Progress In Electromagnetics Research, PIER 51, 219–229, 2005

QUASI-STATIC ANALYSIS OF MATERIALS WITH SMALL TUNABLE STACKED SPLIT RING RESONATORS

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Abstract—This paper presents a quasi-static analysis of a metamaterial consisting of a three-dimensional array of small tunable stacked split ring resonators (SSRRs). The resonance frequency of the proposed resonator structure can be controlled by adjusting the auxiliary lumped elements which are inserted between the split of each ring. In addition, the size of the ring resonator can be reduced to an order of 0.01λ that is one tenth that of the split ring resonator (SRR) by choosing the proper lumped elements. The analysis is based on the quasi-static Lorentz theory, and the generalized matrix representation of the macroscopic constitutive relations of the composite medium is calculated.

1. INTRODUCTION

Since the split ring resonator (SRR) medium was proposed by Pendry et al. [1] it has been extensively investigated in a number of ways [2–4]. The previous research has supported the unique feature of this medium as a metamaterial, that is, it has negative permeability values at a certain frequency range, which is due to its resonating structure. As a result of its resonance dependence, this medium is narrowband and dispersive over the frequency range of interest, and the resonance frequency is determined by the size of the ring ($\approx 0.1\lambda$). These properties limit the practical use of this medium. Although a few other types of metamaterials have been suggested, most of them are theoretical models and have not been realized yet [5, 6].

In this paper, we present a tunable stacked SRR (SSRR) structure with the purpose of improving the material characteristics of the SRR. First, the difference between the SRR and the SSRR will be described, and the advantage of the tunable SSRR will also be discussed. To simulate the above structures, we used an efficient quasi-static approach which we proposed in our recent paper [3]. This approach describes the material characteristics in terms of the physical properties of the inclusions and has been proved to be useful in efficiently investigating metamaterials consisting of a three-dimensional array of electrically small inclusions of arbitrary shapes. In the next section, a brief explanation of the formulation will be given.

2. QUASI-STATIC LORENTZ THEORY

The macroscopic constitutive relations of a composite medium consisting of a three-dimensional array of small inclusions in a host material (Fig. 1), are given by

$$\begin{bmatrix} \bar{D} \\ \bar{B} \end{bmatrix} = \begin{bmatrix} \bar{\bar{\epsilon}} & \bar{\bar{\xi}} \\ \bar{\bar{\zeta}} & \bar{\bar{\mu}} \end{bmatrix} \begin{bmatrix} \bar{E} \\ \bar{H} \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} \bar{D} \\ \bar{H} \end{bmatrix} = \begin{bmatrix} \bar{\bar{\epsilon}}_p & \bar{\bar{\alpha}}_p \\ \bar{\bar{\beta}}_p & \bar{\bar{\mu}}_p \end{bmatrix} \begin{bmatrix} \bar{E} \\ \bar{B} \end{bmatrix}. \tag{2}$$

(1) is the E-H (or Tellegen) representation, and (2) is the E-B (or Boys-Post) representation, respectively. (1) and (2) are equivalent for a linear medium and related through the following:

$$\begin{aligned} \bar{\bar{\epsilon}} &= \bar{\bar{\epsilon}}_p - \bar{\bar{\alpha}}_p \bar{\bar{\mu}}_p \bar{\bar{\beta}}_p, & \bar{\bar{\mu}} &= \bar{\bar{\mu}}_p, \\ \bar{\bar{\xi}} &= \bar{\bar{\alpha}}_p \bar{\bar{\mu}}_p, & \bar{\bar{\zeta}} &= -\bar{\bar{\mu}}_p \bar{\bar{\beta}}_p. \end{aligned} \tag{3}$$

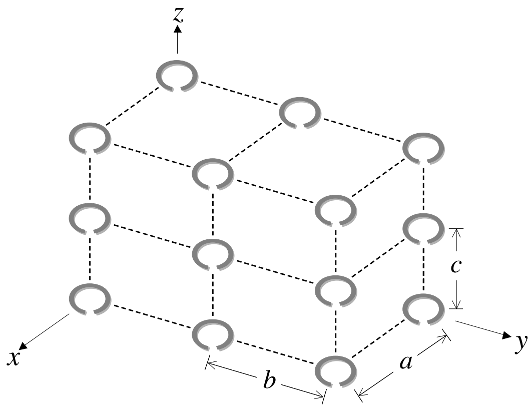


Figure 1. Three-dimensional array of inclusions whose dielectric constant is ϵ_r and conductivity is σ. Host material has a dielectric constant ϵ_b.

In our recent paper, we derived the explicit expressions of the above matrix parameters for a given configuration [3]. The derivation is based on the quasi-static Lorentz theory and, therefore, applicable to inclusions whose sizes and spacings are small compared with a wavelength [7–9]. The detailed derivation and validation are discussed in Ref. [3], and only the final expression of the generalized constitutive relations is given in this paper for completeness. The 6×6 constitutive relation matrix is expressed in terms of the polarizability matrix $\bar{\bar{\alpha}}$ and the interaction matrix $\bar{\bar{\mathrm{C}}}$ which can be numerically calculated.

$$\begin{aligned}\begin{bmatrix} \bar{\bar{\epsilon}}_p & \bar{\bar{\alpha}}_p \\ \bar{\bar{\beta}}_p & \bar{\bar{\mu}}_p \end{bmatrix} &= \begin{bmatrix} \epsilon_0\epsilon_b\bar{\mathrm{U}} & \bar{0} \\ \bar{0} & \dfrac{1}{\mu_0}\bar{\mathrm{U}} \end{bmatrix} + \\ &\quad \mathrm{N}\begin{bmatrix} \bar{\mathrm{U}} & \bar{0} \\ \bar{0} & -\bar{\mathrm{U}} \end{bmatrix}\left[\bar{\bar{\alpha}}\right]\left[\bar{\bar{\mathrm{U}}} - \mathrm{N}\left[\bar{\bar{\mathrm{C}}}\right]\left[\bar{\bar{\alpha}}\right]\right]^{-1}\end{aligned} \tag{4}$$

where ϵ_b is the relative dielectric constant of the host material and N is the number of particles per unit volume.

The polarizability matrix $\bar{\bar{\alpha}}$ and the interaction matrix $\bar{\bar{\mathrm{C}}}$ are as follows.

$$[\bar{\bar{\alpha}}] = \begin{bmatrix} \bar{\bar{\alpha}}_{ee} & \bar{\bar{\alpha}}_{em} \\ \bar{\bar{\alpha}}_{me} & \bar{\bar{\alpha}}_{mm} \end{bmatrix},$$

$$\begin{aligned}\bar{\bar{\alpha}}_{ee} &= \frac{1}{j\omega}\int dv\ \left[\bar{J}_e\right]\\ \bar{\bar{\alpha}}_{me} &= \frac{1}{2}\int dv\ \bar{r}\times\left[\bar{J}_e\right]\\ \bar{\bar{\alpha}}_{em} &= \frac{1}{j\omega}\int dv\ \left[\bar{J}_m\right]\\ \bar{\bar{\alpha}}_{mm} &= \frac{1}{2}\int dv\ \bar{r}\times\left[\bar{J}_m\right].\end{aligned}\tag{5}$$

where $\bar{J}_e$ and $\bar{J}_m$ are the currents produced by the electric and magnetic effective fields [7].

$$\left[\bar{\bar{\mathrm{C}}}\right] = \begin{bmatrix}\frac{1}{\epsilon_0\epsilon_b}\bar{\mathrm{C}} & \bar{0}\\ \bar{0} & \mu_0\bar{\mathrm{C}}\end{bmatrix},\quad \bar{\mathrm{C}} = \begin{bmatrix}\mathrm{C}_x & 0 & 0\\ 0 & \mathrm{C}_y & 0\\ 0 & 0 & \mathrm{C}_z\end{bmatrix},$$

$$\begin{aligned}\mathrm{C}_x &= f\left(\frac{b}{a},\frac{c}{a}\right) = \left(\frac{b}{a}\right)\left(\frac{c}{a}\right)\left[\frac{\zeta(3)}{\pi} - S\left(\frac{b}{a},\frac{c}{a}\right)\right]\\ \mathrm{C}_y &= f\left(\frac{c}{b},\frac{a}{b}\right)\\ \mathrm{C}_z &= f\left(\frac{a}{c},\frac{b}{c}\right)\end{aligned}\tag{6}$$

where

$$\zeta(z) = \sum_{k=1}^{\infty} k^{-z},\ \mathrm{Re}\{z\} > 1 \text{ (Riemann Zeta function)},$$

$$\begin{aligned}S\left(\frac{b}{a},\frac{c}{a}\right) &= \frac{1}{\pi}\sum_{n=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\sum_{m=1}^{\infty}(2m\pi)^2\\ &\qquad \mathrm{K}_0\left(2m\pi\left[\left(\frac{nb}{a}\right)^2 + \left(\frac{sc}{a}\right)^2\right]^{1/2}\right).\end{aligned}$$

K_0 is the modified Bessel function, and the term with $n = s = 0$ is excluded.

3. TUNABLE STACKED SPLIT RING RESONATOR WITH LUMPED ELEMENTS

In our recent work, we presented the quasi-static analysis results of a SRR medium and a SSRR medium in which one split ring is placed

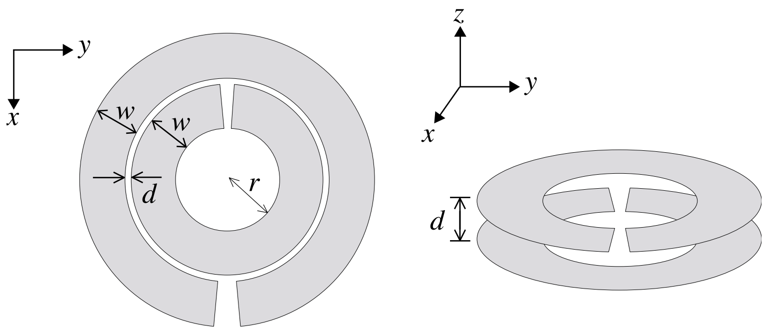


Figure 2. Split Ring Resonator (SRR) and Stacked Split Ring Resonator (SSRR).

on the top of another (Fig. 2) [4]. The calculation results in Ref. [4] show that both mediums have similar characteristics except that one component of the dielectric constants of the SRR medium shows an analogous resonance curve to that of the permeability, whereas, all components of the dielectric constants of the SSRR medium are almost constant over the frequency range of interest.

We further modified the SSRR by inserting lumped elements such as a capacitor and/or an inductor between the split of each ring to alternate the electric and magnetic dipole moments (Fig. 3). Although an ideal capacitor does not contain other elements, an equivalent circuit of most capacitors at microwave frequency is a series L and C due to a parasitic inductance as shown in Fig. 3. A parallel L and C circuit which is an equivalent circuit of most inductors may have limited applications.

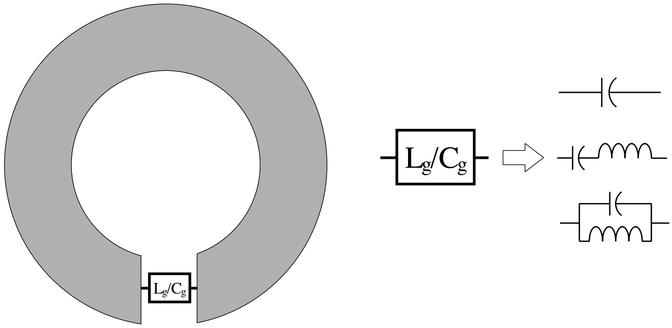


Figure 3. Split ring with lumped elements.

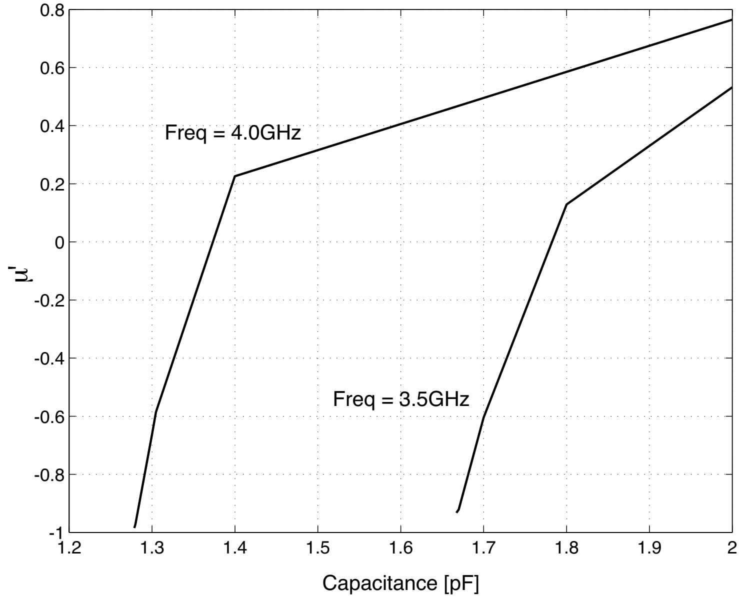
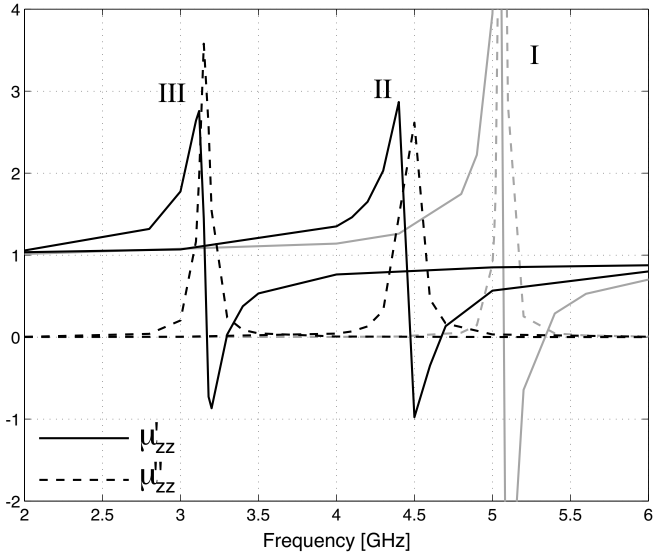


Figure 4. Permeabilities of the tunable SSRR medium with different capacitance ($r = 0.15\,\mathrm{mm}$, $w = 0.08\,\mathrm{mm}$, $d = 0.02\,\mathrm{mm}$, $a = b = 0.8\,\mathrm{mm}$, $c = 0.37\,\mathrm{mm}$).

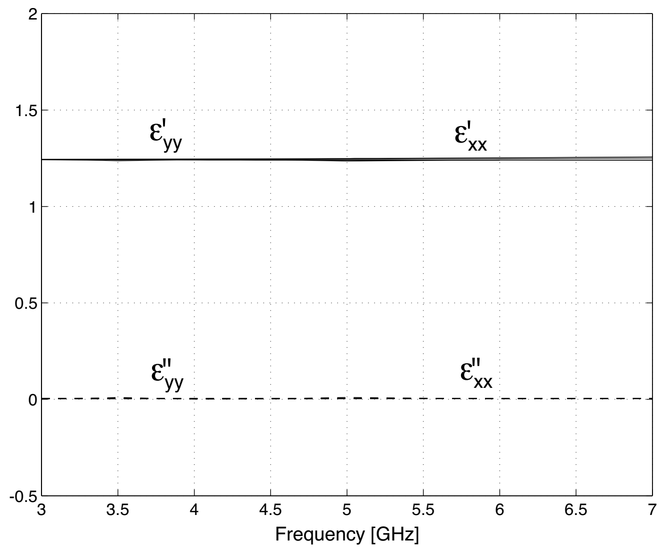
Through a number of numerical experiments, it is found that the resonance frequency of this modified tunable SSRR depends on the capacitance value or the negative reactance value. In Fig. 4, the permeabilities of the tunable SSRR medium with different capacitance at two frequencies are plotted. It is evident that the permeability is a function of a capacitor value. Since it is more desirable for an inclusion to have a smaller size, we tried to reduce the size of the SSRR while keeping the resonance frequency unchanged.

In Fig. 5, the analysis results of the original SSRR and the tunable SSRR with two different lumped elements are shown. In this example, only $\mu_{zz} = \mu'_{zz} - j\mu''_{zz}$, $\epsilon_{xx} = \epsilon'_{xx} - j\epsilon''_{xx}$, and $\epsilon_{yy} = \epsilon'_{yy} - j\epsilon''_{yy}$ are considered. Note that the size of the tunable SSRR is one tenth that of the SSRR without lumped elements. It is obvious from the results that the same order of the resonance frequency is obtained with a much smaller size of a split ring, and the resonance frequency depends on the capacitance value, i.e. the former is decreased as the latter is increased.

It is useful to note that the permittivities, ϵ_{xx} and ϵ_{yy} of the three different mediums are almost same, which means the permittivity mainly depends on volume fraction.



(a)



(b)

Figure 5. (a) Permeabilities and (b) Permittivities of the tunable SSRR mediums. I: without lumped elements ($r = 1.5\,\mathrm{mm}$, $w = 0.8\,\mathrm{mm}$, $d = 0.2\,\mathrm{mm}$, $a = b = 8\,\mathrm{mm}$, $c = 3.7\,\mathrm{mm}$) II: $\mathrm{C_g} = 1\,\mathrm{pF}$, III: $\mathrm{C_g} = 2\,\mathrm{pF}$ ($r = 0.15\,\mathrm{mm}$, $w = 0.08\,\mathrm{mm}$, $d = 0.02\,\mathrm{mm}$, $a = b = 0.8\,\mathrm{mm}$, $c = 0.37\,\mathrm{mm}$).

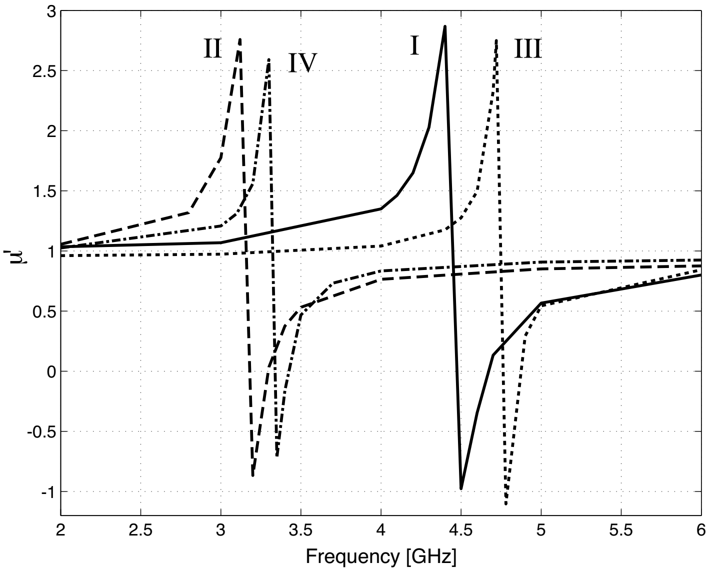


Figure 6. Permeabilities of the tunable SSRR mediums with different lumped elements. I: $C_g = 1\,\mathrm{pF}$, II: $C_g = 2\,\mathrm{pF}$, III: $L_g = 1\,\mathrm{nH}$, $C_g = 2\,\mathrm{pF}$ (parallel), IV: $L_g = 1\mathrm{nH}$, $C_g = 1\,\mathrm{pF}$ (series) ($r = 0.15\,\mathrm{mm}$, $w = 0.08\,\mathrm{mm}$, $d = 0.02\,\mathrm{mm}$, $a = b = 0.8\,\mathrm{mm}$, $c = 0.37\,\mathrm{mm}$).

Fig. 6 shows that SSRR mediums with a series or parallel connection of an inductor and a capacitor can have similar values of the resonance frequencies compared with that of the SSRR with a capacitor only. Note that the reactance values of case I and II are close to those of case III and IV, respectively. This fact means the resonancy frequency is affected by the reactance value, and it can be observed that when a series or parallel connection of an inductor and a capacitor is used, the reactance value changes more rapidly near the resonant frequency, it has the effect of reducing the bandwidth of a tunable SSRR medium.

4. CONCLUSIONS

A tunable SSRR is proposed, and it is shown that the resonant frequency of a metamaterial composed of a three-dimensional array of these modified tunable SSRRs can be controlled by adjusting the auxiliary lumped elements, particularly the capacitance.

A quasi-static analysis of several numerical examples is performed and it is shown that a tunable SSRR of much smaller size (order of 0.01λ) can have a comparable resonance frequency to that of an SSRR without lumped elements whose size is about one tenth of the wavelength. Further improvements such as a wider bandwidth may be able to be achieved by utilizing a variable capacitor. Although a voltage controlled capacitor (varactor) is widely used in RF/microwave circuits, it may not be applicable for the proposed structures because of its requirement of DC control wires. The excitation methods which do not require DC control wires such as a pressure-controlled capacitor, an optically controlled capacitor or a device based on shape memory alloy should be investigated.

One of the problems with the traditional SRR or SSRR is its physical size which is about $0.1\,\lambda$. The element size is simply too large for many applications including as a substrate material of VHF-UHF antennas. With the proposed technique, we will be able to reduce the size of inclusions and create an almost homogeneous material. One application of the proposed structure may be *microwave plasmon*. Unlike the SRR, we can make a negative permeability material while keeping permittivity constant using the SSRR. This will create a plasmon similar to a surface wave excitation on a thin gold layer at an optical wavelength. In this paper, we have used Ansoft HFSS (FEM code) to obtain $\bar{J}_e$ and $\bar{J}_m$ but the present method is complicated and time-consuming. Recently, a highly efficient numerical method was developed to obtain $\bar{J}_e$ and $\bar{J}_m$ [10]. Then Equations (4)–(6) were applied to obtain permeabilities and permittivities. Preliminary results were close to the one presented in this paper.

ACKNOWLEDGMENT

This work was supported by the National Science Foundation (ECS-9908849).

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