Progress In Electromagnetics Research, PIER 48, 201–231, 2004

BATEMAN CONFORMAL TRANSFORMATIONS WITHIN THE FRAMEWORK OF THE BIDIRECTIONAL SPECTRAL REPRESENTATION†

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Abstract—Four-dimensional conformal transformations due originally to Bateman have been used in the past by Hillion as alternative approaches to focus wave mode solutions to the 3D scalar wave equation. More recently, more extended families of focus wave mode solutions to the 3D scalar wave equation have been derived by Borisov and Utkin, as well as Kiselev, based on Bateman transformations together with a dimension-reduction approach, whereby the wave function is separated incompletely into a product of two functions. One particular goal in this exposition is to comment on and extend the work of Borisov and Utkin and simplify and extend the method used by Kiselev. More generally, however, the aim is to show that an already existing method, known as the bidirectional spectral representation, when examined in conjunction with Bateman conformal transformations, encompasses the Borisov-Utkin-Kiselev theories as special cases and allows a systematic derivation of extended families of FWM-type localized waves beyond the ranges of their applicability.

† Dedicated to the memory of Frederick D. Tappert

1. INTRODUCTION

In 1909, Bateman ([1] also, [2, 3]) introduced the following four-dimensional conformal transformations:

$$\begin{aligned} X_{\pm} &= \frac{x_0}{z_0 \pm \tau_0}, & Y_{\pm} &= \frac{y_0}{z_0 \pm \tau_0}, \\ Z_{\pm} &= \frac{x_0^2 + y_0^2 + z_0^2 - \tau_0^2 - 1}{2(z_0 \pm \tau_0)}, & T_{\pm} &= \frac{x_0^2 + y_0^2 + z_0^2 - \tau_0^2 + 1}{2(z_0 \pm \tau_0)}. \end{aligned} \tag{1}$$

Here, (x_0, y_0, z_0, τ_0) are dimensionless coordinates linked to the usual space and time variables by the relationships $(x_0, y_0, z_0, \tau_0) = q(x, y, z, ct)$, where q is a parameter with units of m^{-1} and c has the units of speed; in the sequel, the latter will denote the speed of light in *vacuo*. An important feature of these transformations is that if $u(\vec{r}, t)$ is a solution of the homogeneous 3D scalar wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) u(\vec{r}, t) = 0; \quad c = 1/\sqrt{\varepsilon_0 \mu_0}, \quad \vec{r} = \hat{x}x + \hat{y}y + \hat{z}z, \tag{2}$$

the functions

$$\begin{aligned} &U_{\pm}(x, y, z, t) = B_{\pm}\left\{u(x_0, y_0, z_0, \tau_0)\right\} \\ &\equiv \frac{1}{z_0 \pm \tau_0} u\left[X_{\pm}, Y_{\pm}, Z_{\pm}, T_{\pm}\right] |(x_0, y_0, z_0, \tau_0) \to q(x, y, z, ct), \end{aligned} \tag{3}$$

are also solutions. An additional property of the transformations is that if $\theta(\vec{r},t)$ is a solution to the nonlinear characteristic equation

$$\left[\frac{\partial}{\partial x}\theta(\vec{r},t)\right]^2+\left[\frac{\partial}{\partial y}\theta(\vec{r},t)\right]^2+\left[\frac{\partial}{\partial z}\theta(\vec{r},t)\right]^2-\frac{1}{c^2}\left[\frac{\partial}{\partial t}\theta(\vec{r},t)\right]^2=0 \quad (4)$$

associated with the 3D scalar wave equation (2), the functions $\theta(X_\pm,Y_\pm,Z_\pm,T_\pm)|(x_0,y_0,z_0,\tau_0)\to q(x,y,z,ct)$ are also solutions.

For a simple application of Bateman's conformal transformations, consider the wave function $u(x_0,y_0,z_0,\tau_0)=-iq\exp[i(z_0+\tau_0)]$, then, a Bateman-transformed solution, with $q=\beta$, is given by

$$U_{\_}(x,y,z,t)=B_{\_}\left\{u(x_0,y_0,z_0,\tau_0\right\}=\frac{-i}{z-ct}e^{i\beta\left[z+ct+\frac{\rho^2}{z-ct}\right]};\rho=\sqrt{x^2+y^2}. \quad (5)$$

It was Hillion [4, 5] who first recognized that a complexification ($z\to z-ia_-$, where a_- is a real positive number) of the solution given in (5) results in the expression

$$U_{FWM}(\rho,z,t)=\frac{1}{a_{\_}+i(z-ct)}e^{i\beta\left[z+ct+i\frac{\rho^2}{a_-+i(z-ct)}\right]}. \quad (6)$$

This is the original axis-symmetric focus wave mode (FWM) solution to the 3D scalar wave equation in unbounded space. In the form shown in Eq. (6), it was first formulated by Ziolkowski [6] and Belanger [7] who were motivated by Brittingham's work in 1983 [8]. The pure FWM consists of an envelope traveling along the positive z-direction with speed c, modulated by a plane wave moving in the negative z-direction with speed c. The entire wave packet sustains only local deformations; more precisely, it regenerates periodically. The FWM is physically unrealizable because it contains infinite energy. Finite energy FWM-type localized waves in an unbounded space have been derived by various means, e.g., by a superposition of pure FWMs, by Ziolkowski [9], Besieris *et al.* [10] and others. Occasionally, such localized waves are referred to in the literature as "Bateman-Hillion" solutions.

It is interesting to note that the FWM solution can be rewritten as

$$\begin{aligned}U_{FWM}(\rho,z,t) &= \frac{1}{a_-+i(z-ct)}e^{i\beta\left[z+ct+i\frac{\rho^2}{a_-+i(z-ct)}\right]}\\ &= v(z-ct)\exp[i\beta\theta(\vec{r},t)], \qquad (7)\\ \theta(\vec{r},t) &= z+ct+i\frac{\rho^2}{a_-+i(z-ct)};\end{aligned}$$

specifically, as a product of two functions: a particular solution to the 1D scalar wave equation and a specific functional of a particular solution $\theta(\vec{r}, t)$ to the nonlinear characteristic equation (4). In this respect, the FWM solution conforms to an *ansatz* expounded by Courant and Hilbert [11], who pointed out that "relatively undistorted" progressive solutions to the homogeneous 3D scalar wave equation in free space assume the form $u(\vec{r}, t) = g(\vec{r}, t)f[\theta(\vec{r}, t)]$, where $\theta(\vec{r}, t)$ is an arbitrary solution of Eq. (4), $f(\cdot)$ is essentially an arbitrary function, and $g(\vec{r}, t)$ is an "attenuation" function; the latter depends on the choice of $\theta(\vec{r}, t)$, but not in a unique manner.

Recently, Borisov and Utkin [12; see also, 28] have constructed extensions to Brittingham's FWM by applying Bateman transformations to a wave function of the form $u(x_0, y_0, z_0, \tau_0) = v(z_0 + \tau_0)w(x_0 \pm iy_0)$, a product of a solution to the dimensionless 1D scalar wave equation and a solution to the dimensionless 2D Laplace equation. Following a different approach, Kiselev [13] has rederived Borisov's and Utkin's extensions of the "Bateman-Hillion" relatively undistorted progressive waves; furthermore, he has generalized a family of localized waves known as Bessel-Gauss FWMs.

One particular goal in this paper is to comment on and extend the work of Borisov and Utkin [12] and simplify and extend the method used by Kiselev [13]. More generally, however, we wish to go beyond the work of both Borisov and Utkin [12] and Kiselev [13] by pointing out a systematic approach to deriving exact extended FWM-type localized wave and paraxial pulsed beam solutions to the 3D scalar wave equation. In the process we shall examine the Bateman conformal transformations within the framework of the bidirectional spectral representation and show that the latter encompasses the Borisov-Utkin and Kiselev theories as special cases.

2. COMMENTS ON AND EXTENSIONS OF THE BORISOV-UTKIN AND KISELEV WORK

Underlying the work of Borisov and Utkin [12], as well as Kiselev's [13], is the method of incomplete separation of variables originally suggested by Smirnov [14]. In the work of Borisov and Utkin, the wave function is written in the incompletely separated form $u(x_0, y_0, z_0, \tau_0) = v(z_0 + \tau_0)w(x_0 \pm iy_0)$, which consists of a product of a function obeying the 1D scalar wave equation and a function governed by the 2D Laplace equation. Application of Bateman transformations [cf. Eq. (3)] yields the following new solutions to the 3D scalar wave equation

$$U_{-}(x,y,z,t) = B_{-}\{u(x_0,y_0,z_0,\tau_0)\} = \frac{1}{z-ct} v\left(z+ct+\frac{\rho^2}{z-ct}\right) w\left(\frac{x\pm iy}{z-ct}\right),$$
$$U_{+}(x,y,z,t) = B_{+}\{u(x_0,y_0,z_0,\tau_0)\} = \frac{1}{z+ct} v\left(z-ct+\frac{\rho^2}{z+ct}\right) w\left(\frac{x\pm iy}{z+ct}\right), \tag{8}$$

modulo constant multiplier terms. Upon complexification of z ($z \to z - ia_{-}$ for the first solution and $z \to z + ia_{+}$, for the second one), one obtains

$$U_{-}(x,y,z,t) = \frac{i}{a_{-}+i(z-ct)} v\left(z+ct+i\frac{\rho^2}{a_{-}+i(z-ct)}\right) w\left(i\frac{x\pm iy}{a_{-}+i(z-ct)}\right),$$
$$U_{+}(x,y,z,t) = \frac{-i}{a_{+}-i(z+ct)} v\left(z-ct-i\frac{\rho^2}{a_{+}-i(z+ct)}\right) w\left(-i\frac{x\pm iy}{a_{+}-i(z+ct)}\right). \tag{9}$$

For $w(\cdot) = 1$, these two solutions simplify to

$$\begin{aligned} U_{-}(x,y,z,t) &= \frac{i}{a_{-}+i(z-ct)} v\left(z+ct+i\frac{\rho^2}{a_{-}+i(z-ct)}\right), \\ U_{+}(x,y,z,t) &= \frac{-i}{a_{+}-i(z+ct)} v\left(z-ct-i\frac{\rho^2}{a_{+}-i(z+ct)}\right). \end{aligned} \tag{10}$$

$U_{-}(x,y,z,t)$ is the most general solution one can obtain by a superposition of axis-symmetric FWMs. For the choice $v(\cdot) = -i\exp[i\beta(\cdot)]$, one recaptures the infinite energy pure FWM solution given in Eq. (6). An example of a finite-energy FWM-type solution to the 3D scalar wave equation in unbounded space is Ziolkowski's [9] modified power spectrum (MPS) pulse arising from the choice $v(\cdot) = -i\exp[i(b/p)(\cdot)][a_2 + (1/p)(\cdot)]^{-q}$, where a_2, b, p and q are free positive real parameters; specifically,

$$U_{-}(\rho,z,t) = \frac{1}{a_{-}+i(z-ct)} \frac{e^{i\frac{b}{p}\left[i\frac{\rho^2}{a_{-}+i(z-ct)}+(z+ct)\right]}}{\left[a_2 + -i\frac{1}{p}\left(i\frac{\rho^2}{a_{-}+i(z-ct)}+(z+ct)\right)\right]^q}. \tag{11}$$

Analogous results can be obtained for $U_+(\rho, z, t)$. The importance of the latter is due to its connection to paraxial pulsed beams. Specifically, if $z+ct$ is formally replaced by $2z$ in $U_+(\rho, z, t)$, one obtains an exact solution to the paraxial pulsed beam equation [15]

$$\left(\nabla_\rho^2 + 2\frac{\partial^2}{\partial\varsigma\partial z}\right) U_{PB}(\rho, z, t) = 0; \quad \varsigma = z - ct. \tag{12}$$

The simplest monochromatic paraxial beam, viz.,

$$U_{PB}(\rho, z, t) = \frac{1}{a_+ - i2z} e^{-ia\left(z - ct - i\frac{\rho^2}{a_+ - i2z}\right)}, \tag{13}$$

arises from Eq. (10b) from the choice $v(\cdot) = i\exp[-i\alpha(\cdot)]$.

Based on the discussion above, the solutions [cf. Eq. (9)] derived by Borisov and Utkin [12] and more recently by Kiselev [13] are, indeed, extensions of the Bateman-Hillion solutions formed by superpositions of pure FWMs. The difference lies in the multiplicative terms in Eq. (9) associated with solutions to the 2D Laplace equation.

An important feature, not addressed by Borisov, Utkin and Kiselev, is that the extended solution $U_+(x, y, z, t)$ in Eq. (9b) retains its property as a pulsed beam solution under the formal substitution $z + ct \to 2z$. The most general pulsed beam solution of this category is given by

$$U_{PB}(x, y, z, t) = \frac{i}{a_+ - i2z} v\left(z - ct - i\frac{\rho^2}{a_+ - i2z)}\right) w\left(-i\frac{x \pm iy}{a_+ - i2z}\right). \tag{14}$$

In particular, the extension of the simplest monochromatic beam solution given in Eq. (13) assumes the azimuthally asymmetric form

$$U_{PB}(x, y, z, t) = \frac{1}{a_+ - i2z} e^{-i\alpha\left(z - ct - i\frac{\rho^2}{a_+ - i2z)}\right)} w\left(-i\frac{x \pm iy}{a_+ - i2z}\right). \tag{15}$$

3. SIMPLIFICATION AND EXTENSION OF KISELEV'S WORK

Our aim in this section is to present a much simpler method than that used by Kiselev [13] and, furthermore, extend his work.

The 3D scalar wave equation (2) is rewritten in terms of the dimensionless coordinates $x_0 = qx,\ y_0 = qy,\ z_0 = qz,\ \tau_0 = qct$, where q has the units m^{-1}. A solution is then chosen in the incompletely separated form $u(x_0, y_0, z_0, \tau_0) = w(x_0, y_0; p_0)v(z_0, \tau_0; p_0)$, with the function $w(x_0, y_0; p_0)$ obeying the 2D Helmholtz equation

$$\left(\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial y_0^2} + p_0^2\right) w(x_0, y_0; p_0) = 0, \tag{16}$$

and the function $v(z_0, \tau_0; p_0)$ governed by the 1D Klein-Gordon equation

$$\left(\frac{\partial^2}{\partial z_0^2} - \frac{\partial^2}{\partial \tau_0^2} - p_0^2\right) v(z_0, \tau_0; p_0) = 0, \tag{17}$$

p_0 being a dimensionless parameter. A very simple proof is outlined below

$$\begin{aligned}
u(\vec{r_0}, \tau_0) &= \int\limits_{R^3} d\vec{k}_0 \int\limits_{R^1} d\omega_0 e^{-i\vec{k}_0\cdot\vec{r}_0} e^{i\omega_0\tau_0} \delta\left[\omega_0^2 - k_0^2\right] \tilde{u}\left(\vec{k}_0, \omega_0\right) \\
&= \left\{ \int\limits_{R^1} dk_{x_0} \int\limits_{R^1} dk_{y0} e^{-i(k_{x_0}x_0 + k_{y_0}y_0)} \delta\left[-k_{x_0}^2 - k_{y_0}^2 + p_0^2\right] \tilde{u}_1(k_{x_0}, k_{y_0}) \right\} \\
&\times \left\{ \int\limits_{R^1} dk_{z_0} \int\limits_{R^1} d\omega_0 e^{-i(k_{z_0}z_0 - \omega_0\tau_0)} \delta\left[\omega_0^2 - k_{z_0}^2 - p_0^2\right] \tilde{u}_2\left(k_{z_0}, \omega_0\right) \right\} \\
&\equiv w(x_0, y_0; p_0)v(z_0, \tau_0; p_0),
\end{aligned} \tag{18}$$

where $\delta(\cdot)$ denotes the Dirac delta function. The parameter p_0 entering into the functions $w(x_0, y_0; p_0)$ and $v(z_0, \tau_0; p_0)$ is free. The discussion in Sec. 2 corresponds to the restriction $p_0 = 0$. The procedure described above was alluded to but not carried out in a paper by Borisov [16].

Below, we shall derive a new extended family of FWM-type localized wave and beam solutions to the 3D scalar wave equation by combining the method of incomplete separation of variables with $p_0 \neq 0$ and Bateman's transformations. First, however, we wish to comment on Kiselev's main result [13]. Specifically, he determined that an extended FWM-type solution to the 3D scalar wave equation is given by

$$\begin{aligned}
U_k(x, y, z, t) &= \frac{1}{z + ct + ia_+} \\
&\times \exp\left[-i\alpha\left[\left(z - ct + \frac{\rho^2}{z + ct + ia_+}\right) - i\frac{p_0^2}{4\alpha(z + ct + ia_+)}\right] w(\bar{x}_+, \bar{y}_+; p_0);\right. \\
\bar{x}_+ &\equiv \frac{x}{z + ct + ia_+}, \qquad \bar{y}_+ \equiv \frac{y}{z + ct + ia_+}.
\end{aligned} \tag{19}$$

(In Kiselev's notation, $\alpha = -p,\ a_+ = -\varepsilon,\ p_0 = K$). A much simpler proof of this result than the one offered by Kiselev [13] is given below.

Within the framework of the incomplete separation of variables, the wave function is chosen as

$$\begin{aligned} u(x_0, y_0, z_0, \tau_0) &= v(z_0, \tau_0; p_0) w(x_0, y_0; p_0) \\ &= e^{-i(z_0+\tau_0)} e^{i\frac{p_0^2}{4}(z_0-\tau_0)} w(x_0, y_0; p_0), \end{aligned} \tag{20}$$

in terms of a specific solution of the dimensionless 1D Klein-Gordon equation (17) and an arbitrary solution to the dimensionless 2D Helmholtz equation (16). Then, Kiselev's solution given in Eq. (19) follows from the Bateman transformation

$$U_+(x, y, z, t) = B_+ \left\{ e^{-i(z_0+\tau_0)} e^{i\frac{p_0^2}{4}(z_0-\tau_0)} w(x_0, y_0; p_0) \right\}, \tag{21}$$

with the substitutions $x_0 = \alpha x,\ y_0 = \alpha y,\ z_0 = \alpha z,\ \tau_0 = \alpha ct$ and the additional complexification $z \to z + ia_+$.

An extension to Kiselev's work can be stated as follows: The most general FWM-type localized wave solutions of this category are given by

$$\begin{aligned} U_-(x, y, z, t) &= B_- \left\{ v(z_0, \tau_0; p_0) w(x_0, y_0; p_0) \right\}, \\ U_+(x, y, z, t) &= B_+ \left\{ v(z_0, \tau_0; p_0) w(x_0, y_0; p_0) \right\}, \end{aligned} \tag{22}$$

for arbitrary functions $v(z_0, \tau_0; p_0)$ and $w(x_0, y_0; p_0)$ obeying the 2D Helmholtz Eq. (16) and the 1D Klein-Gordon Eq. (17), respectively, and upon complexification of z ($z \to z - ia_-$ for the first solution and $z \to z + ia_+$ for the second one).

Let

$$w(z_0, y_0; p_0) = J_m(p_0\rho_0) e^{\pm im\phi}; \qquad \rho_0 \equiv \sqrt{x_0^2 + y_0^2}, \tag{23}$$

where $J_m(\cdot)$ is the mth-order ordinary Bessel function, be selected as a specific solution to the 2D Helmholtz Eq. (16). One, then, has in the place of Eq. (21) a Bateman transform of Durnin's exact mth-order monochromatic Bessel beam [17], viz.,

$$U_+(x, y, z, t) = B_+ \left\{ e^{-i(z_0+\tau_0)} e^{i\frac{p_0^2}{4}(z_0-\tau_0)} J_m(p_0\rho_0) e^{\pm im\phi} \right\}. \tag{24}$$

Carrying the transformation out explicitly results in the expression

$$U_{+}(\rho,\phi,z,t)=\frac{-i}{a_{+}-i(z+ct)}\exp\left[-\alpha\left(i(z-ct)+\frac{\rho^{2}}{a_{+}-i(z+ct)}\right)\right]$$
$$\times\exp\left[-\frac{p_{0}^{2}}{4\alpha(a_{+}-i(z+ct))}\right]I_{m}\left(p_{0}\frac{\rho}{a_{+}-i(z+ct)}\right)e^{\pm im\phi}, \tag{25}$$

where $I_m(\cdot)$ denotes the mth-order modified Bessel function. This solution is a variant of the Bessel-Gauss FWM originally introduced by Shaarawi [18] and subsequently studied by Overfelt [19] and Sedky [20]. It is in this respect that Kiselev [13] stated that his solution given in (19) generalizes the Bessel-Gauss FWM.

Since the Bessel-Gauss FWM is a special case of $U_{+}(x,y,z,t)$ in Eq. (22b), an important question is whether it retains the property of a paraxial pulsed beam if the formal change $z+ct\rightarrow 2z$ is made. It turns out that the answer is affirmative. The expression

$$U_{PB}(\rho,\phi,z,t)=\frac{-i}{a_{+}-i2z}\exp\left[-\alpha\left(i(z-ct)+\frac{\rho^{2}}{a_{+}-i2z}\right)\right]\exp\left[-\frac{p_{0}^{2}}{4\alpha(a_{+}-i2z)}\right]$$
$$\times I_{m}\left(p_{0}\frac{\rho}{a_{+}-i2z}\right)e^{\pm m\phi} \tag{26}$$

is, indeed, a monochromatic paraxial beam, known as the Bessel-Gauss beam, and has been studied extensively in the literature [21].

Staying with the choice for $w(x_0,y_0;p_0)$ given in Eq. (23), the Bessel-Gauss FWM in Eq. (25) can be extended by choosing different solutions to the 1D Klein-Gordon Eq. (17). Consider, for example, the solution

$$v(z_0,\tau_0;p_0) = \frac{1}{\sqrt{a_1+a_2+i(z_0+\tau_0)}}$$
$$\times\exp\left[-p_0\sqrt{-i(z_0-\tau_0)[a_1+a_2+i(z_0+\tau_0)]}\right], \tag{27}$$

where $a_{1,2}$ are dimensionless parameters. This choice gives rise to the nonsingular FWM-type localized wave

$$U_+(\rho,\phi,z,t) = -i\frac{\exp\left[-\frac{p_0}{\sqrt{a_+-i(z+ct)}}\sqrt{a_-+i(z-ct)+\frac{\rho^2}{a_+-i(z+ct)}}\right]}{[a_+-i(z+ct)]\sqrt{a_-+i(z-ct)+\frac{\rho^2}{a_+-i(z+ct)}}}$$
$$\times I_m\left(p_0\frac{\rho}{a_+-i(z+ct)}\right)e^{\pm im\phi}. \tag{28}$$

where $a_+ = a_2/\alpha$ and $a_- = a_1/\alpha$. One interesting aspect of this new solution is that it does not conform to the Courant-Hilbert-type of almost undistorted progressive wave solutions. This property is characteristic of many finite-energy FWM-type localized waves [22]. In the limit $p_0 \to 0$, $m \to 0$, the expression in Eq. (28) simplifies to axi-symmetric solution

$$U_+(\rho,z,t) = -i\frac{1}{[a_+-i(z+ct)]\sqrt{a_-+i(z-ct)+\frac{\rho^2}{a_+-i(z+ct)}}}, \tag{29}$$

which is a variation of Ziolkowski's *splash mode* [9]. The formal substitution $z+ct \to 2z$ in the expressions given in Eq. (28) and Eq. (29) results in new paraxial pulsed beam solutions.

It is clear that a large family of FWM-type localized waves $U_\pm(x,y,z,t)$ and paraxial pulsed beams $U_{PB}(x,y,z,t) = U_+(x,y,z,t)|_{z+ct\to 2z}$ can be derived from known solutions of the 2D Helmholtz equation and the 1D Klein-Gordon equation, in conjunction with Bateman's conformal transformations and a complexification of z.

It should be recalled that the dimensionless parameter p_0 is *free*. Thus far, is has been assumed to be real. In general, however, it may be complex. Consider, in particular, the case where p_0 is purely imaginary, viz., $p_0 = is_0$. Then, by inspection, the solutions given in Eqs. (25) and (26) are modified to the (ordinary) Bessel-Gauss FWM

$$U_+(\rho,\phi,z,t) = \frac{-i}{a_+-i(z+ct)}\exp\left[-\alpha\left(i(z-ct)+\frac{\rho^2}{a_+-i(z+ct)}\right)\right]$$
$$\times\exp\left[\frac{s_0^2}{4\alpha(a_+-i(z+ct))}\right]J_m\left(s_0\frac{\rho}{a_+-i(z+ct)}\right)e^{\pm im\phi}, \tag{30}$$

and the corresponding monochromatic paraxial Bessel-Gauss beam

$$U_{PB}(\rho,\phi,z,t) = \frac{-i}{a_+ - i2z}\exp\left[-\alpha\left(i(z-ct)+\frac{\rho^2}{a_+ - i2z}\right)\right]$$
$$\times\exp\left[\frac{s_0^2}{4\alpha(a_+ - i2z)}\right]J_m\left(s_0\frac{\rho}{a_+ - i2z}\right)e^{\pm im\phi}, \quad (31)$$

respectively. More generally, however, for the case $p_0 = is_0$, FWM-type localized waves $U_\pm(x,y,z,t)$ and pulsed beams $U_{PB}(x,y,z,t)$ can be synthesized from solutions of the dimensionless 2D equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y_0^2}-s_0^2\right)w(x_0,y_0;s_0)=0 \quad (32)$$

and the 1D dimensionless Proca (or de Broglie) equation

$$\left(\frac{\partial^2}{\partial z_0^2}-\frac{\partial^2}{\partial \tau_0^2}+s_0^2\right)v(z_0,\tau_0;s_0)=0, \quad (33)$$

again in conjunction with Bateman transformations and a complexification of z.

4. BATEMAN CONFORMAL TRANSFORMATIONS WITHIN THE FRAMEWORK OF THE BIDIRECTIONAL SPECTRAL REPRESENTATION

In many respects, the Borisov-Utkin and Kiselev theories, as well as their extensions, discussed in the previous two sections, are incomplete due to the nonuniqueness of the incomplete method of separation of variables of a solution to the wave equation. As a consequence, additional categories of FWM-type localized waves and pulsed beams are left out. Our main goal in this section is to go beyond the work of Borisov, Utkin and Kiselev by systematically pointing out some of the missing categories of FWM-type localized waves and pulsed beams. Our contribution will not be exhaustive; only a limited number of important categories will be discussed in detail.

4.1. Brittingham's Decomposition

Brittingham's original work on FWMs [8], as well as its clarification and extension[6, 7, 9, 10], is based on the incompletely separated wave function

$$u_-(x,y,z,t)=e^{i\beta(z+ct)}g_-(x,y,\varsigma);\quad \varsigma\equiv z-ct, \quad (34)$$

with β a real positive parameter and the function $g_-(x,y,\varsigma)$ obeying the complex parabolic (Schrödinger-like) equation

$$4i\beta\frac{\partial}{\partial\varsigma}g_-(x,y,\varsigma) = -\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)g_-(x,y,\varsigma). \tag{35}$$

A simple solution to Eq. (35) used in conjunction with Eq. (34) yields the axi-symmetric pure FWM [cf. Eq. (6)]

$$U^-_{FWM}(\rho,z,t) = \frac{1}{a_- + i(z-ct)}e^{i\beta\left[z+ct+i\frac{\rho^2}{a_-+i(z-ct)}\right]}. \tag{36}$$

A dual incompletely separated wave function is given by

$$u_+(x,y,z,t) = e^{-i\alpha(z-ct)}g_+(x,y,\eta); \quad \eta\equiv z+ct, \tag{37}$$

with α a real positive parameter and the function $g_+(x,y,\eta)$ obeying the complex parabolic equation

$$-4i\alpha\frac{\partial}{\partial\eta}g_+(x,y,\eta) = -\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)g_+(x,y,\eta). \tag{38}$$

The FWM solution dual to the one given in Eq. (36) assumes the form

$$U^+_{FWM}(\rho,z,t) = \frac{1}{a_+ - i(z+ct)}e^{-i\alpha\left[z-ct-i\frac{\rho^2}{a_+-i(z+ct)}\right]}. \tag{39}$$

It should be interesting to examine the properties of the pure FWMs $U^+_{FWM}(\rho,z,t)$ under Bateman transformations. It turns out that the only interesting property is given by

$$B_+\left\{U^-_{FWM}(x,y,z,t)\right\}|_{a_-\to a_+,\beta\to\alpha} = ia_+\left(U^+_{FWM}(x,y,z,t\right)^* e^{-i2\alpha(z-ct)}. \tag{40}$$

The superscript * denotes complex conjugation.

Based on the work in Sec. 2, the expression

$$U^\pm_{FWM}(\rho,z,t)w(\bar{x}_\pm \pm i\bar{y}_\pm); \quad (\bar{x}_\pm,\bar{y}_\pm)\equiv(x,y)/(a_\pm \mp i(z\pm ct)), \tag{41}$$

where $w(x\pm iy)$ is an arbitrary solution to the 2D Laplace equation, is an extended azimuthally asymmetric FWM-type solution to the 3D scalar wave equation. Of course, this is a special case of the Borisov-Utkin theory [cf. Eqs. (9)] which is based on the application of Bateman transformations to the incompletely separated wave function $u(x_0,y_0,z_0,\tau_0) = v(z_0+\tau_0)w(x_0\pm iy_0)$. This theory will always give rise to solutions of the form $U_{\pm}(x,y,z,t) = f\left[U^{\pm}_{FWM}(\rho,z,t)\right] w(\bar{x}_{\pm} \pm i\bar{y}_{\pm})$, the first part of which denotes a superposition of axi-symmetric FWMs. The Borisov-Utkin theory fails under any deviation from this prescription. For example, although any derivative of $U^{\pm}_{FWM}(\rho,z,t)w(\bar{x}_{\pm} \pm i\bar{y}_{\pm})$ with respect to the spatio-temporal variables is still a solution to the 3D scalar wave equation, the product of any derivative of $U^{\pm}_{FWM}(\rho,z,t)$ with the function $w(\bar{x}_{\pm} \pm i\bar{y}_{\pm})$ fails to be a solution. Furthermore, any functions formed of spatio-temporal derivatives of $U^{\pm}_{FWM}(\rho,z,t)$ cannot be obtained *directly* by means of Bateman transformations. These important points will be clarified in the following discussion.

Consider, first, the Laguerre-Gauss FWM localized waves [23] defined as

$$\begin{aligned} U_{-}(x,y,z,t) &= \frac{(-1)^n}{2^{2n}n!}\left(\nabla^2_{\perp}\right)^n U^{-}_{FWM}(\rho,z,t) \\ &= \frac{1}{[a_{-}+i(z-ct)]^{n+1}} e^{\beta(z+ct)} \\ &\quad \exp\left[-\beta\left(\frac{\rho^2}{\alpha_{-}+i(z-ct)}\right)\right] L^0_n\left[\beta\left(\frac{\rho^2}{\alpha_{-}+i(z-ct)}\right)\right]; \end{aligned} \tag{42}$$

$$\begin{aligned} U_{+}(x,y,z,t) &= \frac{(-1)^n}{2^{2n}n!}\left(\nabla^2_{\perp}\right)^n U^{+}_{FWM}(\rho,z,t) \\ &= \frac{1}{[a_{+}-i(z+ct)]^{n+1}} e^{-i\alpha(z-ct)} \\ &\quad \exp\left[-\alpha\left(\frac{\rho^2}{\alpha_{+}-i(z+ct)}\right)\right] L^0_n\left[\alpha\left(\frac{\rho^2}{\alpha_{+}-i(z+ct)}\right)\right], \end{aligned} \tag{43}$$

where $\nabla^2_{\perp}$ denotes the transverse (with respect to z) Laplacian operator and $L^0_n(\cdot)$ is the *nth*-order Laguerre polynomial. They are obtained from a solution to Eq. (35) used in conjunction with Eq. (34) and a solution to Eq. (38) in conjunction with Eq. (37), respectively. Since the solutions $U_{\pm}(x,y,z,t)$ given above are formed of derivatives of the pure FWMs $U^{\pm}_{FWM}(\rho,z,t)$, they can be obtained only *indirectly* from Bateman transformations; specifically,

$$\begin{aligned} U_{-}(x,y,z,t) &= \frac{(-1)^n}{2^{2n}n!}\left(\nabla^2_{\perp}\right)^n U^{-}_{FWM}(\rho,z,t) \\ &= \frac{(-1)^n}{2^{2n}n!}\left(\nabla^2_{\perp}\right)^n B_{-}\{-iq\exp[i(z_0+\tau_0)]\}, \\ & q=\beta, \quad z\rightarrow z-i\alpha_{-}; \end{aligned} \tag{44}$$

$$\begin{aligned} U_+(x,y,z,t) &= \frac{(-1)^n}{2^{2n}n!}\left(\nabla_\perp^2\right)^n U_{FWM}^+(\rho,z,t) \\ &= \frac{-1}{2^{2n}n!}\left(\nabla_\perp^2\right)^n B_+\{iq\exp[i(z_0+\tau_0)]\}, \\ & \quad q=\alpha, \quad z\to z+i\alpha_+. \end{aligned} \tag{45}$$

Of course, Bateman transformations can be applied directly to the Laguerre-Gauss solution $u_\pm(x,y,z,t)$ given in Eqs. (42) and (43) in order to determine other forms of FWM-type localized waves and, hence, pulsed beams.

For the next example, consider the azimuthally asymmetric solution [23]

$$\begin{aligned} U_-(x,y,z,t) = &\frac{1}{[a_1+i(z-ct)]^{1/2}[a_2+i(z-ct)]^{1/2}}e^{i\beta(z+ct)} \\ &\times\exp\left[-\beta\left(\frac{x^2}{a_1+i(z-ct)}\right)\right]\exp\left[-\beta\left(\frac{y^2}{a_2+i(z-ct)}\right)\right], \end{aligned} \tag{46}$$

with an analogous result for $U_+(x,y,z,t)$. This solution can be expressed in terms of a product of two 2D FWMs (FWM solutions to the 2D scalar wave equation) as follows:

$$\begin{aligned} U_-(x,y,z,t) = e^{-i\beta(z+ct)}&\left\{\begin{array}{l}\frac{1}{[a_1+i(z-ct)]^{1/2}}e^{i\beta(z+ct)}\exp\left[-\beta\left(\frac{x^2}{a_1+i(z-ct)}\right)\right] \\ \times\frac{1}{[a_2+i(z-ct)]^{1/2}}e^{i\beta(z+ct)}\exp\left[-\beta\left(\frac{y^2}{a_2+i(z-ct)}\right)\right]\end{array}\right\} \\ \equiv e^{-i\beta(z+ct)}&\left\{U_{2D-FWM}^-(x,z,t)U_{2D-FWM}^-(y,z,t)\right\}. \end{aligned} \tag{47}$$

Two-dimensional Bateman conformal transformations are defined as [1, 3]

$$b_x^\pm\{u(z,t)\} \equiv \frac{1}{\sqrt{z_0\pm\tau_0}}u\left[Z_x^\pm,T_x^\pm\right]/(x_0,z_0,\tau_0)\to q(x,z,ct), \tag{48}$$

with

$$Z_x^\pm = \frac{x_0^2+z_0^2-\tau_0^2-1}{2(z_0\pm\tau_0)}, \quad T_x^\pm = \frac{x_0^2+z_0^2-\tau_0^2+1}{2(z_0\pm\tau_0)}. \tag{49}$$

It follows, then, that

$$\begin{aligned} U_{2D-FWM}^-(x,z,t) &= b_x^-\{-iq\exp[i(z_0+\tau_0)]\}/q=\beta, z\to z-ia_1, \\ U_{2D-FWM}^-(y,z,t) &= b_y^-\{-iq\exp[i(z_0+\tau_0)]]\}/q=\beta, z\to z-ia_2, \end{aligned} \tag{50}$$

As a consequence, the solution $U_{-}(x, y, z, t)$ given in Eq. (47) can be determined by the application of a sequence of two 2D Bateman transformations. Of course, four dimensional Bateman transformations can be applied directly to $U_{-}(x, y, z, t)$ in Eq. (46) and the analogous expression for $U_{+}(x, y, z, t)$ in order to determine other forms of FWM-type localized waves and pulsed beams.

As the final example in this subsection, consider the solution given in Eq. (46) differentiated m times with respect to x and n times with respect to y. As a result, one obtains the azimuthally asymmetric Hermite-Gauss FWM localized wave [23, 24]

$$\begin{aligned} U_{-}(x, y, z, t) &= \frac{1}{[a_1 + i(z-ct)]^{\frac{m+1}{2}} [a_2 + i(z-ct)]^{\frac{n+1}{2}}} e^{i\beta(z+ct)} \\ &\times \exp\left[-\beta\left(\frac{x^2}{a_1 + i(z-ct)}\right)\right] \exp\left[-\beta\left(\frac{y^2}{a_2 + i(z-ct)}\right)\right] \\ &\times i^m H_m\left[\beta^{1/2}\frac{x}{\sqrt{\alpha_1 + i(z-ct)}}\right] i^n H_n\left[\beta^{1/2}\frac{y}{\sqrt{\alpha_2 + i(z-ct)}}\right], \quad (51) \end{aligned}$$

where $H_m(\cdot)$ denotes a Hermite polynomial. In terms of 2D conformal Bateman transformations, this solution can be expressed as

$$\begin{aligned} U_{-}(x,y,z,t) = \frac{\partial}{\partial x^m}\frac{\partial}{\partial y^n} e^{-i\beta(z+ct)} [b_x^{-}\{-iq\exp[i(z_0+\tau_0)]\}/q=\beta, z\rightarrow z-ia_1] \\ \times\left[b_y^{-}\{-iq\exp[i(z_0+\tau_0)]\}/q=\beta, z\rightarrow z-ia_2\right], \quad (52) \end{aligned}$$

with a similar result for $U_{+}(x, y, z, t)$.

4.2. The Bidirectional Spectral Synthesis

Our aim in this subsection is to discuss an already existing method which encompasses all the previously discussed results as special cases and, as a consequence, allows the determination of new types of FWM-type localized waves and pulsed beams.

In 1989, Besieris *et al.* [9] introduced the following spectral representation of a general solution to the homogeneous 3D scalar wave equation (2):

$$\begin{aligned} u(x, y, z, t) = \int_0^\infty d\alpha \int_0^\infty d\beta \int_{-\infty}^\infty dk_x \int_{-\infty}^\infty dk_y e^{-i\alpha(z-ct)} e^{i\beta(z+ct)} e^{-i(k_x x + k_y y)} \\ \times\delta\left[\alpha\beta - (k_x^2 + k_y^2)/4\right] \tilde{u}(k_x, k_y, \alpha, \beta). \quad (53) \end{aligned}$$

This spectral representation has been called *bidirectional* because the wave function is synthesized in terms of the product of two plane waves, one traveling in the positive z-direction with the fixed speed c and another moving backwards with the same speed. This representation has been proven the most natural one for deriving all FWM-type localized waves and pulsed beams. In the sequel, it will be examined in the light of the Bateman conformal transformations, and it will be shown that it encompasses the Borisov-Utkin and Kiselev contributions as special cases.

In Eq. (53), the integration with respect to the variable α is carried out. As result, one obtains

$$u_{-}(x,y,z,t)=\int_0^\infty d\beta e^{i\beta(z+ct)}\int_{-\infty}^{\infty}dk_x\int_{-\infty}^{\infty}dk_y e^{-i(k_x x+k_y y)}e^{-i\frac{\kappa^2}{4\beta}(z-ct)}\tilde{u}_{-}\left(k_x,k_y,\frac{\kappa^2}{4\beta},\beta\right), \tag{54}$$

where $\kappa=\sqrt{k_x^2+k_y^2}$. An analogous result follows by integrating with respect to β in Eq. (53); specifically,

$$u_{+}(x,y,z,t)=\int_0^\infty d\alpha e^{-i\alpha(z-ct)}\int_{-\infty}^{\infty}dk_x\int_{-\infty}^{\infty}dk_y e^{-i(k_x x+k_y y)}e^{i\frac{\kappa^2}{4\alpha}(z+ct)}\tilde{u}_{+}\left(k_x,k_y,\alpha,\frac{\kappa^2}{4\alpha}\right). \tag{55}$$

It should be pointed out that the solutions $u_{-}(x,y,z,t)$ and $u_{+}(x,y,z,t)$ given above are similar to those given in Eqs. (34) and (37) in connection with the Brittingham decomposition, simply because the latter is embedded within the bidirectional spectral representation [cf. Eq. (53)].

Let the spectrum in the original bidirectional spectral representation [cf. Eq. (53)] be restricted to the form

$$\delta\left[\alpha\beta-\kappa^2/4\right]\tilde{u}(k_x,k_y,\alpha,\beta)=\left\{\delta(\alpha\beta-p^2/4)\tilde{u}_1(\alpha,\beta)\right\}\left\{\delta(-\kappa^2+p^2)\tilde{u}_2(k_x,k_y)\right\}, \tag{56}$$

where p is a free parameter with units of m^{-1}. Then, the solution in Eq. (53) is restricted to

$$u_r(x,y,z,t)=\left\{\int_0^\infty d\alpha\int_0^\infty d\beta e^{-i\alpha(z-ct)}e^{i\beta(z+ct)}\delta\left(\alpha\beta-p^2/4\right)\tilde{u}_1(\alpha,\beta)\right\}$$
$$\times\left\{\int_{-\infty}^{\infty}dk_x\int_{-\infty}^{\infty}dk_y e^{-i(k_x x+k_y y)}\left\{\delta(-\kappa^2+p^2)\tilde{u}_2(k_x,k_y)\right\}\right\}. \tag{57}$$

Integrating sequentially, first with respect to α and then with respect to β, one obtains from Eq. (57) the restricted solutions

$$
\begin{aligned}
u_{r-}(x,y,z,t) &= \left\{ \int_0^\infty d\beta e^{i\beta(z+ct)} e^{-i\frac{p^2}{4\beta}(z-ct)} \tilde{u}_{1-}(\beta) \right\} \\
&\quad \times \left\{ \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y e^{-i(k_x x + k_y y)} \left\{ \delta\left(-\kappa^2 + p^2\right) \tilde{u}_2(k_x, k_y) \right\} \right\} \\
&\equiv v_-(z,t;p) w(x,y;p),
\end{aligned} \tag{58}
$$

$$
\begin{aligned}
u_{r+}(x,y,z,t) &= \left\{ \int_0^\infty d\alpha e^{-i\alpha(z-ct)} e^{i\frac{p^2}{4\alpha}(z+ct)} \tilde{u}_{1+}(\alpha) \right\} \\
&\quad \times \left\{ \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y e^{-i(k_x x + k_y y)} \left\{ \delta\left(-\kappa^2 + p^2\right) \tilde{u}_2(k_x, k_y) \right\} \right\} \\
&\equiv v_+(z,t;p) w(x,y;p),
\end{aligned} \tag{59}
$$

respectively. The function $w(x_0, y_0; p_0)$ with $p_0 = p/q$ is a solution to the dimensionless 2D Helmholtz equation (16) and the functions $v_\pm(z_0, \tau_0; p_0)$ are solutions to the dimensionless 1D Klein-Gordon equation (17). Thus, when four-dimensional Bateman transformations are applied to $u_{r\pm}(x_0, y_0, z_0, \tau_0)$, one recaptures all of the work in Sec. 3 (related to Kiselev's paper [13]), and in the case $p_0 = 0$ all of the Borisov-Utkin work in [12] discussed in Sec. 2.

It should be pointed out that the restricted solutions $u_{r\pm}(x,y,z,t)$ given in Eqs. (58) and (59) can be derived, as it was done at the beginning of Sec. 3, directly from an ordinary Fourier synthesis of a solution to the 3D scalar wave equation. However, the general bidirectional spectral synthesis[cf. Eq. (53)] has several distinct advantages. Among them are the following: (1) All the FWM-type localized waves and pulsed beams discussed in Secs. 2, 3 and 4.1 can be derived from bidirectional synthesis without resorting to Bateman transformations; (2) New solutions, lying outside the framework of the Borisov-Utkin and Kiselev's theories, can be determined. This has partially been demonstrated in Sec. 4.1 through the Brittingham decomposition which is intimately related to the bidirectional spectral synthesis; (3) The Fourier spectra of certain FWM-type localized, e.g., the MPS pulse, cannot be derived by direct Fourier inversion; yet they can be determined from a conversion of the bidirectional spectrum.

Next, it will be demonstrated how the Borisov-Utkin and Kiselev results can be obtained directly from the bidirectional representation without resorting to Bateman transformations.

First, the solution $u_-(x,y,z,t)$ in Eq. (54) is specialized as

$$U_-(x,y,z,t) = \int_0^\infty d\beta e^{i\beta(z+ct)} \int_{-\infty}^\infty dk_x \int_{-\infty}^\infty dk_y e^{-i(k_x x+k_y y)} e^{-i\frac{\kappa^2}{4\beta}(z-ct)} e^{\frac{-p_1}{2\beta}(kx\pm ik_y)} \tilde{v}(\beta) e^{-a_-\frac{\kappa^2}{4\beta}}$$
$$\sim \frac{1}{a_- + i(z-ct)} \frac{1}{2\pi} \int_0^\infty d\beta \tilde{v}(\beta) \exp\left[\frac{p_1}{2\beta}\left(-i\frac{\partial}{\partial x} \pm \frac{\partial}{\partial y}\right)\right] e^{i\beta\left(z+ct+\frac{x^2+y^2}{z-ct-ia_-}\right)}, \tag{60}$$

where p_1 is a dimensionless parameter. Use of the translational properties of the exponential operators [e.g., $\exp(-a\partial/\partial x)f(x) = f(x-a)$], as well as their commutativity, yields

$$U_-(x,y,z,t) = \left\{\frac{1}{a_- + i(z-ct)} v\left(z+ct+\frac{x^2+y^2}{z-ct-ia_-}\right)\right\} \times \left\{\exp\left[\frac{-ip_1}{z-ct-ia_-}(x\pm iy)\right]\right\}, \tag{61}$$

where $v(\cdot)$ denotes the complex analytic signal associated with the spectrum $\tilde{v}(\cdot)$. A superposition over the free parameter p_1, with spectrum $\tilde{w}(p_1)$, extends the solution in (61) to

$$U_-(x,y,z,t) \sim \frac{1}{a_- + i(z-ct)} v\left[z+ct+\frac{x^2+y^2}{z-ct-ia_-}\right] w\left[\frac{x\pm iy}{z-ct-ia_-}\right], \tag{62}$$

with an analogous expression for $U_+(x,y,z,t)$. But these are precisely the Borisov-Utkin results [cf. Eq. (9)] They have been derived directly from the bidirectional representation without using Bateman transformations.

Kiselev's result can be derived in a similar manner. The solution $u_+(x,y,z,t)$ in Eq. (55) is specialized as

$$U_+(x,y,z,t) = \int_0^\infty d\alpha e^{-i\alpha(z-ct)} \int_{-\infty}^\infty dk_x \int_{-\infty}^\infty dk_y \, e^{-i(k_x x+k_y y)} e^{i\frac{\kappa^2}{4\alpha}(z+ct)} e^{-i\frac{p_0}{2\alpha}(\cos\theta k_x + \sin\theta k_y)} \tilde{v}(\alpha) e^{-a_+\frac{\kappa^2}{4\alpha}}$$

$$\sim \quad \frac{1}{a_+ - i(z+ct)} \frac{1}{2\pi} \int_0^\infty d\alpha \tilde{v}(\alpha) \exp\left[\frac{p_0}{2\alpha}\left(\cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}\right)\right]$$

$$e^{-i\alpha\left[z - ct - i\frac{x^2+y^2}{a_+ - i(z+ct)}\right]}, \tag{63}$$

where p_0 is a dimensionless free parameter and θ an arbitrary angle. Use of the commutativity and translational properties of the exponential operators yields

$$U_+(x,y,z,t) = \frac{1}{a_+ - i(z+ct)} \frac{1}{2\pi} \int_0^\infty d\alpha \tilde{v}(\alpha) e^{-i\alpha\left[z - ct - i\frac{x^2+y^2}{a_+ - i(z+ct)}\right]}$$

$$\times e^{\frac{p_0^2}{4\alpha} \frac{1}{a_+ - i(z+ct)}} e^{-p_0 \frac{1}{a_+ - i(z+ct)}(x\cos\theta + y\sin\theta)}. \tag{64}$$

For the singular spectrum $\tilde{v}(\alpha) = \delta(\alpha - \bar{\alpha})$ and with an additional superposition over θ, the expression above is converted into

$$U_+(x,y,z,t) = \frac{1}{a_+ - i(z+ct)} e^{-i\bar{\alpha}\left[z - ct - i\frac{x^2+y^2}{a_+ - i(z+ct)}\right]}$$

$$\times e^{-\frac{p_0^2}{4\bar{\alpha}} \frac{1}{a_+ - i(z+ct)}} \int_0^{2\pi} d\theta e^{-p_0 \frac{1}{a_+ - i(z+ct)}(x\cos\theta + y\sin\theta)} \tilde{w}(\theta). \tag{65}$$

The integration over θ is a general solution to the 2D Helmholtz equation

$$\left(\frac{\partial^2}{\partial \bar{x}_+^2} + \frac{\partial^2}{\partial \bar{y}_+^2} + p_0^2\right) w\{\bar{x}_+, \bar{y}_+; p) = 0;$$

$$\bar{x}_+ \equiv \frac{x}{z + ct + ia_+}, \quad \bar{y}_+ \equiv \frac{y}{z + ct + ia_+}. \tag{66}$$

A combination of Eqs. (65) and (66) yields Kiselev's expression[cf. Eq. (19)]

$$U_+(x,y,z,t) = \frac{1}{a_+ - i(z+ct)} e^{-i\bar{\alpha}\left(z - ct + \frac{x^2+y^2}{z+ct+ia_+}\right)} e^{-i\frac{p_0^2}{4\bar{\alpha}} \frac{1}{(z+ct+ia_+)}} w(\bar{x}_+, \bar{y}_+; p_0). \tag{67}$$

An analogous result can be obtained for $U_-(x,y,z,t)$, viz.,

$$U_-(x,y,z,t)=\frac{1}{a_-+i(z-ct)}e^{i\bar{\beta}\left(z+ct+\frac{x^2+y^2}{z-ct-ia_-}\right)}e^{i\frac{p_0^2}{4\beta}\frac{1}{(z-ct-ia_-)}}w(\bar{x}_-,\bar{y}_-;p_0). \tag{68}$$

In the limit $p_0 \to 0$ and upon superposition over the spectrum $\tilde{v}(\tilde{\beta})$ in Eq. (68), one obtains the Borisov-Utkin result given in Eq. (62), with an analogous expression for $U_+(x,y,z,t)$.

We wish to conclude this subsection by using the bidirectional spectral representation to derive FWM-type localized waves that lie outside the range of applicability of the Borisov-Utkin-Kiselev theories and their extensions.

A simple solution in this category can be constructed by specializing the solution $u_+(x,y,z,t)$ in Eq. (55) as

$$\begin{aligned} U_+(x,y,z,t) &= \int_0^\infty d\alpha e^{-i\alpha(z-ct)}\int_{-\infty}^{\infty}dk_x\int_{-\infty}^{\infty}dk_y e^{-i(k_x x+k_y y)} \\ &\quad e^{i\frac{\kappa^2}{4\alpha}(z+ct)}e^{-i\frac{p_0}{2\alpha}\frac{1}{\sqrt{2}}(\gamma_1 k_x+\gamma_2 k_y)}\tilde{v}(\alpha)e^{-a_+\frac{\kappa^2}{4\alpha}} \\ &\sim \frac{1}{a_+-i(z+ct)}\frac{1}{2\pi}\int_0^\infty d\alpha\tilde{v}(\alpha)\exp\left[\frac{p_0}{2\alpha}\frac{1}{\sqrt{2}}\left(\gamma_1\frac{\partial}{\partial x}+\gamma_2\frac{\partial}{\partial y}\right)\right] \\ &\quad e^{-i\alpha\left[z-ct-i\frac{x^2+y^2}{a_+-i(z+ct)}\right]}, \end{aligned} \tag{69}$$

where $\gamma_{1,2}$ are dimensionless free parameters. Use of the commutativity and translational properties of the exponential operators yields

$$\begin{aligned} U_+(x,y,z,t) &= \frac{1}{a_+-i(z+ct)}\frac{1}{2\pi}\int_0^\infty d\alpha\tilde{v}(\alpha)e^{-i\alpha\left[z-ct-i\frac{x^2+y^2}{a_+-i(z+ct)}\right]} \\ &\quad\times e^{-\frac{p_0^2}{8\alpha}\frac{\left(\gamma_1^2+\gamma_2^2\right)}{a_+-i(z+ct)}}e^{-p_0\frac{1}{\sqrt{2}}\frac{1}{a_+-i(z+ct)}(\gamma_1 x+\gamma_2 y)}. \end{aligned} \tag{70}$$

For the singular spectrum $\tilde{v}(\alpha) = \delta(\alpha-\bar{\alpha})$ and an additional superposition over the dimensionless parameters k_{x0} and k_{y0}, as shown below, the expression above is converted into

$$\begin{aligned} U_+(x,y,z,t) &= \frac{1}{a_+-i(z+ct)}e^{-i\bar{\alpha}\left[z-ct-i\frac{x^2+y^2}{a_+-i(z+ct)}\right]} \\ &\quad\times\ e^{-\frac{p_0^2}{8\bar{\alpha}}\frac{\left(\gamma_1^2+\gamma_2^2\right)}{a_+-i(z+ct)}}\int_{-\infty}^{\infty}dk_{x0}\int_{-\infty}^{\infty}dk_{y0}e^{-i\frac{1}{a_+-i(z+ct)}(k_{x0}x+k_{y0}y)} \end{aligned}$$

$$\times w(k_{x0}, k_{y0})\delta\left(-\frac{1}{\gamma_1^2}k_{x0}^2 - \frac{1}{\gamma_1^2}k_{y0}^2 + p_0^2\right) \tag{71}$$

The double integration gives rise to a general solution of the anisotropic 2D Helmholtz equation

$$\begin{gathered}\left(\frac{1}{\gamma_1^2}\frac{\partial^2}{\partial \bar{x}_+^2} + \frac{1}{\gamma_2^2}\frac{\partial^2}{\partial \bar{y}_+^2} + p_0^2\right) w\{\bar{x}_+, \bar{y}_+; p_0; \gamma_1, \gamma_2) = 0;\\ \bar{x}_+ \equiv \frac{x}{z+ct+ia_+}, \qquad \bar{y}_+ \equiv \frac{y}{z+ct+ia_+}.\end{gathered} \tag{72}$$

A combination of Eqs. (71) and (72) yields

$$\begin{aligned} U_+(x,y,z,t) &= \frac{1}{a_+ - i(z+ct)} e^{-i\bar{\alpha}\left[z-ct+\frac{x^2+y^2}{z+ct+ia_+}\right]} e^{-i\frac{p_0^2}{8\bar{\alpha}}\frac{(\gamma_1^2+\gamma_2^2)}{(z+cy+ia_+)}}\\ &\quad \times w\{\bar{x}_+, \bar{y}_+; p_0; \gamma_1, \gamma_2), \end{aligned} \tag{73}$$

an illustrative example of an extended FWM-type localized wave derived directly from the bidirectional spectral synthesis and lying beyond the range of applicability of Kiselev's theory.

As a second example, if the spectrum in the expression for $u_+(x,y,z,t)$ given in Eq. (55) is chosen as

$$\tilde{u}_+\left(k_x, k_y, \alpha, \frac{\kappa^2}{4\alpha}\right) = \frac{1}{2}\delta(\alpha - \bar{\alpha})e^{-\alpha_+\frac{\kappa^2}{4\alpha}} J_0\left(\frac{\gamma}{4}\kappa^2\right), \tag{74}$$

the integration can be carried out exactly [cf. Ref. 25, § 6.651.6], resulting in the axi-symmetric solution

$$\begin{aligned} U_+(\rho,z,t) &= \frac{1}{\sqrt{\gamma^2 + \left[\frac{a_+ - i(z+ct)}{\bar{\alpha}}\right]^2}} e^{-i\alpha(z-ct)}\\ \exp&\left[-\rho^2\frac{[a_+ - i(z+ct)]}{\bar{\alpha}}\frac{1}{\gamma^2 + \left[\frac{a_+ - i(z+ct)}{\bar{\alpha}}\right]^2}\right] J_0\left[\frac{\gamma\rho^2}{\gamma^2 + \left[\frac{a_+ - i(z+ct)}{4\bar{\alpha}}\right]^2}\right]. \end{aligned} \tag{75}$$

This is a *quadratic*-Bessel-Gauss FWM. The only restriction for the validity of this solution is that $a_+ > |\mathrm{Im}\{\gamma\}|$. If the formal substitution $z + ct \to 2z$ is made in the expression for $u_+(\rho, z, t)$, one obtains the monochromatic *quadratic*-Bessel-Gauss (QBG) beam [26]

$$U_{PB}(\rho,z,t)=\frac{1}{\sqrt{\gamma^2+\left(\frac{a_+-i2z}{\bar{\alpha}}\right)^2}}e^{-i\alpha(z-ct)}$$
$$\exp\left(-\rho^2\frac{[a_+-i2z]}{\bar{\alpha}}\frac{1}{\gamma^2+\left(\frac{a_+-i2z}{\bar{\alpha}}\right)^2}\right]\times J_0\left[\frac{\gamma\rho^2}{\gamma^2+\left(\frac{a_+-i2z}{\bar{\alpha}}\right)^2}\right]. \tag{76}$$

A similar, but not identical, result can be obtained from Kiselev's solution, viz..

$$U_K(x,y,z,t)=\frac{1}{a_+-i(z+ct)}e^{-i\bar{\alpha}\left[z-ct-i\frac{x^2+y^2}{a_+-i(z+ct)}\right]}e^{\frac{p_0^2}{4\bar{\alpha}}\frac{1}{a_+-i(z+ct)}}w(\bar{x}_+,\bar{y}_+;p_0). \tag{77}$$

In this expression, let $w(\bar{x}_+,\bar{y}_+;p_0)=J_0\left(p_0\sqrt{\bar{x}_+^2+\bar{y}_+^2}\right)$. Then, the superposition

$$U_+(x,y,z,t)=\frac{1}{a_+-i(z+ct)}e^{-i\bar{\alpha}\left[z-ct-i\frac{x^2+y^2}{a_+-i(z+ct)}\right]}$$
$$\times\int_0^\infty dp_0 e^{-\frac{p_0^2}{4\bar{\alpha}}\frac{1}{a_+-i(z+ct)}}(1/2)J_0\left(p_0\sqrt{\bar{x}_+^2+\bar{y}_+^2}\right)J_0\left(\frac{\gamma}{4}p_0^2\right) \tag{78}$$

results in a different type of quadratic Bessel-Gauss FWM localized wave; specifically,

$$U_+(x,y,z,t)=\frac{1}{a_+-i(z+ct)}e^{-i\bar{\alpha}\left[z-ct-i\frac{x^2+y^2}{a_+-i(z+ct)}\right]}\frac{1}{\sqrt{\gamma^2+\left[\frac{1}{4\bar{\alpha}(a_+-i(z+ct)}\right]^2}}$$
$$\times\exp\left\{-\frac{1}{4\bar{\alpha}(a_+-i(z+ct)}\frac{\rho^2}{(z+ct+ia_+)^2}\frac{1}{\gamma^2+\left[\frac{1}{4\bar{\alpha}(a_+-i(z+ct)}\right]^2}\right\}$$

$$\times J_0 \left\{ \gamma \frac{\rho^2}{(z+ct+ia_+)^2} \frac{1}{\gamma^2 + \left[\frac{1}{4\bar{\alpha}(a_+ - i(z+ct))}\right]^2} \right\}, \tag{79}$$

with a corresponding paraxial pulsed beam.

4.3. The Formulation of a Novel Global *Ansatz*

Recall the extensions of Kiselev's work discussed in Sec 3; specifically, that a broad class of extended FWMs is given by

$$\begin{aligned} U_-(x,y,z,t) &= B_- \left\{ v\,(z_0, \tau_0; p_0)\, w(x_0, y_0; p_0) \right\}, \\ U_+(x,y,z,t) &= B_+ \left\{ v\,(z_0, \tau_0; p_0)\, w(x_0, y_0; p_0) \right\}, \end{aligned} \tag{80}$$

for arbitrary functions $v\,(z_0, \tau_0; p_0)$ and $w(x_0, y_0; p_0)$ obeying the 2D Helmholtz (16) and the 1D Klein-Gordon Eq. (17), respectively, and upon complexification of z ($z \to z - ia_-$ for the first solution and $z \to z + ia_+$ for the second one). Our purpose is to recast these "local" extensions into "global" forms.

Since $\bar{\alpha}$ in Eq. (67) is a free parameter, a more general solution to the 3D scalar wave equation is given by

$$U_+(x,y,z,t) = \frac{1}{a_+ - i(z+ct)}$$
$$\left\{ \int_0^\infty d\bar{\alpha}\tilde{v}(\bar{\alpha}) e^{-i\bar{\alpha}\left[z-ct-i\frac{x^2+y^2}{a_+-i(z+ct)}\right]} e^{-\frac{p_0^2}{4\bar{\alpha}}\frac{1}{a_+-i(z+ct)}} \right\} w(\bar{x}_+, \bar{y}_+; p_0). \tag{81}$$

On the other hand, a general solution to the dimensionless 1D Klein-Gordon equation

$$\left(\frac{\partial^2}{\partial z_+^2} - \frac{\partial^2}{\partial \tau_+^2} - p_0^2 \right) v(z_+, \tau_+; p_0) = 0 \tag{82}$$

can be written out explicitly in the form

$$v(z_+, \tau_+; p_0) = \int_0^\infty d\alpha_0 e^{-i\alpha_0(z_+ + \tau_+)} e^{-i\frac{p_0^2}{4\alpha_0}(\tau_+ - z_+)} \tilde{v}\left(\alpha_0, \frac{p_0^2}{4\alpha_0}\right). \tag{83}$$

A comparison of Eq. (83) with the integral within the curly brackets in Eq. (81) suggests the transformations

$$z_+ + \tau_+ = \bar{\alpha}\left[z - ct - i\frac{x^2+y^2}{a_+ - i(z+ct)}\right], \quad \tau_+ - z_+ = \frac{1}{\bar{\alpha}}\frac{1}{z+ct+ia_+}, \tag{84}$$

or, alternatively,

$$\begin{aligned} z_+ &= \frac{1}{2}\left[-\frac{1}{\bar{\alpha}}\frac{1}{z+ct+ia_+}+\bar{\alpha}\left(z-ct+\frac{x^2+y^2}{z+ct+ia_+}\right)\right], \\ \tau_+ &= \frac{1}{2}\left[\frac{1}{\bar{\alpha}}\frac{1}{z+ct+ia_+}+\bar{\alpha}\left(z-ct+\frac{x^2+y^2}{z+ct+ia_+}\right)\right]. \end{aligned} \tag{85}$$

This formulation leads to the following novel *ansatz*. A broad category of extended FWM-type localized wave solutions to the 3D scalar wave equation can be expressed as

$$U_+(x,y,z,t) = \frac{1}{a_+ - i(z+ct)} v(z_+, \tau_+; p_0; \bar{\alpha}) w(\bar{x}_+, \bar{y}_+; p_0); \tag{86}$$

in other words, as a product of an "attenuation factor", an arbitrary solution of the 1D Klein-Gordon equation (82), with z_+ and τ_+ replaced as shown in Eq. (85), and an arbitrary solution to the 2D Helmholtz equation (66). Proceeding similarly, an expression analogous to that given in (86) can be formulated for $U_-(x,y,z,t)$; specifically,

$$U_-(x,y,z,t) = \frac{1}{a_- + i(z-ct)} v(z_-, \tau_-; p_0; \bar{\beta}) w(\bar{x}_-, \bar{y}_-; p_0), \tag{87}$$

where

$$\begin{aligned} z_- &= \frac{1}{2}\left[-\frac{1}{\bar{\beta}}\frac{1}{z-ct-ia_-}+\bar{\beta}\left(z+ct+\frac{x^2+y^2}{z-ct-ia_-}\right)\right], \\ \tau_- &= \frac{1}{2}\left[\frac{1}{\bar{\beta}}\frac{1}{z-ct-ia_-}+\bar{\beta}\left(z+ct+\frac{x^2+y^2}{z-ct-ia_-}\right)\right]. \end{aligned} \tag{88}$$

In the limit $p_0 \to 0$, $w(\bar{x}_\pm, \bar{y}_\pm; 0)$ is a solution to the 2D Laplace equation and $v\left(z_\pm, \tau_\pm; 0; \bar{\alpha}, \bar{\beta}\right)$ is a solution to the dimensionless 1D scalar wave equation. All the Borison-Utkin results are recovered provided that the specific solutions $v\left(z_\pm + \tau_\pm; 0; \bar{\alpha}, \bar{\beta}\right)$ are used.

The *ansatze* embodied in the expressions given in Eqs. (86) and (87) are tantamount to dimension-reduction methods. For $p_0 \neq 0$ an extended FWM localized solution to the 3D scalar wave equation is found from a general solution to a 1D Klein-Gordon equation and a general solution to a 2D Helmholtz equation. For $p_0 = 0$, on the other hand, an extended FWM localized solution to the 3D scalar wave equation is determined from a general solution to a 1D scalar wave equation and an arbitrary solution to a 2D Laplace equation. The two *ansatze* formulated in this subsection extend previous work along these lines by Besieris *et al.* [27].

5. CONCLUDING REMARKS

It has been demonstrated that both the Borisov-Utkin and Kiselev results, as well as their extensions, can be obtained as special cases directly from the bidirectional spectral representation without using Bateman transformations. This does not detract from the elegance and usefulness of the Borisov-Utkin-Kiselev theories and their extensions, which are based on the method of incomplete separation of variables and the application of Bateman transformations (cf. Secs. 2 and 3.) The main goal in this paper has been to systematize the process for deriving FWM-type localized waves and, hence, paraxial pulsed beams. In this respect, the bidirectional spectral synthesis is *all-inclusive*.

The Bateman conformal transformations discussed in Sec. 1 are pertinent only to the 2D or 3D scalar wave equations. Extensions to n-dimensional scalar wave equations is straightforward (see Appendix A). Consider, however, other types of equations, e.g., the 3D Klein-Gordon equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\omega_p^2}{c^2}\right) u(\vec{r}, t) = 0 \tag{89}$$

where ω_p denotes the plasma frequency in the case of a collisionless (cold) plasma, for which the derivation of localized waves would be a physically desirable task. Bateman conformal transformations for the Klein-Gordon equation, analogous to those introduced for the scalar wave equation, would be very useful in several areas of physics. Unfortunately, they have not been discovered. It should be mentioned parenthetically that the Bateman transformations encompass the Lorentz transformation as a special case. At this level of restriction, both the 3D scalar wave and Klein-Gordon equations are Lorentz invariant.

The importance of the bidirectional spectral representation as a general method for deriving FWM-type localized waves becomes more evident for physical problems lacking Bateman-like conformal invariance. Consider, specifically, the 3D Klein-Gordon equation (89). It turns out that an *ansatz* analogous to the one introduced in Sec. 4.3 for the 3D scalar wave equation can be formulated. Specifically, a broad family of FWM-type localized solutions to Eq. (89) is given by

$$U_{\pm}(x, y, z, t) = \frac{1}{a_{\pm} \mp i(z \pm ct)} v_{\pm}(z_{\pm}, \tau_{\pm}; p_0) w_{\pm}(\bar{x}_{\pm}, \bar{y}_{\pm}; p_1), \tag{90}$$

with $v_{\pm}(z_{\pm}, \tau_{\pm}; p_0)$ an arbitrary solution to the dimensionless 1D Klein-

Gordon equation

$$\left(\frac{\partial^2}{\partial z_\pm^2} - \frac{\partial^2}{\partial \tau_\pm^2} - p_0^2\right) v_\pm(z_\pm, \tau_\pm; p_0) = 0, \tag{91}$$

$w_\pm(\bar{x}_\pm, \bar{y}_\pm; p_1)$ satisfying the dimensionless 2D Helmholtz equation

$$\left(\frac{\partial^2}{\partial \bar{x}_\pm^2} + \frac{\partial^2}{\partial \bar{y}_\pm^2} + p_1^2\right) w_\pm(\bar{x}_\pm, \bar{y}_\pm; p_1) = 0, \tag{92}$$

and the definitions

$$\begin{aligned}
&(\bar{x}_\pm, \bar{y}_\pm) = \left(\frac{x}{z \pm ct \pm ia_\pm}, \frac{y}{z \pm ct \pm ia_\pm}\right);\\
&z_\pm = \alpha\left(z \mp ct + \frac{\rho^2}{z \pm ct \pm ia_\pm}\right)\\
&\times \frac{1}{2p_0^2}\left[\frac{p_1^2}{\alpha(z \pm ct \pm ia_\pm)} - \alpha p_0^2\left(z \mp ct + \frac{\rho^2}{z \pm ct \pm ia_\pm}\right) + \frac{\omega_p^2}{\alpha c^2}(z \pm ct \pm ia_\pm)\right],\\
&\tau_\pm = -\frac{1}{2p_0^2}\left[\frac{p_1^2}{\alpha(z \pm ct \pm ia_\pm)} - \alpha p_0^2\left(z \mp ct + \frac{\rho^2}{z \pm ct \pm ia_\pm}\right) + \frac{\omega_p^2}{\alpha c^2}(z \pm ct \pm ia_\pm)\right].
\end{aligned} \tag{93}$$

A detailed derivation of this *ansatz* with specific physical applications will be presented elsewhere.

APPENDIX A. $m + 1$-DIMENSIONAL BATEMAN CONFORMAL TRANSFORMATIONS

Consider the $m + 1$-dimensional homogeneous scalar wave equation

$$\left(\sum_{j=1}^{m-1} \frac{\partial^2}{\partial x_{0j}^2} + \frac{\partial^2}{\partial z_0^2} - \frac{\partial^2}{\partial \tau_0^2}\right) u(x_{01}, x_{02}, \ldots, x_{0m-1}, z_0, \tau_0) = 0, \tag{A1}$$

where (x_{0j}, z_0, τ_0), $j = 1, 2, \ldots, m-1$, are dimensionless coordinates linked to the usual space and time variables by the relationships $(x_{0j}, z_0, \tau_0) = \beta(x_j, z, ct)$, β being a parameter with units of m^{-1} and c is the speed of light *in vacuo*. An extension of the 3 + 1-dimensional Bateman conformal transformations embodied in Eq. (1) can be stated as follows: Given a solution $u(x_{01}, x_{02}, \ldots, x_{0m-1}, z_0, \tau_0)$ to Eq. (A1), the functions

$$\begin{aligned}
&U_\pm(x_{01}, x_{02}, \ldots, x_{0m-1}, z_0, \tau_0) \\
&\quad = B_\pm\{u(x_{01}, x_{02}, \ldots, x_{0m-1}, z_0, \tau_0)\} \\
&\quad = \frac{1}{(z_0 \pm \tau_0)^{(m-1)/2}} u\left(\frac{x_{01}}{z_0 \pm \tau_0}, \frac{x_{02}}{z_0 \pm \tau_0}, \ldots, \frac{x_{0m-1}}{z_0 \pm \tau_0}, \frac{1}{2(z_0 \pm \tau_0)}\right. \\
&\qquad \left. \times \sum_{j=1}^{m-1} \left(x_{0j}^2 + z_0^2 - \tau_0^2 - 1\right), \frac{1}{2(z_0 \pm \tau_0)} \sum_{j=1}^{m-1} \left(x_{0j}^2 + z_0^2 - \tau_0^2 + 1\right)\right) \quad \text{(A2)}
\end{aligned}$$

Another, distinct, formulation are also solutions.

Another, distinct, formulation of multi-dimensional Bateman conformal transformations is the following: Let $u(x_{0j}, z_0, \tau_0)$ be a solution to the 2 + 1-dimensional scalar wave equation

$$\left(\frac{\partial^2}{\partial x_{0j}^2} + \frac{\partial^2}{\partial z_0^2} - \frac{\partial^2}{\partial \tau_0^2}\right) u(x_{0j}, z_0, \tau_0) = 0, \; j = 1, 2, \ldots, m-1. \quad \text{(A3)}$$

Then, the functions $U_\pm(x_{01}, x_{02}, \ldots, x_{0m-1}, z_0, \tau_0)$, defined as

$$U_\pm(x_{01}, x_{02}, \ldots, x_{0m-1}, z_0, \tau_0) = e^{-i(m-2)(z_0 \mp \tau_0)} \prod_{j=1}^{m-1} U_j^\pm(x_{0j}, z_0, \tau_0);$$

$$\begin{aligned}
U_j^\pm(x_{0j}, z_0, \tau_0) \equiv{}& \frac{1}{(z_0 \pm \tau_0)^{1/2}} u\left(\frac{x_{0j}}{z_0 \pm \tau_0}, \frac{1}{2(z_0 \pm \tau_0)}\right. \\
&\left. \times \sum_{j=1}^{m-1} \left(x_{0j}^2 + z_0^2 - \tau_0^2 - 1\right), \frac{1}{2(z_0 \pm \tau_0)} \sum_{j=1}^{m-1} \left(x_{0j}^2 + z_0^2 - \tau_0^2 + 1\right)\right), \quad \text{(A4)}
\end{aligned}$$

are solutions to the $m + 1$-dimensional scalar wave equation (A1). As an illustrative example, we choose $u(x_{0j}, z_0, \tau_0) = \exp[i(z_0 + \tau_0)]$, $\forall\, j = 1, 2, \ldots, m-1$. It follows, then, that

$$U_j^-(x_{0j}, z_0, \tau_0) = \frac{1}{\sqrt{z_0 - \tau_0}} e^{i(z_0+\tau_0)} e^{i\frac{x_{0j}^2}{z_0 - \tau_0}}, \; \forall\, j = 1, 2, \ldots, m-1, \quad \text{(A5)}$$

and, by virtue of Eq. (A4),

$$U_-(x_{01},x_{02},\ldots,x_{0m-1},z_0,\tau_0) = e^{-i(m-2)(z_0+\tau_0)} \prod_{j=1}^{m-1} U_j^-(x_{0j},z_0,\tau_0)$$
$$= e^{i(z_0+\tau_0)} \frac{1}{(z_0-\tau_0)^{(m-1)/2}} \exp\left(i \frac{\sum_{j=1}^{m-1} x_{0j}^2}{z_0-\tau_0} \right). \tag{A6}$$

This is indeed, a solution to Eq. (A1). It should be noted, however, that the functions $U_j^-(x_{0j},z_0,\tau_0)$ in Eq. (A5) are solutions to the 2 + 1-dimensional scalar wave equation (A3). In terms of the conventional space-time variables, we introduce the functions

$$K_j^-(x_j,z,t) \equiv U_j^-(x_{0j},z_0,\tau_0)\Big|\{x_{0j}\to\beta x_j,\ z_0\to\beta(z-ia_j),\ \tau_0\to\beta ct\}$$
$$= \frac{1}{\sqrt{a_j+i(z-ct)}} e^{i\beta(z+ct)} e^{-\beta\frac{x_j^2}{a_j+i(z-ct)}}, \tag{A7}$$

The equality holds *modulo* a constant multiplier term. The free parameters a_j, arising from the complexification of z, are positive. The functions $K_j^-(x_j,z,t)$ are governed by the 2 + 1-dimensional scalar wave equation

$$\left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) K_j^-(x_j,z,t) = 0,\ \ j=1,2,\ldots,m-1. \tag{A8}$$

By analogy to Eq. (A6), a composite function is created as follows:

$$K_-(x_1,x_2,\ldots,x_{m-1},z,t) = e^{-i(m-2)(z_0+\tau_0)} \prod_{j=1}^{m-1} K_j^-(x_j,z,t)$$
$$= \prod_{j=1}^{m-1} \frac{1}{\sqrt{a_j+i(z-ct)}} e^{i\beta(z+ct)} e^{-\beta\frac{x_j^2}{a_j+i(z-ct)}}. \tag{A9}$$

This function satisfies the $m+1$-dimensional scalar wave equation

$$\left(\sum_{j=1}^{m-1} \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) K_-(x_1,x_2,\ldots,x_{m-1},z,t) = 0. \tag{A10}$$

Solutions to this equation of form given in Eq. (A9) have been reported by Kiselev and Perel [29] recently. Here, they have been derived as a direct application of the extended Bateman conformal transformations.

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