim_{z=cK+i0} (\overset{\curvearrowright}{\bullet\text{---}\bullet})^p \right) - \exp(icKt) \left(\lim_{z=-cK+i0} (\overset{\curvearrowright}{\bullet\text{---}\bullet})^p \right) \right] \tag{95}$$

$$\lim z = cK + i0 \Leftrightarrow \lim_{\eta \rightarrow 0, \eta > 0} z = cK + i\eta$$

The asymptotic time behavior of (94) depends thus only on

$$\lim_{z=\pm cK+i0} (\overset{\curvearrowright}{\bullet\text{---}\bullet})$$

and not on the whole z -dependence of $\overset{\curvearrowright}{\bullet\text{---}\bullet}$. This is immediately generalized to any product of connected diagrams, belonging or not to the Bourret series. Accordingly the asymptotic time dependence of the whole perturbation series is only governed by $\lim_{z=\pm cK+i0} M(k; z)$; M being the mass operator. In order to show that the Bourret approximation is uniform, we only need prove that for $z = \pm cK + i0$, $\overset{\curvearrowright}{\bullet\text{---}\bullet}$ is a good approximation of the mass operator. We shall just give a dimensional justification of this for connected diagrams without terminals and consisting more than two vertices. A connected diagram in k -space is written as the F. T. of the corresponding r -space diagram. The correlation functions corresponding to the different clusters of this diagram are non-dimensionalized by introducing a new position vector $r' = r/l$. It is then easily found that for $z = \pm cK + i0$, a connected diagram L_p with p vertices and without terminals has the dimensional dependence

$$L_p(\varepsilon, c, K, \ell) \sim \varepsilon^p K^{2p} \ell^{2(p-1)} \Lambda_p(\ell K) \tag{96}$$

where $\Lambda_p(\ell K)$ being a non-dimensional function of the non-dimensional quantity ℓK . For small values of ℓK we can write

$$L_p \sim \varepsilon^p K^{2p} \ell^{2(p-1)} (1 + O(\ell K) + \dots) \tag{97}$$

For $\ell K \ll 1$ and $p > 2$, L_p is small compared to both $L_2 \sim \varepsilon^2 K^4 \ell^2$ and its first $O(1K)$ correction. This first order correction is necessary, because it is found that $\varepsilon^2 K^4 \ell^2$ does not contribute to the damping time, but only to the real frequency shift of the renormalized frequencies according to (91).

If the condition $\ell K \ll 1$ is violated, the contributions arising from diagrams not belonging to the Bourret series become more and more important, and for $\ell K \gg 1$ Tatarskii et al. [16–18] have shown that all diagrams with the same number of vertices are almost equal (this is only true for $t \ll t_{\text{int}}$, but as in this case $t_d \ll t_{\text{int}}$, it does not matter). If all these dimensional considerations are not very convincing, it is

still possible to check the validity of Bourret's approximation on the 1D-model (see Section 4). The exact solution is (65)

$$\langle \Psi(x) \rangle = Y(x) \exp[ik_0 x] \exp \left\{ -\varepsilon^2 k_0^2 \ell^2 \left(\frac{x}{\ell} + e^{-\frac{x}{\ell}} - 1 \right) \right\} \quad (98)$$

The solution of Bourret's equation, which is an ordinary equation in k -space is:

$$\langle \Psi(k) \rangle_B = \left[ik - ik_0 + \frac{\varepsilon^2 k_0^2}{ik - ik_0 + \frac{x}{\ell}} \right]^{-1} \quad (99)$$

Taking the inverse F. T. we find

$$\begin{aligned} \langle \Psi(k) \rangle_B &= Y(x) \frac{e^{ik_0 x}}{2\sqrt{\Delta}} \left\{ (1 + \sqrt{\Delta}) \exp \left[-\frac{1}{2\ell} (1 - \sqrt{\Delta}) x \right] \right. \\ &\quad \left. - (1 - \sqrt{\Delta}) \exp \left[-\frac{1}{2\ell} (1 + \sqrt{\Delta}) x \right] \right\} \\ \Delta &= 1 - 4\varepsilon^2 k_0^2 \ell^2 \end{aligned} \quad (100)$$

Expanding (100) for $\varepsilon k_0 \ell \ll 1$, we find

$$\langle \Psi(k) \rangle_B = Y(x) e^{ik_0 x} \exp \left\{ -\varepsilon^2 k_0^2 \ell x \right\} \quad (101)$$

which is the uniform approximation (66), already found for the exact solution. If the condition $\varepsilon k_0 \ell \ll 1$ is not satisfied (100) and (98) do not agree. For example, if $\varepsilon k_0 \ell \gg 1$, the Bourret approximation gives

$$\langle \Psi(k) \rangle_B = Y(x) e^{ik_0 x} \cos(\varepsilon k_0 x) \exp \{ -x/2\ell \} \quad (102)$$

whereas the exact solution (70) is

$$\langle \Psi(x) \rangle = Y(x) e^{ik_0 x} \exp \left\{ -\frac{1}{2} \varepsilon^2 k_0^2 x^2 \right\} \quad (103)$$

We conclude that:

The Bourret approximation is a long wave length approximation, uniformly valid for any random perturbation, Gaussian or not, which have a correlation range much shorter than the wavelength ($\ell \ll \lambda$)

This approximation can also be used for more general random equations than the scalar wave equation, because the dimensional

analysis is easily generalized. In Section 8, we apply it to the electromagnetic wave equation and in Section 9 to coupled wave equation.

Let us consider now the Bourret approximation for the double propagator. We have seen in Section 3 that the mean spectral energy propagator $\langle |G(k; z)|^2 \rangle$ and thus the mean spectral energy density $\langle |\Psi(k; z)|^2 \rangle$, satisfy a separate equation. For the radiation problem, using the Bourret approximation (90), this equation is

$$\begin{aligned} \langle |\Psi(k)|^2 \rangle_B &= |\langle G(k) \rangle_B|^2 |j(k)|^2 + |\langle G(k) \rangle_B|^2 \varepsilon^2 k_0^2 k_0^{*2} \\ &\quad \int \Gamma(k - k') \langle |\Psi(k')|^2 \rangle_B d^3k \end{aligned} \quad (104)$$

where $\langle G(k) \rangle_B$ is the Bourret approximation for the single propagator, and $j(k)$ is the F. T. of the source function. Equation (104) is an integral equation for the mean spectral energy density. As follows from [15], the Bourret approximation for $\langle G(k) \rangle$ is only valid for $Kl \ll 1$, but the integral term of (104) relates the region of the spectrum which does satisfy this condition and the other one. Even if we assume that $j(k)$ is vanishing outside $Kl \ll 1$, we do not know if some energy will not be transferred to the other part of the spectrum. There is thus a serious difficulty here, and it may be possible that the Bourret approximation is never uniform for the mean double propagator. This question remains up today open.

Kraichnan's random coupling model [25]. Given a random equation, Kraichnan [23] has constructed an other equation which is related to it, but can be reduced to a nonlinear nonrandom equation. This was achieved through the introduction of an additional random coupling between wave vectors, and called by him the random coupling model. The remarkable point is that its solution can be considered both as the exact solution of the model equation, and as an approximate solution of the original equation. This ensures that if we can solve the model equation, the solution will be physically acceptable. Kraichnan's perturbation diagrams are somewhat different from presented above, but the connection is easy to establish. With our notations the mean propagator of the random coupling model, which we denote by $\langle G(k) \rangle_K$, is the sum of all perturbation diagrams of the *original problem* such that there are

- no clusters of more than two points;
- no intersecting dotted lines.

We give the first diagrams as

$$\begin{aligned}
 \langle G \rangle_K = & \text{---} + \text{---} \overset{\text{dotted}}{\curvearrowright} \text{---} + \text{---} \overset{\text{dotted}}{\curvearrowright} \overset{\text{dotted}}{\curvearrowright} \text{---} + \text{---} \overset{\text{dotted}}{\curvearrowright} \overset{\text{dotted}}{\curvearrowright} \overset{\text{dotted}}{\curvearrowright} \text{---} + \\
 & \text{---} \overset{\text{dotted}}{\curvearrowright} \overset{\text{dotted}}{\curvearrowright} \overset{\text{dotted}}{\curvearrowright} \overset{\text{dotted}}{\curvearrowright} \text{---} + \text{---} \overset{\text{dotted}}{\curvearrowright} \overset{\text{dotted}}{\curvearrowright} \overset{\text{dotted}}{\curvearrowright} \overset{\text{dotted}}{\curvearrowright} \overset{\text{dotted}}{\curvearrowright} \text{---} + \dots \quad (105)
 \end{aligned}$$

This expansion is easily found to be equivalent to a nonlinear but nonrandom equation (we use the symbol \equiv for $\langle G \rangle_K$)

$$\equiv = \text{---} + \text{---} \overset{\text{dotted}}{\curvearrowright} \equiv \quad (106)$$

The only difference with Bourret's equation (89) is that the mean propagator, and not unperturbed propagator appears under the dotted line. The equation for the mean propagator of the random coupling model is

$$\langle G \otimes G^* \rangle_K = \equiv\equiv + \boxed{\equiv\equiv} \langle G \otimes G^* \rangle_K \quad (107)$$

This linear equation can only be solved after the nonlinear equation (106). All Bourret diagrams are included in the random coupling model. Accordingly, we expect that it will give a better approximation. But the main reason for studying this model is that there is no a priori limitation to its validity such as $Kl \ll 1$. The solution of the random coupling model for the wave equation would give at least a partial answer to important questions such as

- Is there always a damping of the mean propagator?
- How does the damping time vary with k ?
- What is the spectral mean energy distribution corresponding to a given excitation or source function?

Let us write down the random coupling model equation for our 1D-model and the scalar wave equation. A one-dimension model gives:

$$\langle G(k; z) \rangle_K = \frac{c}{ick - iz} + \frac{(\varepsilon iz)^2}{c(ick - iz)} \langle G(k; z) \rangle_K \int \Gamma(k - k') \langle G(k'; z) \rangle_K d^3 k' \quad (108)$$

At the same time the scalar wave equation is

$$\langle G(k; z) \rangle_K = \frac{c^2}{z^2 - c^2 k^2} + \frac{\varepsilon^2 z^4}{c^2(z^2 - c^2 k^2)} \langle G(k; z) \rangle_K \int \Gamma(k - k') \langle G(k'; z) \rangle_K d^3 k' \quad (109)$$

If $\Gamma(k - k') = \delta(k - k')$ equations (108) and (109) are ordinary nonlinear equations which can be solved analytically. This case corresponds to a covariance function in r -space which is a constant. In other terms,

the random function μ degenerates into a random value. As in this case, the equation for the original problem can be solved without perturbation method (see Section 6).

In the general case we show that such nonlinear integral equations constitute a very difficult problem. They must be solved for $\text{Im}(z) > 0$, but the asymptotic behavior of $\langle G \rangle_K$ is determined by the poles of the analytic continuation of this solution in the lower half plane $\text{Im}(z) < 0$. Even if we find an approximation method which converge in the upper half plane, it will generally be very difficult, if not impossible, to extend it by analytic continuation.

There are two possible iterations methods for the random coupling model equation: given the equation (106)

$$\text{====} = \text{---} + \text{---} \overset{\text{---}}{\text{---}} \text{---}$$

we can iterate it, considering the second term on the random homogeneous space as a perturbation. This gives

$$\text{====} = \text{---} + \text{---} \overset{\text{---}}{\text{---}} \text{---} + \text{---} \overset{\text{---}}{\text{---}} \overset{\text{---}}{\text{---}} \text{---} + \text{---} \overset{\text{---}}{\text{---}} \overset{\text{---}}{\text{---}} \overset{\text{---}}{\text{---}} \text{---} + \dots \tag{110}$$

which cannot be used for long times because of the secular form. It more interesting method is to write equation (106) as

$$\langle G \rangle_K = \frac{G^{(0)}}{1 - G^{(0)}L(\langle G \rangle_K)} \tag{111}$$

where L is the linear operator

$$\text{====} \xrightarrow{L} \text{---} \overset{\text{---}}{\text{---}} \text{---} \tag{112}$$

Equation (111) is then iterated giving

$$\langle G \rangle_K = \frac{G^{(0)}}{1 - G^{(0)}L \left(\frac{G^{(0)}}{1 - G^{(0)}L(\dots)} \right)} \tag{113}$$

This is the operator analogue of a continued fraction. It corresponds to successive self-consistent approximation of the mass operator. For the 1D-model and the covariance function $\sim \exp(-|x|/1)$ it was possible to show that this iteration process converges for $|\varepsilon lz/c| < 1/2$, and to find its analytic continuation. The proof is somewhat artificial because we used the fact that for any function $f(k)$ which is bounded and analytic

in the half plane $\text{Im}(z) < 0$

$$\int \Gamma(k - k') f(k') dk' = f(k - i/1) \quad (114)$$

We get thus a nonlinear finite difference equation which is solved by means of a continued fraction. Unfortunately, it is not possible to extend this method to the scalar wave equation.

6. RANDOM TAYLOR EXPANSION AT SHORT WAVELENGTHS

In Section 5, we have found that in the limiting case $\varepsilon k_0 \ell \gg 1$ the random refractive index behaves as a mere random value (r. v.) and not as a random function (r. f.). This is easily understood, because when ℓ is very large ($\ell \gg \lambda$), each realization (or sample) of the random index is a very slowly varying function, which can be approximated by a constant. At an intermediate level between general r. f. and r. v., we could try to approximate a r. f. by a linear function or a quadratic function with r. v. as coefficients, i.e., make a limited random Taylor expansion of the r. f. We shall not give a rigorous justification of this procedure, but only check its validity according to [7, 22] on the 1D-model. The random equation for this model is

$$\frac{\partial \Psi(x)}{\partial x} - ik_0 [1 + \varepsilon \mu(x)] \Psi(x) = \delta(x) \quad (115)$$

where $\mu(x)$ is a random function, $\delta(x)$ is Dirac's distribution at the origin, the wave number $k_0 = 2\pi/\lambda = 2\pi f/c$ is taken, as in Section 4, positive. Here λ is the wavelength, f is the radiated frequency and c is the velocity of light. We want to approximate the r. f. $\mu(x)$ by its random Taylor expansion [9, 24]

$$\mu(x) = \mu(0) + x\mu'(0) + \frac{x^2}{2}\mu''(0) \dots \quad (116)$$

where $\mu(0)$, $\mu'(0)$, $\mu''(0)$, ... are r. v. (not independent). We cannot keep the covariance function $\exp\{-|x|/\ell\}$ because the corresponding r. f. is not mean square differentiable (see Section 2). As we do not need to specify the covariance Γ , we shall only assume that it has derivatives of all orders at $x = 0$ and that $\Gamma(0) = 1$. Approximate equation for $\Psi(x)$ is

$$\frac{\partial \Psi(x)}{\partial x} - ik_0 \left[1 + \varepsilon \mu(0) + \varepsilon x \mu'(0) + \varepsilon \frac{x^2}{2} \mu''(0) \right] \Psi(x) = \delta(x) \quad (117)$$

It is solved for the mean wave function

$$\langle \Psi(x) \rangle = Y(x)e^{ik_0x} \left\langle \exp \left[ik_0\varepsilon x \left(\mu(0) + \varepsilon x \mu'(0) + \varepsilon \frac{x^2}{2} \mu''(0) \right) \right] \right\rangle \quad (118)$$

As $\mu(x)$ is a Gaussian random function (g. r. f.), the multivariate distribution of $\mu(0)$, $\mu'(0)$, $\mu''(0)$ is also Gaussian, it is thus determined by its second order moment such as $\langle (\mu(0))^2 \rangle$, $\langle (\mu'(0))^2 \rangle$, $\langle \mu(0)\mu'(0) \rangle$ etc. They are easily calculated in terms of covariance function, for example

$$\langle \mu(0)\mu'(0) \rangle = \lim_{h \rightarrow 0} \frac{\langle \mu(0)\mu(h) \rangle - \langle \mu(0)\mu(0) \rangle}{h} = \Gamma'(0) \equiv 0 \quad (119)$$

because Γ is an even function.

The mean value in (118) is easily related to the characteristic function of $\mu(0)$, $\mu'(0)$, $\mu''(0)$, and can be calculated in terms of Γ ; this gives

$$\langle \Psi(x) \rangle = Y(x)e^{ik_0x} \exp \left[-\frac{k_0^2\varepsilon^2x^2}{2} \left(1 + \frac{x^2}{12}\Gamma''(0) + O(x^3) \right) \right] \quad (120)$$

Let us compare this to the exact solution (63)

$$\langle \Psi(x) \rangle = Y(x)e^{ik_0x} \exp \left[-\frac{k_0^2\varepsilon^2}{2} \int_0^x \int_0^y \Gamma(y-y') dy dy' \right] \quad (121)$$

Expanding the covariance function in power of x and integrating we get exactly the same result as in (120). If the condition

$$\left| \frac{k_0^2\varepsilon^2}{\Gamma''(0)} \right| \gg 1 \quad (122)$$

is satisfied we can use the random variable (r. v.) approximation

$$\langle \Psi(x) \rangle = Y(x)e^{ik_0x} \exp \left[-\frac{k_0^2\varepsilon^2x^2}{2} \right] \quad (123)$$

An equivalent condition is that the damping length $x_d = 1/k_0\varepsilon = \lambda/\varepsilon$ corresponding to this approximation should be much shorter than the range of random correlations $\ell = |\Gamma''(0)|^{-1/2}$. If it is satisfied, the wave cannot escape the region where the random index is properly approximated by a r. v. The r. v. approximation is easily applied to

any propagation equation because we only need to solve a partial differential equation with constant coefficients and average afterwards. If we want higher order approximation, we must solve a partial differential equation with linear or quadratic coefficients. The case of linear coefficients can in principle be solved by means of a generalized Laplace transformation, but this is rather complicated. In Section 9 we shall apply the r. v. method to coupled wave equations at short wavelength.

To end this section we derive the short wave approximation for the scalar wave equation with point source, which is actual for wireless communications

$$\Delta\Psi(r) + k_0^2[1 + \varepsilon\mu]\Psi(r) = \delta(r) \quad (124)$$

We assume μ to be a centered Gaussian r. v. and $\langle\mu^2\rangle = 1$. Solving (124) we get

$$\Psi(r) = \frac{\exp\{ik_0(1 + \varepsilon\mu)R\}}{-4\pi R}, \quad R = |r| \quad (125)$$

Taking the mean value of this wave function we find

$$\langle\Psi(r)\rangle = \frac{\exp(ik_0R) \exp\left(-\frac{1}{2}\varepsilon^2k_0^2R^2\right)}{-4\pi R} \quad (126)$$

The damping due to phase mixing is thus exponential with a damping length

$$R_d = (\varepsilon|k_0|)^{-1} > \lambda \quad (127)$$

The condition $R \ll 1$ can be written as

$$\varepsilon|k_0|\ell \gg 1 \quad (128)$$

It is thus a short wavelength condition ($\ell \gg \lambda$). The result (126) disagrees with a result derived by Tatarskii [16–18] for $|k_0|\ell \gg 1$. His mean wave function

$$\langle\Psi(r)\rangle = \frac{\exp(ik_0R)}{-4\pi R} \cdot \frac{1}{(1 + \varepsilon^2k_0^2R\ell)^{1/2}} \quad (129)$$

has not an exponential decrease, and a damping length

$$R_d = \left(\varepsilon^2k_0^2\ell\right)^{-1} \ll \lambda \quad (130)$$

His result is expressed as a certain integral over the solution of the Bourret equation (89), and this integral is calculated by the method

of stationary phase. The expansion of the solution of the Bourret equation used by Tatarskii is only valid for $|k_0|\ell \ll 1$. If the proper expansion is used the result becomes identical with mentioned above. This can also be checked on the 1D-model for which the calculation is easier.

7. EXACT SOLUTION OF THE SCALAR WAVE EQUATION

In [18–20] was suggested that functional space integration might be used to solve random equations. The authors thought that it would be necessary to generalize the Wiener measure to more general stochastic processes than the Brownian motion. We recall however that the success of functional integration methods is largely due to the fact that the Brownian motion enjoys the Markov property, and that the stochastic processes which are used in presented here problem do not. Nevertheless we shall show that the reduced scalar wave equation can be related to a complex version of the heat equation and solved via a complex Wiener measure. The method indicated here can also be used for the Schrodinger equation with random potential. We want to solve the reduced scalar wave equation with random refractive index

$$\Delta\Psi(r) + k_0^2[1 + \mu(r)]\Psi(r) = \delta(r) \quad (131)$$

where, once more, $\delta(r)$ is Dirac's distribution at the origin. As usually, we shall assume that the wave number k_0 has a small positive imaginary part. $\mu(r)$ is a centered Gaussian random function (g. r. f.) with covariance $\Gamma(r, r')$ which need not be stationary. In order to relate this equation to the heat equation, we introduce a new unknown function $\tilde{\Psi}(r, \theta)$ such that

$$\Psi(r) = -\frac{i}{k_0} \int_0^\infty \exp(ik_0\theta) \tilde{\Psi}(r, \theta) d\theta \quad (132)$$

Because of the positive imaginary part of k_0 , this integral is convergent if $\tilde{\Psi}(r, \theta)$ is not increasing too fast at infinity. Equation (132) is now multiplied by k_0^2 and integrated by parts

$$k_0^2\Psi(r) = -\int_0^\infty \tilde{\Psi}(r, \theta) \frac{\partial}{\partial\theta} \exp(ik_0\theta) d\theta = \tilde{\Psi}(r, 0) + \int_0^\infty \frac{\partial\tilde{\Psi}(r, \theta)}{\partial\theta} \exp(ik_0\theta) d\theta \quad (133)$$

Using alternatively (132) and (133) we write

$$\begin{aligned} \Delta\Psi(r) + k_0^2\Psi(r) + k_0^2\mu(r)\Psi(r) &\equiv -\frac{i}{k_0}\int_0^\infty \exp(ik_0\theta) \left[\Delta\tilde{\Psi} + k_0^2\mu(r)\tilde{\Psi}\right] d\theta \\ &\quad + \tilde{\Psi}(r, 0) + \int_0^\infty \frac{\partial\tilde{\Psi}(r, \theta)}{\partial\theta} \exp(ik_0\theta) d\theta \\ &= \delta(r) \end{aligned} \quad (134)$$

This is satisfied if we take

$$\frac{\partial\tilde{\Psi}(r, \theta)}{\partial\theta} = \frac{i}{k_0}\Delta\tilde{\Psi}(r, \theta) + ik_0\mu(r)\tilde{\Psi}(r, \theta) \quad (135)$$

Let us compared (135) to the perturbed heat equation

$$\frac{\partial\tilde{\Psi}(r, \theta)}{\partial\theta} = \Delta\tilde{\Psi}(r, \theta) + V(r)\tilde{\Psi}(r, \theta) \quad (136)$$

and the Schrodinger equation

$$\frac{\partial\tilde{\Psi}(r, \theta)}{\partial\theta} = \alpha\Delta\tilde{\Psi}(r, \theta) + V(r)\tilde{\Psi}(r, \theta) \quad (137)$$

The first one (136) can be solved by functional integration for all functions $V(r)$ continuous and bounded from above, using the Wiener measure of the Brownian motion process. For the second one (137) there is no Wiener measure, and it is well known that for the Schrodinger equation this solution through functional integration given first by Feynman [1–3] is only (mathematically speaking) a formal extension of the heat equation case. Fortunately it can be shown that all equations such as (137) where has a positive real part, can be rigorously solved with a complex Wiener measure. This is the case here, because $\text{Re}(i/k_0) > 0$. The solution of (134) is

$$\tilde{\Psi}(r, \theta) = \int_{\Omega} \exp \left[ik_0 \int_0^\theta \mu(\rho(\tau)) d\tau \right] dW \left(\theta, r, \frac{i}{k_0} \right) \quad (138)$$

where Ω is the space of continuous function $\rho(\tau)$ such that $\rho(0) = 0$ and $\rho(\theta) = r$, and $dW(\theta, r, i/k_0)$ is the complex Wiener measure corresponding to the complex heat equation

$$\frac{\partial\tilde{\Psi}(r, \theta)}{\partial\theta} = \frac{i}{k_0}\Delta\tilde{\Psi}(r, \theta) \quad (139)$$

Expression (138) can also be written more explicitly as the limit of ordinary multiple integrals

$$\begin{aligned} \tilde{\Psi}(r, \theta) = \lim_{n \rightarrow \infty} & \left(\frac{4\pi i \Delta \tau}{k_0} \right)^{-\frac{3}{2}n} \int \exp \left\{ \frac{ik_0}{4\Delta \tau} \left[r_1^2 + (r_2 - r_1)^2 + (r - r_{n-1})^2 \right] \right\} \\ & \times \exp \{ ik_0 \Delta \tau [\mu(r_1) + \mu(r_2) + \dots + \mu(r_{n-1})] \} d^3 r_1 d^3 r_2 \dots d^3 r_{n-1} \end{aligned} \quad (140)$$

where $\Delta \tau = \theta/n$, $(4\pi i \Delta \tau/k_0)^{-\frac{3}{2}n}$ is the $3n$ -th power of the square root of $4\pi i \Delta \tau/k_0$ which has a positive real part. A formal proof of (140) can be based upon the following formula

$$\exp[(A + B) \cdot \theta] = \lim_{n \rightarrow \infty} [\exp(B \cdot \theta/n) \cdot \exp(A \cdot \theta/n)]^n \quad (141)$$

where A and B need not commute. This is applies to (135) written as

$$\frac{\partial \tilde{\Psi}}{\partial \theta} = A \cdot \tilde{\Psi} + B \cdot \tilde{\Psi} \quad (142)$$

with $A = i\Delta/k_0$ and $B = ik_0\mu(r)$; $\exp(A \cdot \theta)$ is the r -convolution operator with $(4\pi i\theta/k_0)^{-\frac{3}{2}} \exp\{ik_0r^2/4\theta\}$ which is the elementary solution of the following equation

$$\frac{\partial \tilde{\Psi}(r, \theta)}{\partial \theta} = \frac{i}{k_0} \Delta \tilde{\Psi}(r, \theta), \quad \tilde{\Psi}(r, 0) = \delta(r) \quad (143)$$

and

$$\exp(B \cdot \theta) = \exp\{ik_0\mu(r)\theta\} \quad (144)$$

acting as a multiplication. Reordering the exponentials in (141) we get (140).

We calculate now the mean value of $\tilde{\Psi}(r, \theta)$ using (138) and interchanging the functional integration and the mean value:

$$\langle \tilde{\Psi}(r, \theta) \rangle = \int_{\Omega} \left\langle \exp \left[ik_0 \int_0^{\theta} \mu(\rho(\tau)) d\tau \right] \right\rangle dW \left(\theta, r, \frac{i}{k_0} \right) \quad (145)$$

We shall now make use of the fact that $\mu(r)$ is a centered g. r. f., but the Gaussian assumption can be dropped, because we actually only need to know the characteristically functional $F(\varphi(r)) = \langle \exp i \int \varphi(r) \mu(r) d^3 r \rangle$ of $\mu(r)$.

The following calculations are almost the same as in Section 4 for the 1D-model. For a fixed curve $\rho(\tau)$, $\varphi \int_0^{\theta} \mu(\rho(\tau)) d\tau$ is a linear

functional of $\mu(r)$, it is thus a centered Gaussian random value (r. v.) and $\langle \exp[ik_0\varphi] \rangle$ is its characteristic function

$$\langle \exp[ik_0\varphi] \rangle = \exp \left\{ -\frac{1}{2} k_0^2 \langle \varphi^2 \rangle \right\} \quad (146)$$

where

$$\langle \varphi^2 \rangle = \int_0^\theta \int_0^\tau \langle \mu(\rho(\tau)) \mu(\rho(\tau')) \rangle d\tau d\tau' = \int_0^\theta \int_0^\tau \Gamma(\rho(\tau), \rho(\tau')) d\tau d\tau' \quad (147)$$

Turning back to the initial equation (145), we get

$$\begin{aligned} \langle \tilde{\Psi}(r) \rangle &= -\frac{i}{k_0} \int_0^\infty d\theta \exp(ik_0\theta) \int_\Omega \left\langle \exp \left[-\frac{1}{2} k_0^2 \int_0^\theta \int_0^\tau \Gamma(\rho(\tau), \rho(\tau')) d\tau d\tau' \right] \right\rangle \\ &\quad \cdot dW \left(\theta, r, \frac{i}{k_0} \right) \end{aligned} \quad (148)$$

which solves the problem. This functional integral can also be approximated for numerical purposes, for example, by multiple integrals:

$$\begin{aligned} \langle \tilde{\Psi}(r) \rangle_n &= -\frac{i}{k_0} \int_0^\infty d\theta \exp(ik_0\theta) \left(\frac{4\pi i \Delta\tau}{k_0} \right)^{-\frac{3}{2}n} \\ &\quad \times \int \exp \left\{ \frac{ik_0}{4\Delta\tau} \left[r_1^2 + (r_2 - r_1)^2 + (r - r_{n-1})^2 \right] \right\} \\ &\quad \times \exp \left\{ -\frac{1}{2} k_0^2 (\Delta\tau)^2 \sum_{i,j=1}^{n-1} \Gamma_{i,j} \right\} d^3 r_1 d^3 r_2 \dots d^3 r_{n-1} \end{aligned} \quad (149)$$

where $\Gamma_{i,j} = \Gamma(r_i, r_j)$ and $\Delta\tau = \theta/n$.

The extension to higher order moments is straightforward, using characteristic functions of multivariate Gaussian distributions. The rigorous proof of (148) should not be too difficult due to the positive imaginary part of k_0 which makes all integrals convergent. There is perhaps a restriction on the covariance function (not too negative values).

Approximate evaluations of the functional integral (148). Let us just give indications, further details are published elsewhere.

a) *Short wavelength approximation.* If the range of the covariance function is much longer than the wavelength, we use a functional saddle

point method, approximating the functional

$$\exp \left[-\frac{1}{2} k_0^2 \int_0^\theta \int_0^\tau \Gamma(\rho(\tau), \rho(\tau')) d\tau d\tau' \right]$$

by a quadratic functional of $\rho(\tau) - \rho_0(\tau)$, where $\rho_0(\tau)$ is the function which makes the exponent stationary. It is then possible to calculate exactly this approximate functional integral.

b) *Long wavelength expansion.* The multiple integral (149) reminds us of the formula for the partition function of a gas in thermodynamic equilibrium. We can write

$$\exp \left\{ -\frac{1}{2} k_0^2 (\Delta\tau)^2 \sum_{i,j=1}^{n-1} \Gamma_{i,j} \right\} = \prod_{i,j=1}^{n-1} (1 + F_{ij}) = 1 + \sum F_{ij} + \sum \sum F_{ij} F_{ikl} + \dots \tag{150}$$

with

$$F_{ij} = \exp \left\{ -\frac{1}{2} k_0^2 (\Delta\tau)^2 \Gamma_{i,j} \right\} - 1 \tag{151}$$

The resulting integrals are then represented by Mayer diagrams [1–3]. This method can also be related to the perturbation method of Section 3.

8. ELECTROMAGNETIC WAVE EQUATION

In this section we study the full electromagnetic wave equation with random refractive index

$$\Delta \mathbf{E}(\mathbf{r}) - \nabla(\nabla \cdot \mathbf{E}(\mathbf{r})) + k_0^2 [1 + \varepsilon\mu(\mathbf{r})] \mathbf{E}(\mathbf{r}) = \mathbf{j}(\mathbf{r}) \tag{152}$$

$\mathbf{j}(\mathbf{r})$ is related to the actual current density $\mathbf{j}^*(\mathbf{r})$ by $\mathbf{j}(\mathbf{r}) = -i\omega\mu_0\mathbf{j}^*(\mathbf{r})$; ω is the angular frequency $\omega = 2\pi f$ and $\mu_0 = 4\pi \cdot 10^{-7}$. This equation is not equivalent to the reduced scalar wave equation because of the term $\nabla(\nabla \cdot \mathbf{E}(\mathbf{r}))$ which is important when the refractive index changes much over a wavelength. We shall therefore only consider the case of long wavelengths such that $|k_0|\ell \ll 1$, and use the Bourret approximation. This problem has already been treated by Tatarskii [18–20] but the results presented here do not agree. Taking the F. T. of (152) we get

$$\left[(k_0^2 - k^2) \delta_{ij} + k_i k_j \right] \mathbf{E}_j(\mathbf{k}) + \varepsilon k_0^2 \int \mu(\mathbf{k} - \mathbf{k}') \mathbf{E}_i(\mathbf{k}') d\mathbf{k} = \mathbf{j}_i(\mathbf{k}) \tag{153}$$

The unperturbed propagator $G_{ij}^{(0)}(k)$ satisfies the following equation

$$\left[(k_0^2 - k^2) \delta_{ij} + k_i k_j \right] G_{ji}^{(0)}(k) = \delta_{il} \quad (154)$$

This equation is easily solved

$$G_{ji}^{(0)}(k) = \frac{1}{[k_0^2 - k^2]} \left(\delta_{ij} - \frac{k_i k_j}{k_0^2} \right) \quad (155)$$

The Bourret equation for the mean perturbed propagator $\langle G_{jl}(k) \rangle$ is

$$\langle G \rangle = \text{---} + \text{---} \overset{\text{---}}{\text{---}} \langle G \rangle \quad (156)$$

or

$$\langle G(k) \rangle = G^{(0)}(k) + G^{(0)}(k) \varepsilon^2 k_0^4 \left[\int \Gamma(k - k') G^{(0)}(k') d^3 k' \right] \langle G(k) \rangle \quad (157)$$

where $\Gamma(k)$ is the F. T. of the covariance function. After a few transformations, equation (157) becomes

$$\left[(k_0^2 - k^2) \delta_{ij} + k_i k_j - \varepsilon^2 k_0^4 \int \frac{\Gamma(k - k')}{k_0^2 - k'^2} \left(\delta_{ij} - \frac{k'_i k'_j}{k_0^2} \right) d^3 k' \right] \langle G_{jl}(k) \rangle = \delta_{il} \quad (158)$$

Let us now denote tensor $T_{ij}(k)$

$$T_{ij}(k) = \int \frac{\Gamma(k - k')}{k_0^2 - k'^2} \left(\delta_{ij} - \frac{k'_i k'_j}{k_0^2} \right) d^3 k'$$

and assume that the covariance function is isotropic. Then the tensor $T_{ij}(k)$ is the convolution product of an isotropic tensor and an isotropic function; it is thus an isotropic tensor and can be written as

$$T_{ij}(k) = \chi(k) \delta_{ij} + \mu(k) \frac{k_i k_j}{k_0^2} \quad (159)$$

The Bourret equation for the mean propagator becomes now

$$\left[(k_0^2 - k^2 - \varepsilon^2 k_0^4 \chi(k)) \delta_{ij} + \left(1 - \varepsilon^2 k_0^2 \mu(k) \right) k_i k_j \right] \langle G_{jl}(k) \rangle = \delta_{il} \quad (160)$$

Let us now find the free oscillations which satisfy

$$\left[(k_0^2 - k^2 - \varepsilon^2 k_0^4 \chi(k)) \delta_{ij} + \left(1 - \varepsilon^2 k_0^2 \mu(k) \right) k_i k_j \right] \langle E_j(k) \rangle = 0 \quad (161)$$

There are two kinds of oscillations:

a) Transverse oscillations. Here $\langle \varepsilon \rangle$ and \mathbf{k} are perpendicular. The dispersion equation is

$$k_0^2 - k^2 - \varepsilon^2 k_0^4 \chi(k) = 0 \quad (162)$$

b) Longitudinal oscillations. Here $\langle \varepsilon \rangle$ and \mathbf{k} are parallel. The dispersion equation is

$$1 - \varepsilon^2 \left(k_0^2 \chi(k) + k^2 \mu(k) \right) = 0 \quad (163)$$

Let us also find the renormalized wave number K_\perp for transverse waves. We take for this purpose the correlation function $\exp(-R/\ell)$. After straightforward calculations we find that for $k\ell \ll 1$ ($\ell \ll \lambda$)

$$\chi(K_\perp) = -\frac{2}{3}\ell^2(1 + 2iK_\perp\ell) + \frac{1}{3k_0^2} + O(\ell^4 K_\perp^2) \quad (164)$$

The dispersion equation for transverse oscillations is solved for the renormalized wave number

$$K_\perp = \left[k_0^2 \left(1 - \varepsilon^2 k_0^2 \chi(k) \right) \right]^{1/2} \approx k_0 \left[1 - \frac{1}{6}\varepsilon^2 + \frac{1}{3}\varepsilon^2 k_0^2 \ell^2 (1 + 2ik_0\ell) \right] \quad (165)$$

We compare this result to the corresponding formula for the scalar wave equation which can be found in [18–20] or deduced from Keller's result [16] with the covariance function $\exp(-R/\ell)$

$$K_\perp = k_0 \left[1 + \frac{1}{2}\varepsilon^2 k_0^2 \ell^2 (1 + 2ik_0\ell) \right] \quad (166)$$

First of all, the imaginary part of for the electromagnetic wave equation has been reduced by factor $\sim (1/3)$, the damping length of the mean wave being thus increased by 50%. Secondly, due to the additional negative term $\sim \frac{1}{6}k_0\varepsilon^2$, the real part of K_\perp is less than the real part of k_0 if $2k_0^2\ell^2 < 1$. As we assumed that $k\ell \ll 1$ ($\ell \ll \lambda$) this is satisfied.

We conclude that the effective phase velocity of transverse waves is increased at long wave lengths, and not decreased as for the scalar wave equation. This needs some explanation. There are two wave modes actually in this medium: the transverse mode, whose phase velocity is approximately ω/k_0 , and the longitudinal wave mode, whose phase velocity is much longer (infinite in the nonrandom case). Due to the term $k_i k_j E_j$ of (153), the wave modes are coupled and some of the mean transverse wave has travelled part of its way as a longitudinal wave. The travelling time being thus decreased, the phase velocity is

increased. Without this coupling it would be impossible to explain the increase of the phase velocity. As the additional term $\frac{1}{6}k_0\varepsilon^2$ does not depend on ℓ , it is possible that it corresponds rather to a diffraction effect by the scattering blobs (whose sizes are small compared to the wavelength), than to a volume scattering effect.

9. PROPAGATION IN STATISTICALLY INHOMOGENEOUS MEDIA

In this section we assume that the mean refractive index is constant through space, but that its random part is not strictly stationary with respect to space translations. The correlation functions $\Gamma(x, x')$ such as are only functions of the difference $x - x'$ but also of $(x + x')/2$. We shall assume that this additional space dependence has a scale of variations h which is large compared to the wavelength. As there is no strictly homogeneous turbulence in nature, this is a very common situation. Strictly speaking, the F. T. $\mu(k)$ of such a slowly varying random function (r. f.) does not satisfy the wave vector conservation condition

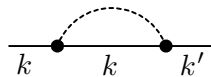
$$\langle \mu(k_1)\mu(k_2)\cdots\mu(k_p) \rangle = 0, \quad \text{if } k_1 + k_2 + \cdots + k_p \neq 0 \quad (167)$$

and does not give rise to any secular terms in the perturbation series. All arguments based upon extraction of leading secular terms seem to disappear suddenly. we shall show however, that if the condition $\varepsilon^2 K^4 \ell^3 h \gg 1$ is satisfied in addition to the usual condition $K\ell \ll 1$, nothing is changed, because we have pseudosecular terms which behave like secular terms.

We assume that the additional space variation of the correlation functions $X(r_1, r_2, \dots, r_n)$ is given by a factor $\exp\left[\frac{i\mathbf{s}(r_1+r_2+\cdots+r_n)}{r_1}\right]$, where \mathbf{s} is given vector. This is not of course the most general case, but it will be sufficient for our purpose. The scale of variation of this additional factor is $h = 1/|\mathbf{s}|$. The Fourier transform (F. T.) of the function $\exp\left[\frac{i\mathbf{s}(r_1+r_2+\cdots+r_n)}{r_1}\right] X(r_1, r_2, \dots, r_n)$ is $X(k_1 + \frac{\mathbf{s}}{n}, k_2 + \frac{\mathbf{s}}{n}, \dots, r_n + \frac{\mathbf{s}}{n})$. The wave vector conservation condition becomes thus


$$k_1 + k_2 + \cdots + k_n + \mathbf{s} = 0 \quad (168)$$

If we apply this to a connected diagram in k -space, such as



we find that

$$k - k' = \mathbf{s} \quad (169)$$

Because of the condition $h \gg \lambda$, which can also be written $|\mathbf{s}| < K$, the wave vectors at the terminals of a connected diagram are almost equal. Instead of a squared unperturbed propagator, the terminals of  introduce a factor

$$\frac{c^2}{z^2 - c^2 K^2} \frac{c^2}{z^2 - c^2 K'^2} = \frac{c}{2(K^2 - K'^2)} \left[\frac{1}{K} \left(\frac{1}{z - cK} - \frac{1}{z + cK} \right) - \frac{1}{K'} \left(\frac{1}{z - cK'} - \frac{1}{z + cK'} \right) \right] \quad (170)$$

Let us find the corresponding contribution to the inverse L. T. It is proportional to

$$\frac{1}{(K^2 - K'^2)} \left[\frac{(e^{-icKt} - e^{ic'Kt})}{K} - \frac{(e^{-icK't} - e^{icK't})}{K'} \right] \quad (171)$$

then, using the fact that $K - K'$ is small compared to K , we approximate (171) by

$$\frac{1}{2K^2} \left[\frac{e^{-icKt}(1 - e^{ic(K-K')t})}{K - K'} - \frac{e^{-icKt}(1 - e^{-ic(K-K')t})}{K - K'} \right] \quad (172)$$

For $c(K - K') \ll 1$, we can make a Taylor expansion of (172) and find

$$-\frac{ic}{2K^2} (te^{-icKt} + te^{icKt}) \quad (173)$$

This expression is not really secular, because it is only valid for $c|K - K'|t \ll 1$; this condition can also be written $t \ll h/c$. If the damping time t_d corresponding to the Bourret approximation in the stationary case is small compared to h/c , such an expression behaves exactly as a secular term; we call it a pseudosecular term. As $t_d \sim \frac{1}{\varepsilon^2 c K^4 \ell^3}$, the condition $t_d \ll h/c$ can be written as

$$\varepsilon^2 K^4 \ell^3 h \gg 1 \quad (174)$$

because of $K\ell \ll 1$ and $\varepsilon^2 < 1$, h must be very large compared to the wavelength.

Two examples on how to resolve equations for scattering wave from the obstructions randomly distributed according to Poisson law at the ground surface were shown in [34] for description of multiple scattering from trees in forested environment and in [35] for description of single scattering and diffraction from houses and trees placed at the rough terrain. In these works, by use instead of the Feynman

diagrams expansion (156), the symbolic operator form of Green functions presentation (157), that is, in the symbolic operator form based on integro-differential operators, a very simple formulas have been obtained and successfully compared with experimental data based on corresponding simulations of the obtained engineering formulas for future prediction of propagation loss characteristics in the land wireless communication channels.

10. PROPAGATION IN HOMOGENEOUS ANISOTROPIC MEDIA

Waves in anisotropic media, such as a crystal or plasma with magnetic field, satisfy partial differential equations which may be much more involved than the scalar wave equations. The wave function may have several components corresponding to a perturbed density, a perturbed velocity, a perturbed magnetic field etc. Instead of a single dispersion equation we may have several corresponding to different wave modes (e.g., ordinary and extraordinary). The wave modes are defined to be the time harmonic eigenfunctions of the propagation equations, with the boundary conditions taken into account. If we change the shape of the boundaries (e.g., from spherical to cylindrical) we change also the nature of the wave modes.

10.1. Coupling Between Wave Modes

Here we shall only consider waves in free space because the eigenfunctions are easily found by means of a Fourier transformation (F. T.). Let us first consider the nonrandom case in order to introduce some definitions and notations.

A. Nonrandom (deterministic) case. If the medium has constant parameters the propagation equations have constant coefficients. We assume that they can be written as a system of first order partial differential equations; this is the most frequent case

$$\frac{\partial \Psi_j(r; t)}{\partial t} = b_{jlm} \nabla_l \Psi_m(r; t), \quad l = 1, 2, 3, \quad j, m = 1, 2, \dots, n \quad (175)$$

where n being the number of unknowns. Introducing now the F. T.

$$\Psi_j(k; t) = \int \exp(-ikr) \Psi_j(r, t) d^3r \quad (176)$$

equation (175) becomes

$$\frac{\partial \Psi_j(k; t)}{\partial t} = ib_{jlm} k_l \Psi_m(k; t) \quad (177)$$

or in matrix notations

$$-i \frac{\partial \Psi(k; t)}{\partial t} = A(k) \Psi(k; t) \quad (178)$$

with

$$A_{jm}(k) = b_{jlm} k_l \quad (179)$$

We assume that $A(k)$ is diagonalizable, i.e. that there exists a matrix $S(k)$ such that

$$S^{-1}(k) A(k) S(k) = D(k) \quad (180)$$

$D(k)$ being a diagonal matrix whose elements are the solution of the eigenvalue equation

$$\det[\omega - A(k)] = 0 \quad (181)$$

This equation has n solutions (distinct or not)

$$\omega = \omega_j(k), \quad j = 1, 2, \dots, n \quad (182)$$

We call equation (182) the dispersion equation of the j th mode in an anisotropic medium. For mathematical convenience, we shall take n modes even if some of them not physically distinct, such as x and y polarization for an electromagnetic wave $\sim E \exp(ikz)$ in an isotropic medium. We introduce the wave mode amplitude vector

$$Q(k; t) = S^{-1}(k) \Psi(k; t) \quad (183)$$

which satisfies the following diagonal equation

$$-i \frac{\partial Q(k; t)}{\partial t} = D(k) Q(k; t) \quad (184)$$

The j th component of $Q(k; t)$ is called the complex amplitude of the j th wave mode; it satisfies a separate propagation equation

$$-i \frac{\partial Q_j(k; t)}{\partial t} = \omega_j(k) Q_j(k; t) \quad (185)$$

without summation on j .

We may conclude that in a nonrandom anisotropic medium different wave modes are uncoupled. We define also the spectral energy density of the j th wave mode

$$E_j(k; t) = |Q_j(k; t)|^2 \quad (186)$$

If the medium is lossless, the frequencies $\omega_j(k)$ are real and the spectral energy densities remain constant. We shall see that in the random case there are energy transfers from one mode to another.

B. Random case. Let us assume now that the parameters of the medium are stationary random functions (r. f.) of position, i.e., that the medium is statistically homogeneous.

Separating as usually the mean values from the random part we write the random propagation equation as

$$\frac{\partial \Psi_j(r; t)}{\partial t} = b_{jlm} \nabla_l \Psi_m(r; t) + \varepsilon \delta b_{jlm}(r) \nabla_l \Psi_m(r; t) \quad (187)$$

ε is a small parameter; $\delta b_{jlm}(r)$ are stationary centered r. f. of r . Taking the F. T. of (187) and using matrix notations, we have

$$-i \frac{\partial \Psi(k; t)}{\partial t} = A(k) \Psi(k; t) + \varepsilon \int \delta A(k - k') \Psi(k') d^3 k' \quad (188)$$

$A(k)$ is defined by (179) and

$$\delta A_{jm}(k) = \delta b_{jlm} k_l \quad (189)$$

(188) is a random integral equation which could be used as starting point of a perturbation expansion. We prefer to diagonalize $A(k)$ first, in order to obtain a set of coupled equations for the wave mode amplitudes. The wave modes are defined in the same way as for the nonrandom case, i.e., as for $\varepsilon = 0$. This is perhaps somewhat artificial, but is well justified if the parameter fluctuations are not too strong. The wave mode amplitude vector being defined by (183), we obtain a set of coupled integral equations

$$-i \frac{\partial Q(k; t)}{\partial t} = D(k) Q(k; t) + \varepsilon \int C(k, k') Q(k'; t) d^3 k' \quad (190)$$

with

$$C(k, k') = S^{-1}(k) \delta A(k - k') S(k) \quad (191)$$

Or using tensor notations

$$-i \frac{\partial Q_j(k; t)}{\partial t} = \omega_l(k) \delta_{lj} Q_j(k; t) + \varepsilon \int C_{jl}(k, k') Q_l(k'; t) d^3 k' \quad (192)$$

The random integral operator $C_{jl}(k, k')$ give the coupling between wave modes for $j \neq l$, and the scattering (or self coupling) for $j = l$.

Equations (190) and (192) are fundamental equations for wave mode coupling which we shall use as starting point for the perturbation expansion and the random variable approximation (lowest order of a random Taylor expansion) in energy transfer.

10.2. Energy Transfer Between Wave Modes

a) Let us begin with the perturbation method at *long wavelengths* ($K\ell \ll 1$). The Laplace transform (L. T.) of $Q(k; t)$

$$\tilde{Q}(k; t) = \int_0^{\infty} \exp(izt) Q(k; t) dt \quad (193)$$

satisfies

$$[-iz - iD(k)]\tilde{Q}(k; z) - i\varepsilon \int C(k, k')\tilde{Q}(k'; z)d^3k = Q(k, 0) \quad (194)$$

This equation is of the type

$$(L_0 + \varepsilon L_1)Q = j \quad (195)$$

considered in Section 3. As in this section, we introduce the unperturbed propagator

$$G^{(0)} = L_0^{-1} = [-iz - iD(k)]^{-1} \quad (196)$$

which is diagonal operator, and the perturbed propagator

$$G = (L_0 + \varepsilon L_1)^{-1} \quad (197)$$

The diagram methods for the mean perturbed propagator and the mean double propagator are easily extended to this problem, the only differences being that the solid line and the dot are no more scalar operators, but tensor (or matrix) operators. The only new feature is the coupling between wave modes (see subsection above). As we want to avoid unnecessary complications, we shall only take two wave modes and assume that there are no self-coupling terms and no losses. Without these simplifications, the new features, such as *energy transfer between wave modes*, would be more or less hidden. We study then the following coupled wave equations:

$$-i\frac{\partial Q_1(k; t)}{\partial t} = i\omega_1(k)Q_1(k; t) + \varepsilon \int C_{12}(k, k')Q_2(k'; t)d^3k' \quad (198a)$$

$$-i\frac{\partial Q_2(k; t)}{\partial t} = i\omega_2(k)Q_2(k; t) + \varepsilon \int C_{21}(k, k')Q_1(k'; t)d^3k' \quad (198b)$$

The medium being lossless, the total energy must be constant

$$\int \left[|Q_1(k; t)|^2 + |Q_2(k; t)|^2 \right] d^3k = 0 \quad (199)$$

This implies that $\omega_1(k)$ and $\omega_2(k)$ are real and

$$C_{12}(k, k') = -C_{21}^*(k, k') \quad (200)$$

We rewrite now the Bourret equation (see Section 5) for the mean propagator of equation (198)

$$\langle G \rangle_B = \text{---} + \text{---} \overset{\text{---}}{\curvearrowright} \langle G \rangle_B \quad (201)$$

As there are only two modes, it is more convenient to have scalar operator instead of matrix one. The unperturbed propagator being a diagonal matrix, it is sufficient to introduce over the solid line a superscript indicating the wave mode. Using this convention we write (201) as

$$\langle G_{11} \rangle_B = \frac{1}{\text{---}} + \frac{1}{\text{---}} \overset{2}{\curvearrowright} \langle G_{11} \rangle_B \quad (202a)$$

$$\langle G_{22} \rangle_B = \frac{2}{\text{---}} + \frac{2}{\text{---}} \overset{1}{\curvearrowright} \langle G_{22} \rangle_B \quad (202b)$$

As there are no self-coupling terms, $\langle G_{12} \rangle_B$ and $\langle G_{21} \rangle_B$ vanish. The consequence of the absence of self coupling is thus the absence of coupling between the mean propagators (this is due to the fact that any Bourret diagram has an even number of vertices). But this will not prevent energy transfer between the wave modes, because the energy densities are calculated from the mean double propagator.

Let us find the behaviour of $\langle G_{11} \rangle_B$ for $t \gg t_{\text{int}}$. It is determined by the scalar terms of the Bourret series and thus by

$$\lim_{z = -\omega_1(k) + i0} \overset{2}{\curvearrowright} = i\beta_1(k) \quad (203)$$

In explicit from equation (202a) becomes

$$[-iz - i\omega_1(k) - i\beta_1(k)] \langle G_{11} \rangle_B = 1 \quad (204)$$

To derive this equation we have replaced $\overset{2}{\curvearrowright}$ by $i\beta_1(k)$. Solving this equation we obtain

$$\langle G_{11}(k; t) \rangle = \exp \{ i[\omega_1(k) + \beta_1(k)]t \} \quad (205)$$

The frequency $\omega_1(k)$ has thus been renormalized. For the second wave mode from (202b) we find a similar result with $\beta_2(k) = -\beta_1^*(k)$. Let us calculate $\beta_1(k)$. Writing

$$\langle B_{12}(k, k') B_{21}(k', k'') \rangle = \Gamma(k - k') \delta(k - k'') \quad (206)$$

we obtain

$$\beta_1(k) = \lim_{\eta \rightarrow 0, \eta > 0} \varepsilon \int \frac{\Gamma(k, k')}{\omega_1(k) - \omega_2(k) - i\eta} d^3k' \quad (207a)$$

$\Gamma(k, k')$ is a positive measure (F. T. of a covariance function), accordingly the imaginary part of $\beta_1(k)$ is positive and $\langle G_{11}(k; t) \rangle$ is damped. It is possible to evaluate this imaginary part by a dimensional analysis of (207a)

$$\left| \frac{\text{Im}\beta_1(k)}{\omega_1(k)} \right| \sim \varepsilon^2 K^2 \ell^3 \ll 1 \quad (208)$$

We turn now to the mean double propagator and calculate the mean spectral energy densities

$$\langle |Q_1(k; t)|^2 \rangle, \quad \langle |Q_2(k; t)|^2 \rangle$$

They do not satisfy any propagation equation but are deducible from $\langle \tilde{Q}_1(k; z) \tilde{Q}_1^*(k; z') \rangle$ and $\langle \tilde{Q}_2(k; z) \tilde{Q}_2^*(k; z') \rangle$, which satisfy the following equations of the Bourret approximation

$$\langle \tilde{Q}_1(k; z) \tilde{Q}_1^*(k; z') \rangle = \frac{1}{1} \tilde{Q}_1(k; 0) \tilde{Q}_1^*(k; 0) + \frac{1}{1} \langle \tilde{Q}_2(k; z) \tilde{Q}_2^*(k; z') \rangle \quad (209a)$$

$$\langle \tilde{Q}_2(k; z) \tilde{Q}_2^*(k; z') \rangle = \frac{2}{2} \tilde{Q}_2(k; 0) \tilde{Q}_2^*(k; 0) + \frac{2}{2} \langle \tilde{Q}_1(k; z) \tilde{Q}_1^*(k; z') \rangle \quad (209b)$$

$\langle \tilde{Q}_1(k; z) \tilde{Q}_1^*(k; z') \rangle$ is the F. T. of $\langle \tilde{Q}_1(k; z) \tilde{Q}_1^*(k; z') \rangle$ with respect to t and t' . It is interesting to solve (209a) with the following initial conditions

$$\begin{aligned} \tilde{Q}_1(k; 0) \tilde{Q}_1^*(k; 0) &= E_0 \delta(k - k_0) \\ \tilde{Q}_2(k; 0) \tilde{Q}_2^*(k; 0) &= 0 \end{aligned} \quad (210)$$

for which the total initial energy of the wave is concentrated in the first wave mode with a single spectral line. Eliminating $\langle \tilde{Q}_1(k; z) \tilde{Q}_1^*(k; z') \rangle$ between (209a) and (209b) we obtain

$$\langle \tilde{Q}_2(k; z) \tilde{Q}_2^*(k; z') \rangle = \frac{2}{2} \frac{1}{1} E_0 \delta(k' - k_0) + \frac{2}{2} \frac{1}{1} \langle \tilde{Q}_2(k'; z) \tilde{Q}_2^*(k'; z') \rangle \quad (211)$$

This integral equation can be solved by successive approximations. This method does not give secular terms because they have already been taken into account in the mean (single) propagators

$$\frac{1}{\underline{\quad}} = \frac{i}{z + \omega_1(k) + \beta_1(k)}, \quad \frac{2}{\underline{\quad}} = \frac{i}{z + \omega_2(k) + \beta_2(k)} \quad (212)$$

Let us calculate the lowest order approximation for $\langle \tilde{Q}_2(k; z) \tilde{Q}_2^*(k; z') \rangle$

$$\begin{aligned} \frac{2k}{2k} \frac{1k'}{1k'} E_0 \delta(k' - k_0) &= \frac{i}{[z + \nu_2(k)]} \frac{-i}{[z'^* + \nu_2^*(k)]} \varepsilon^2 E_0 \Gamma(k, k_0) \\ &\times \frac{i}{[z + \nu_1(k_0)]} \frac{-i}{[z'^* + \nu_1^*(k_0)]} \end{aligned} \quad (213)$$

We denote $\nu_1(k)$ and $\nu_2(k)$ the renormalized frequencies

$$\begin{aligned} \nu_1(k) &= \omega_1(k) + \beta_1(k) \\ \nu_2(k) &= \omega_2(k) + \beta_2(k) = \omega_2(k) - \beta_1^*(k) \end{aligned} \quad (214)$$

Taking the inverse L. T. of (213) we obtain the following expression for the spectral energy density of the second mode as a function of time

$$E_2(k; t) = \varepsilon^2 \Gamma(k - k_0) \left| \frac{\exp(i\nu_2(k)t) - \exp(i\nu_1(k_0)t)}{\nu_2(k) - \nu_1(k_0)} \right|^2 \quad (215)$$

The Born approximation would give the same result with the frequencies ω_1 and ω_2 instead of renormalized frequencies ν_1 and ν_2 . Accordingly it would be secular for any wave vector such that

$$\omega_2(k) = \omega_1(k_0) \quad (216)$$

This is now prevented by the imaginary part of $\nu(k)$. Let us assume that the real part of β_1 and β_2 have been incorporated into ω_1 and ω_2 , and recall that the imaginary part of β_1 and β_2 are small compared to ω_1 and ω_2 . It is then easily found that $E_2(k; t)$ can only be important for wave vectors which satisfy the *coupling conditions*:

$$|\omega_2(k) - \omega_1(k_0)| < |\beta_1(k_0)| \quad (217)$$

and that the maximum of $E_2(k; t)$ occurs for $t \sim 1/|\beta_1(k_0)| \equiv t_d$, which is the damping time of the mean propagator. The maximum of $E_2(k; t)$

$$E_2(k; t_d) = \frac{\varepsilon^2 \Gamma(k, k_0)}{|\beta_1^2(k_0)|} \quad (218)$$

If we want to obtain the total energy transfer from the first wave mode to the second one, we must integrate (218) over all wave vectors

satisfying the coupling condition (217). As β_1 is proportional to ε^2 , the total energy transfer is found to be independent of the strength of random fluctuations. A dimensional analysis of the total energy transfer shows that it is of order unity (i.e., independent of $\varepsilon^2, c, K, \ell$, etc.) for $t \sim t_d$. As t_d is proportional to ε^{-2} , the energy transfer may take a very long time for small random fluctuations.

b) We turn now to the other limiting approximation of *short wavelengths* satisfying $\varepsilon K \ell \gg 1$. As we have seen in Section 6, we may treat the random parameters as mere random variables. The coupled wave equations (198) become now ordinary differential equations with a random parameter

$$\frac{\partial Q_1(t)}{\partial t} = i\omega_1 Q_1(t) + i\varepsilon b Q_2(t) \quad (219a)$$

$$\frac{\partial Q_2(t)}{\partial t} = i\omega_2 Q_2(t) + i\varepsilon b Q_1(t) \quad (219b)$$

We have dropped the wave vector dependence because there is no more coupling between different wave vectors. We shall assume that b is a real centered random variable. Equations (219) describe a set of two randomly coupled oscillators.

We solve (219) with initial conditions such that the initial energy of the wave is concentrated in the first mode, that is, $Q_1(0) = 1$ and $Q_2(0) = 0$. Using the L. T. we get the following solutions

$$Q_1(t) = \frac{q_1 + \omega_1}{\sqrt{\Delta}} \exp\{-iq_1 t\} - \frac{q_2 + \omega_2}{\sqrt{\Delta}} \exp\{-iq_2 t\} \quad (220a)$$

$$Q_2(t) = \frac{-ib}{\sqrt{\Delta}} [\exp\{-iq_1 t\} - \exp\{-iq_2 t\}] \quad (220b)$$

with

$$\Delta = (\omega_1 - \omega_2)^2 + 4b^2\varepsilon^2$$

$$q_{1,2} = \frac{-(\omega_1 - \omega_2) \pm \sqrt{\Delta}}{2}$$

The mean energy of the second wave mode at time t is thus

$$E_2(t) = \langle |Q_2(t)|^2 \rangle$$

$$= \left\langle \frac{2\varepsilon b^2}{(\omega_1 - \omega_2)^2 + 4b^2\varepsilon^2} \left(1 - \cos \left(t \sqrt{(\omega_1 - \omega_2)^2 + 4b^2\varepsilon^2} \right) \right) \right\rangle \quad (221)$$

or in terms of the probability density $P(b)$ of the random variable (r. v.) b

$$E_2(t) = \int_{-\infty}^{\infty} |Q_2(t)|^2 P(b) db \quad (222)$$

Let us first assume that $\omega_1 = \omega_2$, then

$$E_2(t) = \frac{1}{2} \int_{-\infty}^{\infty} (1 - \cos 2bt) P(b) db \quad (223)$$

as $t \rightarrow \infty$ and $E_2(t) \rightarrow \frac{1}{2}$. The asymptotic energy distribution is thus equipartition. In the more general case $\omega_1 \neq \omega_2$, we take the probability density as

$$P(b) = \frac{\sigma}{\pi} \frac{1}{(b^2 + \sigma^2)} \quad (224)$$

The asymptotic energy distribution is easily calculated by means of a L. T.

$$\lim_{t \rightarrow \infty} \langle E_2(t) \rangle = \frac{1}{2} \frac{2\varepsilon\sigma}{[2\varepsilon\sigma + |\omega_2 - \omega_1|]} \leq \frac{1}{2} \quad (225)$$

The condition of effective energy transfer is thus

$$|\omega_2 - \omega_1| \leq 2\varepsilon\sigma \quad (226)$$

It is found that the time required to reach the equilibrium energy distribution is of the order of magnitude of $(\varepsilon\sigma)^{-1}$. The situation is the same as in the long wavelength approximation: *we can have an important energy transfer in a medium with very small random fluctuations, but this requires a very long time.*

We conclude that if one is interested in the long time behaviour of a random medium, energy transfer is always important between waves whose frequencies are not too different (see condition (217) at long wavelengths, and (226) at short wavelengths).

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