

NON-RELATIVISTIC SCATTERING: PULSATING INTERFACES

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Abstract—Scattering by pulsating objects is discussed. In the case of the pulsating cylinder, its surface vibrates time-harmonically in the radial direction. The formalism is based on first-order v/c relativistic approximations, and on the assumption that the ambient media are not affected by the mechanical motion of the interface. This is conducive to simpler and amenable approximations.

The cases analyzed display the modulation effect due to the mechanical motion at frequency Ω , creating new spectral components in the scattered wave, peaking at the sideband frequencies $\omega_{ex} \pm n\Omega$ around the excitation frequency. To put such phenomena in a *quasi*-relativistic and electromagnetic context, and account for the boundary-condition problem and the representation of the scattered wave is the subject of the present investigation.

Such effects can be used to remotely sense the properties of the scatterer, especially its motion.

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1. INTRODUCTION

In a series of articles [1–4], the non-relativistic, or *quasi*-relativistic, theory for scattering by moving objects and media has been developed. Insofar as for some simple cases the results can be compared to exact special-relativistic results [5, 6], the new model is consistent with Special Relativity within the first-order approximation in v/c .

Presently the problem of harmonically pulsating surfaces is investigated. It was mentioned before [4, 7], that because of the varying velocity, the classical Special-Relativistic Lorentz transformation [5, 6] becomes inadequate for cases involving varying velocity, hence an appropriate generalization is needed. We use a *quasi*-Lorentzian transformation that takes into account the velocity-dependent kinematics in question

$$\begin{aligned} \mathbf{r}_T &= \mathbf{r} - \int_{\mathbf{R}_0}^{\mathbf{R}} \mathbf{v}(\bar{\mathbf{R}}) d\bar{\mathbf{t}} \\ t_T &= t - c^{-2} \int_{\mathbf{R}_0}^{\mathbf{R}} \mathbf{v}(\bar{\mathbf{R}}) \cdot d\bar{\mathbf{r}} \end{aligned} \quad (1)$$

where in (1) superscript T denotes the reference-frame attached to the boundary, $\mathbf{R} = (\mathbf{r}, ict)$ is a quadruplet of spatiotemporal coordinates indicating a so-called event in the Minkowski space. The bar indicates the integration variable, which is subsequently suppressed, assuming that the integration variable can be identified from the context. In (1) we have path-independent line integrals in the Minkowski space, the velocity field is laminar, i.e., $\partial_{\mathbf{r}} \times \mathbf{v} = 0$ [4], hence the differentials of (1) yield

$$\begin{aligned} d\mathbf{r}_T &= d\mathbf{r} - \mathbf{v}(\mathbf{R}) dt \\ dt_T &= dt - \mathbf{v}(\mathbf{R}) \cdot d\mathbf{r}/c^2 \end{aligned} \quad (2)$$

which is immediately recognized as the first-order in v/c differential form of the global Lorentz transformation [5, 6].

The non-relativistic model also requires a relation between the spectral components

$$\begin{aligned} \mathbf{k}_T &= \mathbf{k} - \mathbf{v}(\mathbf{R})\omega/c^2 \\ \omega_T &= \omega - \mathbf{v}(\mathbf{R}) \cdot \mathbf{k} \end{aligned} \quad (3)$$

which is recognized as the first-order approximation in v/c for the relativistic Doppler effect and the Fresnel drag effect formula [1–4].

To the first-order in v/c manipulation of (3) yields

$$\mathbf{k}_T \cdot \mathbf{k}_T = \mathbf{k} \cdot \mathbf{k} - 2\mathbf{v} \cdot \mathbf{k}\omega/c^2, \quad k_T = k - \mathbf{v} \cdot \hat{\mathbf{k}}\omega/c^2$$

$$\begin{aligned}
 v_{ph,T}^{(1)} &= \omega_T/k_T = v_{ph}^{(1)} \left(1 + \beta_k^{(1)} (A^{(1)} - 1) \right) \\
 v_{ph}^{(1)} &= \omega/k, \quad \beta_k^{(1)} = \beta^{(1)} \hat{\mathbf{k}} \cdot \hat{\mathbf{v}}, \quad \beta^{(1)} = v/v_{ph}^{(1)}, \quad A^{(1)} = (v_{ph}^{(1)}/c)^2
 \end{aligned}
 \tag{4}$$

where in (4) superscript (1) correspond to medium $\{1\}$, the phase velocity $v_{ph}^{(1)}$ is associated with a plane wave observed in medium $\{1\}$ at-rest, and it is displayed how the phase velocity is modified in the presence of a moving medium.

For a plane wave propagating in the direction parallel to the velocity we have $\hat{\mathbf{k}} \cdot \hat{\mathbf{v}} = 1$, $\beta_k^{(1)} = \beta^{(1)}$. For propagation directions normal to the velocity, the effect vanishes. In free space $v_{ph,T}^{(1)} = v_{ph}^{(1)}$, hence $A^{(1)} = 1$ and once again the effect vanishes. Formula (4) will be heuristically exploited below for varying velocities as well.

Boundary conditions corresponding to the relativistically exact relations, appropriate for this class of problems, have been introduced before [1-4]

$$\begin{aligned}
 \hat{\mathbf{n}} \times \left(\mathbf{E}_{eff}^{(1)} - \mathbf{E}^{(2)} \right) &= 0, \quad \hat{\mathbf{n}} \times \left(\mathbf{H}_{eff}^{(1)} - \mathbf{H}^{(2)} \right) = 0 \\
 \mathbf{E}_{eff}^{(1)} &= \mathbf{E}^{(1)} + \mathbf{v} \times \mathbf{B}^{(1)}, \quad \mathbf{H}_{eff}^{(1)} = \mathbf{H}^{(1)} - \mathbf{v} \times \mathbf{D}^{(1)}
 \end{aligned}
 \tag{5}$$

where in (5) superscripts (1),(2), correspond to media $\{1\}, \{2\}$ respectively, and $\mathbf{E}_{eff}^{(1)}, \mathbf{H}_{eff}^{(1)}$ are the effective fields due to motion of medium $\{1\}$ when observed at the boundary, which is at-rest with respect to medium $\{2\}$. The unit vector $\hat{\mathbf{n}}$ is normal to the boundary, and to the first-order in v/c is not affected by the motion.

Unlike previous problems analyzed by this method, here we encounter local spatiotemporally-dependent velocities, e.g., radial in the case of a pulsating cylinder, rather than a uniform lineal motion. This introduces more complexity because a different *quasi*-Lorentz transformation must be assigned at each point on the scatterer.

Similarly to other problems tackled by this model [2-4], the objects are considered to move through the ambient medium without disturbing its mechanical flow, thus violating mechanical fluid-continuity. Consider for example a pulsating cylinder. It will be assumed that in spite of the boundary motion, both the external and the internal media are not compressed or rarified. Admittedly, taking into account the mechanical continuity at the boundary would improve the physical model, but at this stage we cannot solve such problems in general. Some interaction problems of this kind have been considered before [8,9]. In a limited sense, we can imagine cases where the boundary is porous, thus allowing the continuity of the flow, and yet electromagnetically acting as if we are dealing with a smooth surface.

Generally speaking, we have to some extent sacrificed physical reality for mathematical feasibility. The model is still correct for objects in free space (vacuum), and is expected to yield good approximations in the presence of very transparent ambient media, e.g., atmospheric gases.

2. PULSATING PLANE INTERFACE

By way of introducing the present model and the notation used throughout, the problem of the pulsating plane interface with normal incidence is briefly summarized. In this case we are dealing with global lineal motion, as done before [4].

The excitation plane wave, propagating in the ambient medium $\{1\}$, is characterized by material parameters $\varepsilon^{(1)}, \mu^{(1)}$

$$\begin{aligned} \mathbf{E}_{ex} &= \hat{\mathbf{x}}E_{ex}e^{i\theta_{ex}}, \quad \mathbf{H}_{ex} = \hat{\mathbf{y}}H_{ex}e^{i\theta_{ex}}, \quad E_{ex}/H_{ex} = (\mu^{(1)}/\varepsilon^{(1)})^{1/2} = \zeta^{(1)} \\ \theta_{ex} &= k_{ex}z - \omega_{ex}t, \quad k_{ex}/\omega_{ex} = (\mu^{(1)}\varepsilon^{(1)})^{1/2} = 1/v_{ph}^{(1)} \end{aligned} \quad (6)$$

The pulsating medium is terminated by a plane interface moving through medium $\{1\}$ according to

$$z_T = z - z_0 S_{\Omega t}, \quad S_{\Omega t} = \sin \Omega t \quad (7)$$

where in (7) z_T denotes the local coordinate system in which the boundary is at-rest. The interface is located at $z_T = Z$. The local origin moves according to $z_T = 0$. For any point $z_T = \text{const.}$, in particular $z_T = 0$, the associated velocity as observed from the original reference-frame of $\{1\}$ follows from (7) as

$$v(t) = dz/dt = v_0 C_{\Omega t}, \quad v_0 = z_0 \Omega, \quad C_{\Omega t} = \cos \Omega t \quad (8)$$

Substituting $z = z_0 S_{\Omega t}$ in (6) yields the phase at $z_T = 0$

$$\begin{aligned} \theta_{ex0} &= \theta_{ex} \Big|_{z_T=0} = k_{ex} z_0 S_{\Omega t} - \omega_{ex} t, \quad e^{i\theta_{ex0}} = \sum_n I_n e^{-i\omega_n t} \\ \omega_n &= \omega_{ex} - n\Omega, \quad I_n = J_n(k_{ex} z_0), \quad \Sigma_n = \sum_{n=-\infty}^{n=\infty} \end{aligned} \quad (9)$$

In (9) it is assumed that we have an array of instruments, at-rest in medium $\{1\}$, in which we read off the results at positions $z = z_0 S_{\Omega t}$, as a function of time t . From (2) it is clear that to the first-order in v/c we have $dt_T/dt = 1$, i.e., the exact relativistic time dilatation, which is known to be a second-order effect in v/c vanishes here. Therefore the same phase θ_{ex0} in (9) is also measured in terms of the native time t_T

by an observer attached to the position $z_T = 0$. Also note in (9) the representation of the exponential in terms of a series of Bessel functions (e.g., see [10], p. 372).

Now we compute for each frequency ω_n in (9) the phase shift from $z_T = 0$ to the scatterer location $z_T = Z$, using for each $\omega_T = \omega_n$ the appropriate phase velocity $v_{ph,T}^{(1)}$ given in (4). This yields

$$\begin{aligned} \theta_n &= k_n Z - \omega_n t \\ k_n Z &= \omega_n / v_{ph,T}^{(1)} \simeq k_n \left(1 - \beta_0^{(1)} (A^{(1)} - 1) C_{\Omega t} \right) \\ k_n &= \omega_n / v_{ph}^{(1)}, \quad \beta_0^{(1)} = v_0 / v_{ph}^{(1)} \end{aligned} \tag{10}$$

Furthermore, we have to include the amplitude effect prescribed by (5), amounting in the present case to a factor $1 - \beta_0^{(1)} C_{\Omega t}$. Also note that the Fresnel drag effect in (10) is of first-order in v/c , and the exponential can be approximated by its leading terms of the appropriate Taylor series expansion. Thus we obtain at the boundary

$$\begin{aligned} \mathbf{E}_{exT} &= \hat{\mathbf{x}} E_{exT}, \quad \mathbf{H}_{exT} = \hat{\mathbf{y}} H_{exT} = \hat{\mathbf{y}} E_{exT} / \zeta^{(1)} \\ E_{exT} &= E_{ex} \sum_n I_n e^{iK_n(1-\beta_0^{(1)}(A^{(1)}-1)C_{\Omega t})-i\omega_n t} \left(1 - \beta_0^{(1)} C_{\Omega t} \right) \\ &= \sum_n E_{ex;n} e^{-i\omega_n t} \\ E_{ex;n} &= E_{ex} \left(I_n e^{iK_n} + \beta_0^{(1)} \left(B_{n-1} I_{n-1} e^{iK_{n-1}} + B_{n+1} I_{n+1} e^{iK_{n+1}} \right) \right) \\ B_n &= \left(iK_n(1 - A^{(1)}) - 1 \right) / 2, \quad K_n = k_n Z \end{aligned} \tag{11}$$

where in (11) indices have been judiciously raised and lowered in order to end up with a spectrum of sidebands with frequencies ω_n . As a check on (11) consider the free-space case $A^{(1)} = 1$, for which the Fresnel drag effect vanishes and we get plane waves in free space in the excitation wave direction.

The internal medium $\{\mathcal{D}\}$ is assumed to be at-rest with respect to the interface, i.e., the medium and the interface are moving together. Of course, this implies that the medium is accelerated, but this aspect of the problem is considered negligible for practical examples. It follows that the internal field is a solution of the wave equation and must possess the same frequencies prescribed by (11)

$$\begin{aligned} \mathbf{E}_{in} &= \hat{\mathbf{x}} E_{in}, \quad \mathbf{H}_{in} = \hat{\mathbf{y}} H_{in} = \hat{\mathbf{y}} E_{in} / \zeta^{(2)}, \quad \zeta^{(2)} = (\mu^{(2)} / \varepsilon^{(2)})^{1/2} \\ E_{in} &= \sum_n E_{in;n} e^{\kappa_n z_T - i\omega_n t}, \quad \kappa_n / \omega_n = (\mu^{(2)} \varepsilon^{(2)})^{1/2} = 1 / v_{ph}^{(2)} \end{aligned} \tag{12}$$

where the coefficients $E_{in;n}$ in (12) have to be determined by the boundary conditions at $z_T = Z$.

The scattered (reflected) wave propagating in medium $\{1\}$ at-rest must be stipulated with the same spectral structure, hence we choose

$$\begin{aligned}
 \mathbf{E}_{sc} &= \hat{\mathbf{x}}E_{sc}, \quad \mathbf{H}_{sc} = -\hat{\mathbf{y}}H_{sc} = -\hat{\mathbf{y}}E_{sc}/\zeta^{(1)} \\
 E_{sc} &= \sum_{\nu} E_{sc;\nu} e^{-ik_{sc;\nu}z - i\omega_{sc;\nu}t} \\
 \omega_{sc;\nu} &= \omega_{ex} - \nu\Omega, \quad k_{sc;\nu}/\omega_{sc;\nu} = (\mu^{(1)}\epsilon^{(1)})^{1/2} = 1/v_{ph}^{(1)}
 \end{aligned}
 \tag{13}$$

Upon substituting $z = z_0 S_{\Omega t}$, (13) becomes at $z_T = 0$ a double sum

$$\begin{aligned}
 E_{sc} \Big|_{z_T=0} &= \sum_{\nu} E_{sc;\nu} e^{-ik_{sc;\nu}z_0 S_{\Omega t} - i\omega_{sc;\nu}t} \\
 &= \sum_{\nu\mu} E_{sc;\nu} e^{-i\omega_{sc;\nu-\mu}t} J_{\mu}(k_{sc;\nu}z_0) \\
 &= \sum_n e^{-i\omega_n t} E'_{sc;n} \\
 E'_{sc;n} &= \sum_{\nu} E_{sc;\nu} J_{\nu-n}(k_{sc;\nu}z_0), \quad \omega_n = \omega_{sc;\nu-\mu}
 \end{aligned}
 \tag{14}$$

where in (14) we have included a constraint $\nu - \mu = n$, so that frequencies at the boundary must coincide with the same frequencies ω_n prescribed by the excitation wave (11). The constraint amounts to a Kronecker delta function $\delta_{\mu;\nu-n}$, and the double summation collapses into a single summation.

Similarly to (11), we include the amplitude effect, expressed now by a factor $1 + \beta_0^{(1)} C_{\Omega t}$, where the sign change compared to (11) is due to the reversed direction of \mathbf{H}_{scT} in (5). Noting that in (4) now $\hat{\mathbf{k}}$ points in the opposite direction, i.e., compared to the excitation wave, the scattered wave now propagates in the opposite direction relative to the velocity, the phase shift from $z_T = 0$ to $z_T = Z$ is modified (cf. (11)) yielding

$$\begin{aligned}
 \mathbf{E}_{scT} &= \hat{\mathbf{x}}E_{scT}, \quad \mathbf{H}_{scT} = -\hat{\mathbf{y}}H_{scT} = -\hat{\mathbf{y}}E_{scT}/\zeta^{(1)} \\
 E_{scT} &= \sum_n E'_{sc;n} e^{-iK_n(1+\beta_0^{(1)}(A^{(1)}-1)C_{\Omega t}) - i\omega_n t} \left(1 + \beta_0^{(1)} C_{\Omega t}\right) \\
 &= \sum_n e^{-i\omega_n t} E_{scT;n} \\
 E_{scT;n} &= E'_{sc;n} e^{-iK_n} + \beta_0^{(1)} \left(B'_{n-1} E'_{sc;n-1} e^{-iK_{n-1}} + B'_{n+1} E'_{sc;n+1} e^{-iK_{n+1}} \right) \\
 B'_n &= \left(iK_n(1 - A^{(1)}) + 1 \right) / 2
 \end{aligned}
 \tag{15}$$

As a check, once again consider in (15) the free-space case $A^{(1)} = 1$, which shows that the Fresnel drag effect vanishes and we get simple reflected waves propagating in the reflection direction.

In a similar manner the associated magnetic fields are derived, and the coefficients are computed from the boundary conditions

$\mathbf{E}_{exT} + \mathbf{E}_{scT} - \mathbf{E}_{inT} = 0|_Z$, $\mathbf{H}_{exT} + \mathbf{H}_{scT} - \mathbf{H}_{inT} = 0|_Z$. Thus the boundary-value problem is considered as solved.

From the above analysis the characteristics of this class of problems emerge: We start with an excitation wave and derive its time-dependent phase at an arbitrary point, at-rest with respect to the boundary. The time signal in question is recast in a series (or in general that would lead to an integral) of harmonic spectral components. Then the phase shifts to points on the boundary are computed. The field amplitudes at the boundary are derived from the Lorentz force formulas or quasi-relativistic relations for the effective fields observed in the presence of motion (5). In the cases discussed here, first-order in v/c approximations further simplify the results, facilitating the computation of the pertinent scattering and transmission coefficients. As in (15), all results of such problems contain terms of first-order in v/c , in which coefficients can be exploited from the zero-order approximation, i.e., from the velocity-independent solution of the scattering problem, for the frequencies in question. Finally as in (15) it is typical for such problems to show interaction of terms of various orders. This has been observed for cases of uniform motion as well [1, 3], even in free space [11].

3. PULSATING CYLINDER: THE BOUNDARY-VALUE PROBLEM

In this example we consider a medium $\{1\}$ with given parameters $\varepsilon^{(1)}$, $\mu^{(1)}$, in which a circular cylinder, characterized by medium $\{2\}$ with parameters $\varepsilon^{(2)}$, $\mu^{(2)}$, is pulsating. Similarly to the plane interface problem, it is assumed that the motion does not disturb medium $\{1\}$, and the internal medium $\{2\}$ remains at-rest relative to the boundary.

The scatterer is chosen as a circular cylinder of quiescent radius $r_T = \mathfrak{R}$. We choose the center of the cylinder $\mathbf{r}_T = 0$ as the origin of the ensemble of local coordinate systems, relevant to various points on the boundary.

The interface moves radially through medium $\{1\}$ according to

$$\begin{aligned} r_T &= r - r_0 S_{\Omega t}, & C_\varphi &= \cos \varphi, & S_\varphi &= \sin \varphi \\ y_T &= -(r - r_0 S_{\Omega t}) S_\varphi = y - y_0 S_{\Omega t} \\ z_T &= (r - r_0 S_{\Omega t}) C_\varphi = z - z_0 S_{\Omega t} \\ \mathbf{r}_T &= \mathbf{y}_T + z_T = \hat{\mathbf{y}}(y - y_0 S_{\Omega t}) + \hat{\mathbf{z}}(z - z_0 S_{\Omega t}) = \mathbf{r} - \mathbf{r}_0 S_{\Omega t} \end{aligned} \quad (16)$$

where the vector expression in (16) is very simple, due to the choice of the origin. The angle φ is measured off the z -axis in a right

handed screw direction towards the *negative* direction of the y -axis. Inasmuch as the motion is radial the angles $\varphi = \varphi_T$ are identical, whether observed in the initial reference-frame \mathbf{r} , or from the boundary reference-frame \mathbf{r}_T . It is obvious from (16) that for each angle φ we have to use a different Cartesian coordinate transformation similar to (1), (7).

Similarly to (8) we now have at $\mathbf{r}_T = 0$, for each local coordinate system

$$\mathbf{v} = \hat{\mathbf{r}}dr/dt = \hat{\mathbf{r}}v_0C_{\Omega t}, \quad v_0 = r_0\Omega \tag{17}$$

displaying, for each point on the rim, the velocity of the associated local origin.

At $r_T = 0$ (16) prescribes $z = z_0S_{\Omega t}$, hence the phase of the incident wave at this point is given by (9), and in cylindrical coordinates we have

$$\begin{aligned} \theta_{ex0} &= \theta_{ex} \Big|_{r_T=0} = k_{ex}r_0C_{\varphi}S_{\Omega t} - \omega_{ex}t, \quad \varphi' = \varphi + \pi/2 \\ e^{i\theta_{ex0}} &= \Sigma_n I_n e^{-i\omega_n t}, \quad I_n = J_n(k_{ex}r_0C_{\varphi}) = J_n(k_{ex}r_0S_{\varphi'}) \end{aligned} \tag{18}$$

Inasmuch as the Bessel functions can be represented in terms of power series of the argument, it is clear that in (18) I_n is periodic in φ and φ' , with a period of 2π , hence it can be represented in terms of a Fourier series

$$\begin{aligned} I_n(\varphi') &= \Sigma_m I'_{nm} e^{im\varphi'} = \Sigma_m I_{nm} e^{im\varphi} \\ I_{nm} &= i^m I'_{nm}, \quad I'_{nm} = \frac{1}{2\pi} \int_{-\pi}^{\pi} I_n(\varphi') e^{-im\varphi'} d\varphi' \end{aligned} \tag{19}$$

In the present case the coefficients I_{nm}, I'_{nm} can be represented explicitly. We start with the integrals (see [12], referring the reader to [13])

$$\begin{aligned} \int_0^{\pi} \sin(2\mu\chi) J_{2\nu}(2a \sin \chi) d\chi &= \pi \sin(\mu\pi) J_{\nu-\mu}(a) J_{\nu+\mu}(a), \quad \text{Re}(\nu) > -1 \\ \int_0^{\pi} \cos(2\mu\chi) J_{2\nu}(2a \sin \chi) d\chi &= \pi \cos(\mu\pi) J_{\nu-\mu}(a) J_{\nu+\mu}(a), \quad \text{Re}(\nu) > -1/2 \end{aligned} \tag{20}$$

Choosing in (20) the stronger condition $\text{Re} \nu > -1/2$, defining $m = 2\mu, \varphi' = \chi, n = 2\nu, \text{Re}(n) > -1, a = k_{ex}r_0/2$, and adding and subtracting in (20) according to $C_{\gamma} \pm iS_{\gamma} = e^{\pm i\gamma}$, yields after some

manipulation

$$\begin{aligned} \int_0^\pi e^{im\varphi'} J_n(2aS_{\varphi'})d\varphi' &= \pi i^m J_{(n-m)/2}(a)J_{(n+m)/2}(a) \\ \int_0^\pi e^{-im\varphi'} J_n(2aS_{\varphi'})d\varphi' &= \pi i^{-m} J_{(n-m)/2}(a)J_{(n+m)/2}(a) \end{aligned} \tag{21}$$

We wish to adapt (21) and the integration limits to the Fourier series format (19), requiring a relation for Bessel functions with negative arguments and integer order (see for example [14]). This also requires formulas for negative integer n [15]

$$\begin{aligned} J_\alpha(ze^{i\beta\pi}) &= e^{i\alpha\beta\pi} J_\alpha(z), \quad \beta = 1 \\ J_n(-z) &= e^{in\pi} J_n(z) = (-1)^n J_n(z) \\ J_{-n}(z) &= (-1)^n J_n(z) \end{aligned} \tag{22}$$

Copying the second integral (21) and manipulating the first one now yields

$$\begin{aligned} \int_0^\pi e^{-im\varphi'} J_n(2aS_{\varphi'})d\varphi' &= \pi i^{-m} J_{(n-m)/2}(a)J_{(n+m)/2}(a) \\ \int_{-\pi}^0 e^{-im\varphi'} J_n(2aS_{\varphi'})d\varphi' &= (-1)^n \pi i^m J_{(n-m)/2}(a)J_{(n+m)/2}(a) \\ I_{nm} &= \frac{1}{2\pi} \int_{-\pi}^\pi e^{-im\varphi'} I_n(\varphi') = \frac{1}{2\pi} \int_{-\pi}^\pi e^{-im\varphi'} J_n(2a \sin \varphi')d\varphi' \\ &= [(-1)^n i^m + i^{-m}] J_{(n-m)/2}(a)J_{(n+m)/2}(a)/2, \quad a = k_{ex}r_0/2 \end{aligned} \tag{23}$$

where the second line (23) is obtained from the first line (21) by inverting the sign of the integration variable and adjusting the sign of the integral by interchanging the limits. Thus (23) takes into account positive and negative values of n, m . It can be easily verified that the last integral in (23) vanishes when $n \pm m$ is an odd integer, hence for I_{nm} with n, m integers, we are only dealing with integer order $(n \pm m)/2$ Bessel functions.

Returning to (18), we have to compute the phase at the rim $r_T = \Re$ of the cylinder. To that end we have to include the Doppler effect and Fresnel drag effect (3), and the resulting velocity $v_{ph,T}^{(1)}$ (4), for each frequency ω_n included in (11). The motion is radial, and the excitation wave propagates in the z -direction, therefore like $\hat{\mathbf{k}} \cdot \hat{\mathbf{v}}$ in (4), we have here $\beta_{0k}^{(1)} = \beta_0^{(1)} C_\varphi$. The analog of (10), including the

approximation of the exponential as used in (11), is therefore

$$\begin{aligned}
 \theta_{exnT} &= \mathbf{k}_{exnT} \cdot \hat{\mathbf{r}}\mathfrak{R} - \omega_n t = k_{exnT}\mathfrak{R}C_\varphi - \omega_n t \\
 k_{exnT} &= k_n \left(1 - \beta_{0k}^{(1)}(A^{(1)} - 1)C_{\Omega t} \right) \\
 \theta_{exnT} &= \theta_{exn} - K_n\beta_0^{(1)}(A^{(1)} - 1)C_\varphi^2 C_{\Omega t} \\
 \theta_{exn} &= K_n C_\varphi - \omega_n t, \quad K_n = k_n \mathfrak{R}, \quad k_n = \omega_n / v_{ph}^{(1)} \\
 e^{i\theta_{exnT}} &\simeq e^{i\theta_{exn}} \left(1 - \beta_0^{(1)} i K_n (A^{(1)} - 1) C_\varphi^2 C_{\Omega t} \right)
 \end{aligned} \tag{24}$$

The technique demonstrated in (24) will be used subsequently for plane waves propagating in arbitrary directions: First decompose the velocity into components parallel and normal with respect to direction of propagation. Then apply (4) with the parallel velocity component. Finally separate the velocity-dependent term and approximate the exponential, keeping only terms of first-order in v/c .

We also need to include the amplitude effect prescribed by (5), similarly to what has been done in (11). This amounts here to a factor $(1 - \beta_0^{(1)} C_{\Omega t} C_\varphi)$. Accordingly (cf. (11)) we have

$$\begin{aligned}
 \mathbf{E}_{exT} &= \hat{\mathbf{x}}E_{exT}, \quad \mathbf{H}_{exT} = \hat{\mathbf{y}}H_{exT} = \hat{\mathbf{y}}E_{exT}/\zeta^{(1)} \\
 E_{exT} &\simeq E_{ex}\Sigma_n I_n e^{i\theta_{exn}} \left(1 - \beta_0^{(1)} C_{\Omega t} C_\varphi \right) \left(1 - \beta_0^{(1)} i K_n (A^{(1)} - 1) C_\varphi^2 C_{\Omega t} \right) \\
 &\simeq E_{ex}\Sigma_n I_n e^{iK_n C_\varphi - i\omega_n t} \left(1 - \beta_0^{(1)} B_n (e^{i\Omega t} + e^{-i\Omega t}) \right) \\
 &= E_{ex}\Sigma_n e^{-i\omega_n t} \left(I_n e^{iK_n C_\varphi} - \beta_0^{(1)} \Sigma_{\sigma=n\pm 1} I_\sigma B_{\sigma\partial} e^{iK_\sigma C_\varphi} \right) \\
 &= E_{ex}\Sigma_{n\mu} e^{-i\omega_n t} e^{i\mu\varphi} i^\mu \left(I_n J_{n\mu} - \beta_0^{(1)} \Sigma_{\sigma=n\pm 1} I_\sigma B_{\sigma\partial} J_{\sigma\mu} \right) \\
 &= E_{ex}\Sigma_{nm\mu} e^{im\varphi - i\omega_n t} i^\mu \left(I_{n;m-\mu} J_{n\mu} - \beta_0^{(1)} \Sigma_{\sigma=n\pm 1} I_{\sigma;m-\mu} B_{\sigma\partial} J_{\sigma\mu} \right) \\
 J_{n\mu} &= J_\mu(K_n) \\
 B_n &= \left(iK_n(A^{(1)} - 1)C_\varphi^2 + C_\varphi \right) / 2 \\
 &= -i \left(K_n(A^{(1)} - 1)\partial_{K_n}^2 + \partial_{K_n} \right) / 2 = B_{n\partial}
 \end{aligned} \tag{25}$$

where in (25) $\Sigma_{\sigma=n\pm 1}$ means that only expressions with $\sigma = n + 1, \sigma = n - 1$ are considered, and indices have been judiciously raised and lowered.

In (25) the new differential operator $B_{n\partial}$ is defined by exploiting the relation

$$(C_\varphi + i\partial_{K_n})e^{iK_n C_\varphi} = 0 \tag{26}$$

We could also replace in (25) $I_n = J_n(k_{ex}r_0C_\varphi) = J_n(-ik_{ex}r_0\partial_{K_n})$ and recast J_n in differential operator power series, but this seems to complicate the result. The exponentials in (25) are recast in a Bessel function series, which requires us to add a summation over an index μ , as shown. Furthermore, in order to deal with constant coefficients, we substitute from (19), introducing another index. Thus we deal with triple infinite sums. In (25) indices have been shifted so that the summation is on a new index m , rather than $m + \mu$. This yields the same results because for a fixed μ both indices scan the range $-\infty$ to $+\infty$.

We have demonstrated that (25) is expressible in terms of a spectral orthogonal series of discrete frequencies ω_n , and a discrete spatial orthogonal series in terms of φ , facilitating the computation of the coefficients prescribed by the boundary conditions. Such forms will serve us below for the scattered and internal fields as well. The problem is therefore very complicated, and decisions on truncating the sums must be based on further investigation.

The internal field is a solution of the wave equation and must contain the frequencies prescribed by (25). Therefore at the boundary $r_T = \Re$ we have

$$\begin{aligned} \mathbf{E}_{inT} &= \hat{\mathbf{x}}E_{inT}, \quad E_{inT} = \sum_n E_{in;n} e^{-i\omega_n t} \\ E_{in;n} &= \sum_m i^m E_{in;nm} J_m(K_n) e^{im\varphi} \\ \kappa_n/\omega_n &= (\mu^{(2)}\varepsilon^{(2)})^{1/2} = 1/v_{ph}^{(2)}, \quad K_n = \kappa_n \Re \end{aligned} \tag{27}$$

The corresponding field \mathbf{H}_{inT} can be found directly from Maxwell's equations

$$\mathbf{H}_{inT} = \left(\hat{\mathbf{r}}_T \Re^{-1} \partial_\varphi - \hat{\varphi} \partial_\Re \right) \sum_n E_{in;n} e^{-i\omega_n t} / (i\omega_n \mu^{(2)}) \tag{28}$$

For evaluation of the boundary-value problem we need the component of (28) tangential to the surface, given by

$$\begin{aligned} \hat{\mathbf{r}}_T \times \mathbf{H}_{inT} &= -\hat{\mathbf{x}} \sum_n \kappa_n \partial_{\kappa_n \Re} E_{in;n} / (i\omega_n \mu^{(2)}) = \hat{\mathbf{x}} \sum_n i \partial_{K_n} E_{in;n} / \zeta^{(2)} \\ &= \hat{\mathbf{x}} \sum_{nm} i^{m+1} E_{in;nm} J'_m(K_n) e^{im\varphi} / \zeta^{(2)}, \quad K_n = \kappa_n \Re \end{aligned} \tag{29}$$

The coefficients $E_{in;nm}$ (27), (29) are to be derived from the solution of the boundary conditions equations at \Re , discussed below.

The scattered field is now constructed as a superposition of plane waves that satisfy the spatiotemporal conditions prescribed by the incident wave at the boundary. It is anticipated that each constituent plane wave of this superposition, propagating in an arbitrary direction

α , will include the new frequencies produced by the scattering at the pulsating boundary, therefore we choose (cf. (13))

$$\begin{aligned} \mathbf{E}_\alpha &= \hat{\mathbf{x}}E_\alpha, \quad \mathbf{H}_\alpha = \hat{\mathbf{k}}_\alpha \times \hat{\mathbf{x}}H_\alpha = \hat{\mathbf{k}}_\alpha \times \hat{\mathbf{x}}E_\alpha/\zeta^{(1)}, \quad E_\alpha = \sum_\nu E_{\alpha\nu} e^{i\theta_{\alpha\nu}} \\ \theta_{\alpha\nu} &= \mathbf{k}_{\alpha\nu} \cdot \mathbf{r} - \omega_\nu t = k_\nu r C_{\varphi-\alpha} - \omega_\nu t \\ \omega_\nu &= \omega_{ex} - \nu\Omega, \quad k_\nu/\omega_\nu = (\mu^{(1)}\varepsilon^{(1)})^{1/2} = 1/v_{ph}^{(1)} \end{aligned} \quad (30)$$

Similarly to (9), (14), (18), we first evaluate the phase of (30) at $r_T = 0$ according to (16) (cf. (9))

$$\begin{aligned} \theta_{\alpha\nu 0} &= \theta_{\alpha\nu} \Big|_{r_T=0} = k_\nu r_0 S_{\Omega t} C_{\varphi-\alpha} - \omega_\nu t \\ e^{i\theta_{\alpha\nu 0}} &= \sum_l J_{\alpha\nu l} e^{-i\omega_\nu + lt}, \quad J_{\alpha\nu l} = J_l(k_\nu r_0 C_{\varphi-\alpha}) \end{aligned} \quad (31)$$

Like I_n in (18), $J_{\alpha\nu l}$ in (31) is periodic in φ and can be recast in a Fourier series similarly to (19).

The time variation of all waves at the boundary must be identical, hence in (31) a constraint $\delta_{n;\nu+l}$ is prescribed, which for a constant n collapses the double summation into a single series. When summing over all n , we have again a double summation

$$\begin{aligned} E_{\alpha 0} &= E_\alpha \Big|_{r_T=0} = \sum_{n\nu} E_{\alpha\nu} e^{i\theta_{\alpha\nu 0}} = \sum_{\nu l} E_{\alpha\nu} J_{\alpha\nu l} e^{-i\omega_\nu + lt} \\ &= \sum_{\nu l} E_{\alpha\nu} J_{\alpha\nu l} e^{-i\omega_n t} = \sum_{\nu; n-\nu} E_{\alpha\nu} J_{\alpha\nu; n-\nu} e^{-i\omega_n t} \\ &= \sum_{n\nu} E_{\alpha n\nu} e^{-i\omega_n t} \\ E_{\alpha n\nu} &= E_{\alpha\nu} J_{\alpha\nu; n-\nu}, \quad J_{\alpha\nu; n-\nu} = J_{n-\nu}(k_\nu r_0 C_{\varphi-\alpha}) \end{aligned} \quad (32)$$

Another way of looking at it, as in (32), is to realize that if both n and ν are in the range $-\infty$ to $+\infty$, so does $n - \nu$.

Similarly to (24), the phase at the rim $r_T = \mathfrak{R}$ is computed for each frequency ω_n , essentially by replacing φ by $\varphi - \alpha$, and with the appropriate indexing

$$\begin{aligned} \theta_{\alpha n T} &= \mathbf{k}_{\alpha n T} \cdot \hat{\mathbf{r}}\mathfrak{R} - \omega_n t = k_{\alpha n T} \mathfrak{R} C_{\varphi-\alpha} - \omega_n t \\ k_{\alpha n T} &= k_n \left(1 - \beta_0^{(1)}(A^{(1)} - 1)C_{\varphi-\alpha}C_{\Omega t}\right) \\ \theta_{\alpha n T} &= \theta_{\alpha n} - K_n \beta_0^{(1)}(A^{(1)} - 1)C_{\varphi-\alpha}^2 C_{\Omega t} \\ \theta_{\alpha n} &= K_n C_{\varphi-\alpha} - \omega_n t, \quad K_n = k_n \mathfrak{R}, \quad k_n = \omega_n/v_{ph}^{(1)} \\ e^{i\theta_{\alpha n T}} &\simeq e^{i\theta_{\alpha n}} \left(1 - \beta_0^{(1)}iK_n(A^{(1)} - 1)C_{\varphi-\alpha}^2 C_{\Omega t}\right) \end{aligned} \quad (33)$$

Including the amplitude effect, we have similarly to (25)

$$\begin{aligned}
 \mathbf{E}_{\alpha T} &= \hat{\mathbf{x}} E_{\alpha T} \\
 E_{\alpha T} &= \sum_{n\nu} E'_{\alpha n\nu} e^{i\theta_{\alpha n T}} = \sum_{n\nu} E_{\alpha n\nu} \left(1 - \beta_0^{(1)} C_{\Omega t} C_{\varphi-\alpha} \right) e^{i\theta_{\alpha n T}} \\
 &\simeq \sum_{n\nu} E_{\alpha n\nu} e^{iK_n C_{\varphi-\alpha} - i\omega n t} \left(1 - \beta_0^{(1)} B_{\alpha n} \left(e^{i\Omega t} + e^{-i\Omega t} \right) \right) \quad (34) \\
 &= \sum_{n\nu} e^{-i\omega n t} \left(E_{\alpha n\nu} e^{iK_n C_{\varphi-\alpha}} - \beta_0^{(1)} \sum_{\sigma=n\pm 1} E_{\alpha\sigma\nu} B_{\sigma\partial} e^{iK_{\sigma} C_{\varphi-\alpha}} \right) \\
 B_{\alpha n} &= \left(iK_n (A^{(1)} - 1) C_{\varphi-\alpha}^2 + C_{\varphi-\alpha} \right) / 2 \\
 &= \left(-iK_n (A^{(1)} - 1) \partial_{K_n}^2 + \partial_{K_n} \right) / 2 = B_{n\partial}
 \end{aligned}$$

where in (34) it is noted that the differential operator $B_{n\partial}$ is the same as in (25), and independent of the index α . Only the coefficients $E_{\alpha n\nu}$, $E_{\alpha\sigma\nu}$ are dependent on α , cf. (32). Also note that (34), unlike (25), is left here in terms of exponential functions.

Now, a superposition (integral) of plane waves is constructed, and the integration path is chosen such that we get cylindrical functions associated with outgoing waves

$$\begin{aligned}
 \mathbf{E}_{scT} &= \hat{\mathbf{x}} \sum_{n\nu} e^{-i\omega n t} \frac{1}{\pi} \int E''_{\alpha n\nu} d\alpha \\
 E''_{\alpha n\nu} &= E_{\alpha n\nu} e^{iK_n C_{\varphi-\alpha}} - \beta_0^{(1)} \sum_{\sigma=n\pm 1} E_{\alpha\sigma\nu} B_{\sigma\partial} e^{iK_{\sigma} C_{\varphi-\alpha}} \quad (35) \\
 &= \sum_{m'} \left(E_{n\nu m'} e^{iK_n C_{\varphi-\alpha} + im'\alpha} - \beta_0^{(1)} \sum_{\sigma=n\pm 1} E_{\sigma\nu m'} B_{\sigma\partial} e^{iK_{\sigma} C_{\varphi-\alpha} + im'\alpha} \right) \\
 \mathbf{E}_{scT} &= \hat{\mathbf{x}} \sum_{n\nu m'} e^{-i\omega n t} i^{m'} e^{im'\varphi} \left(E_{n\nu m'} H_{nm'} - \beta_0^{(1)} \sum_{\sigma=n\pm 1} E_{\sigma\nu m'} B_{\sigma\partial} H_{\sigma m'} \right) \\
 \int &= \int_{\alpha=\varphi-(\pi/2)+i\infty}^{\alpha=\varphi+(\pi/2)-i\infty}, \quad E_{\alpha n\nu} = \sum_{m'} E_{n\nu m'} e^{im'\alpha}, \quad H_{nm'} = H_{m'}(K_n)
 \end{aligned}$$

In (35) $H_{m'}$ denotes the Hankel function of the first kind and order m' . Due to the fact that for bounded objects $E_{\alpha n\nu}$ is periodic in α , with a period of 2π , it can be represented as a Fourier series summed over m' with coefficients $E_{n\nu m'}$ independent of α . However, it must be noted that these coefficients are still dependent on φ through $J_{\alpha\nu;n-\nu}$, see (32), hence another Fourier expansion summed over m'' was effected in (35), yielding

$$\begin{aligned}
 \mathbf{E}_{scT} &= \hat{\mathbf{x}} \sum_{n\nu m' m''} e^{-i\omega n t} i^{m'} e^{i(m'+m'')\varphi} \\
 &\quad \cdot \left(E_{n\nu m' m''} H_{nm''} - \beta_0^{(1)} \sum_{\sigma=n\pm 1} E_{\sigma\nu m' m''} B_{\sigma\partial} H_{\sigma m''} \right) \\
 &= \hat{\mathbf{x}} \sum_{n\nu m m'} e^{im\varphi - i\omega n t} i^{m'} \quad (36)
 \end{aligned}$$

$$\cdot \left(E_{n\nu m'; m-m'} H_{nm'} - \beta_0^{(1)} \Sigma_{\sigma=n\pm 1} E_{\sigma\nu m'; m-m'} B_{\sigma\partial} H_{\sigma m'} \right)$$

$$E_{n\nu m'} = \Sigma_{m''} E_{n\nu m' m''} e^{im''\varphi}, \quad m'' = m - m'$$

A constraint $m' + m'' = m$ is necessary in (36) in order to exploit the orthogonality with respect to $e^{im\varphi}$ in (25), (27).

Thus (25), (27), (36) are all represented as orthogonal series in terms of ω_n , φ , facilitating the evaluation of the coefficients in the equations prescribed by the boundary conditions $\mathbf{E}_{exT} + \mathbf{E}_{scT} - \mathbf{E}_{inT} = 0|_{\mathfrak{R}}$.

Associated with the \mathbf{E} fields are \mathbf{H} fields, whose tangential component is continuous across the boundary thus prescribing the boundary condition $\hat{\mathbf{r}} \times (\mathbf{H}_{exT} + \mathbf{H}_{scT} - \mathbf{H}_{inT}) = 0|_{\mathfrak{R}}$. From (25)

$$\begin{aligned} \hat{\mathbf{r}} \times \mathbf{H}_{exT} &= \hat{\mathbf{r}} \times \hat{\mathbf{y}} E_{exT} / \zeta^{(1)} = -\hat{\mathbf{x}} C_\varphi E_{exT} / \zeta^{(1)} \\ &= \hat{\mathbf{x}} E_{ex} \Sigma_{nm\mu} e^{im\varphi - i\omega_n t} j_{\mu+1} \\ &\quad \cdot \left(I_{n; m-\mu} J'_{n\mu} - \beta_0^{(1)} \Sigma_{\sigma=n\pm 1} I_{\sigma; m-\mu} B_\sigma J'_{\sigma\mu} \right) / \zeta^{(1)} \quad (37) \\ J'_{n\mu} &= \partial_{K_n} J_\mu(K_n) \end{aligned}$$

where in (37) we have exploited (26), prescribing here $(C_\varphi + i\partial_{K_n})E_{exT} = 0$, where only the Bessel functions depend on K_n and are affected by the differential operator.

The corresponding expression for the internal field is already given by (29). From (5), (17), (30) we have

$$\begin{aligned} \hat{\mathbf{r}} \times \mathbf{H}_{\alpha T} &= \hat{\mathbf{r}} \times (\mathbf{H}_\alpha - \mathbf{v} \times \mathbf{D}_\alpha) = \hat{\mathbf{r}} \times \left(\hat{\mathbf{k}}_\alpha \times \hat{\mathbf{x}} H_\alpha - \hat{\mathbf{r}} v_0 C_{\Omega t} \times \hat{\mathbf{x}} \varepsilon^{(1)} E_\alpha \right) \\ &= E_\alpha \hat{\mathbf{r}} \times \left(\hat{\mathbf{k}}_\alpha \times \hat{\mathbf{x}} - \hat{\mathbf{r}} \beta_0^{(1)} C_{\Omega t} \times \hat{\mathbf{x}} \right) / \zeta^{(1)} = -\hat{\mathbf{x}} E_\alpha \left(C_{\varphi-\alpha} - \beta_0^{(1)} C_{\Omega t} \right) / \zeta^{(1)} \quad (38) \end{aligned}$$

The amplitude effect (38) must now replace the corresponding factor $1 - \beta_0^{(1)} C_{\Omega t} C_{\varphi-\alpha}$ in (34). This yields

$$\begin{aligned} \hat{\mathbf{r}} \times \mathbf{H}_{\alpha T} &= -\hat{\mathbf{x}} \Sigma_{n\nu} E_{\alpha n\nu} e^{i\theta_{\alpha n T}} \left(C_{\varphi-\alpha} - \beta_0^{(1)} C_{\Omega t} \right) / \zeta^{(1)} \\ &\simeq -\hat{\mathbf{x}} \Sigma_{n\nu} E_{\alpha n\nu} e^{i\theta_{\alpha n}} \left(C_{\varphi-\alpha} - \beta_0^{(1)} C_{\Omega t} \right) \\ &\quad \cdot \left(1 - \beta_0^{(1)} i K_n (A^{(1)} - 1) C_{\varphi-\alpha}^2 C_{\Omega t} \right) / \zeta^{(1)} \\ &\simeq -\hat{\mathbf{x}} \Sigma_{n\nu} E_{\alpha n\nu} e^{iK_n C_{\varphi-\alpha} - i\omega_n t} \\ &\quad \cdot \left(C_{\varphi-\alpha} - \beta_0^{(1)} P_{\alpha n} \left(e^{i\Omega t} + e^{-i\Omega t} \right) \right) \quad (39) \end{aligned}$$

$$\begin{aligned}
 &= \hat{\mathbf{x}}\Sigma_{n\nu}e^{-i\omega_n t} \left(E_{\alpha n\nu}i\partial_{K_n}e^{iK_n C_\varphi-\alpha} \right. \\
 &\quad \left. + \beta_0^{(1)}\Sigma_{\sigma=n\pm 1}E_{\alpha\sigma\nu}P_{\sigma\partial}e^{iK_\sigma C_\varphi-\alpha} \right) \\
 P_{\alpha n} &= \left(1 + iK_n(A^{(1)} - 1)C_{\varphi-\alpha}^3 \right) / 2 \\
 &= \left(1 - K_n(A^{(1)} - 1)\partial_{K_n}^3 \right) / 2 = P_{n\partial}
 \end{aligned}$$

As in (35), (36), we construct now $\hat{\mathbf{r}} \times \mathbf{H}_{scT}$ in the form

$$\hat{\mathbf{r}} \times \mathbf{H}_{scT} = \hat{\mathbf{x}}\Sigma_{n\nu}e^{-i\omega_n t} \frac{1}{\pi} \int E'''_{\alpha n\nu} d\alpha$$

$$\begin{aligned}
 E'''_{\alpha n\nu} &= E_{\alpha n\nu}i\partial_{K_n}e^{iK_n C_\varphi-\alpha} + \beta_0^{(1)}\Sigma_{\sigma=n\pm 1}E_{\alpha\sigma\nu}P_{\sigma\partial}e^{iK_\sigma C_\varphi-\alpha} \\
 &= \Sigma_{m'} \left(E_{n\nu m'}i\partial_{K_n}e^{iK_n C_\varphi-\alpha+im'\alpha} \right. \\
 &\quad \left. + \beta_0^{(1)}\Sigma_{\sigma=n\pm 1}E_{\sigma\nu m'}P_{\sigma\partial}e^{iK_\sigma C_\varphi-\alpha+im'\alpha} \right) \\
 \hat{\mathbf{r}} \times \mathbf{H}_{scT} &= \hat{\mathbf{x}}\Sigma_{n\nu m'}e^{-i\omega_n t} i^{m'} e^{im'\varphi} \tag{40} \\
 &\quad \cdot \left(E_{n\nu m'}i\partial_{K_n}H_{nm'} + \beta_0^{(1)}\Sigma_{\sigma=n\pm 1}E_{\sigma\nu m'}P_{\sigma\partial}H_{\sigma m'} \right) \\
 &= \hat{\mathbf{x}}\Sigma_{n\nu m m'}e^{im'\varphi-i\omega_n t} i^{m'} \left(E_{n\nu m';m-m'}i\partial_{K_n}H_{nm'} \right. \\
 &\quad \left. + \beta_0^{(1)}\Sigma_{\sigma=n\pm 1}E_{\sigma\nu m';m-m'}P_{\sigma\partial}H_{\sigma m'} \right)
 \end{aligned}$$

As in (36), the last two lines of (40) provide series which are orthogonal in terms of ω_n, φ , thus finally facilitating the solution of the boundary-value equations.

4. PULSATING CYLINDER: THE SCATTERED FIELD

With the boundary-value problem supposedly solved, we turn our attention to the scattered field in the initial reference-frame, where medium $\{1\}$ is at-rest. Let us review what was done: We dealt with plane waves whose phase was computed for an observer attached to the boundary, then included the amplitude effect, and finally constructed first-order approximations. See (11), (15), (25), and (34). These approximation, adequate for solving the boundary-value problem, are based on K_n being small, i.e., at small distances from the boundary. With this proviso, also t and t_T are interchangeable. These approximations cannot serve us now when the scattered wave is sought for arbitrary distances.

The problem of the pulsating plane interface is straightforward, because the scattered wave was already stated in (13). Once the boundary-value problem is solved, $E_{scT;n}$ in (15) are known, and $E'_{sc;n}$ can be computed, e.g., by using the velocity-independent $E'_{sc;n}$ in all terms already multiplied by $\beta_0^{(1)}$. Then from (14) $E_{sc;\nu}$ are found, thus the scattered wave (13) is available.

The problem of the pulsating circular-cylinder interface is more complicated. Here we started with individual plane waves in (30), and synthesized a field \mathbf{E}_{scT} , (35), (36), (40), at the boundary. Now we need to express the scattered field as observed in the initial reference-frame where medium $\{1\}$ is at-rest.

Returning to (2), (3), it is easy to verify that to the first-order in v/c the differential phase is conserved

$$d\theta = \mathbf{k} \cdot d\mathbf{r} - \omega dt \sim d\theta_T = \mathbf{k}_T \cdot d\mathbf{r}_T - \omega_T dt_T \quad (41)$$

This so-called principle of the invariance of the phase [4, 16], (41), becomes exact in Special-Relativity theory which involves inertial reference-frames moving at constant velocities. Therefore, taking off the amplitude effect from the waves in question, and avoiding the approximations of the velocity-dependent exponentials, yields the waves as measured by an observer at-rest with respect to medium $\{1\}$, but expressed in terms of \mathbf{r}_T, t_T coordinates.

Thusly for the pulsating cylindrical interface we return to (33), but for arbitrary distances r_T we obviate the approximation of the $\beta_0^{(1)}$ -dependent exponential by its Taylor-series leading terms. furthermore t_T is now used explicitly, yielding

$$\begin{aligned} \theta_{\alpha n T} &= \mathbf{k}_{\alpha n T} \cdot \mathbf{r}_T - \omega_n t_T = k_{\alpha n T} r_T C_{\varphi-\alpha} - \omega_n t_T \\ k_{\alpha n T} &= k_n \left(1 - \beta_0^{(1)} (A^{(1)} - 1) C_{\varphi-\alpha} C_{\Omega t_T} \right) \\ \theta_{\alpha n T} &= \theta_{\alpha n} - \bar{K}_n \beta_0^{(1)} (A^{(1)} - 1) C_{\varphi-\alpha}^2 C_{\Omega t_T} \\ \theta_{\alpha n} &= \bar{K}_n C_{\varphi-\alpha} - \omega_n t_T, \quad \bar{K}_n = k_n r_T, \quad k_n = \omega_n / v_{ph}^{(1)} \end{aligned} \quad (42)$$

where in (42), and according to the phase invariance (41), this is the same phase for an observer at-rest with respect to medium $\{1\}$, given in (30).

Instead of (34) we now discard the amplitude effect, hence (30) becomes

$$\mathbf{E}_\alpha = \hat{\mathbf{x}} E_\alpha, \quad E_\alpha = \Sigma_{n\nu} E_{\alpha n \nu} e^{i\theta_{\alpha n T}} \quad (43)$$

With the coefficients $E_{\alpha n \nu}$ supposedly known from the solution of the boundary-value problem.

Similarly to (35), the waves (43) are combined into a Sommerfeld-type plane-wave integral

$$\begin{aligned} \mathbf{E}_{sc} &= \hat{\mathbf{x}} \Sigma_{n\nu} e^{-i\omega_n t_T} \frac{1}{\pi} \int e^{i\bar{K}_n C_{\varphi-\alpha}} \bar{E}_{\alpha n \nu} d\alpha \quad (44) \\ \bar{E}_{\alpha n \nu} &= E_{\alpha n \nu} e^{-i\bar{K}_n \beta_0^{(1)} (A^{(1)}-1) C_{\varphi-\alpha}^2 C_{\Omega t_T}}, \quad E_{\alpha n \nu} = \Sigma_{m'} E_{n\nu m'} e^{im'\alpha} \end{aligned}$$

where in (44) $E_{n\nu m'}$ are already known from the solution of the boundary-value problem, hence to proceed, recast $\bar{E}_{\alpha n \nu}$ in a Fourier series and exploit the Sommerfeld integral representation to derive the solution as a series of Hankel functions

$$\begin{aligned} \mathbf{E}_{sc} &= \hat{\mathbf{x}} \Sigma_{n\nu} e^{-i\omega_n t_T} \frac{1}{\pi} \int e^{i\bar{K}_n C_{\varphi-\alpha}} \bar{E}_{\alpha n \nu} d\alpha \\ &= \hat{\mathbf{x}} \Sigma_{n\nu m} \bar{E}_{n\nu m} e^{-i\omega_n t_T} \frac{1}{\pi} \int e^{i\bar{K}_n C_{\varphi-\alpha}} e^{im\alpha} d\alpha \quad (45) \\ &= \hat{\mathbf{x}} \Sigma_{n\nu m} \bar{E}_{n\nu m} i^m e^{im\varphi - i\omega_n t_T} H_m(\bar{K}_n) \\ \bar{E}_{\alpha n \nu} &= E_{\alpha n \nu} e^{-i\bar{K}_n \beta_0^{(1)} (A^{(1)}-1) C_{\varphi-\alpha}^2 C_{\Omega t_T}} = \Sigma_m \bar{E}_{n\nu m} e^{im\alpha} \end{aligned}$$

where in (45) $\bar{E}_{n\nu m}$ are known functions, independent of, but still dependent on φ and t_T . Inasmuch as we do not seek orthogonal series and discrete frequency spectra, at this stage this is no impediment.

One can also recast the scattered wave in terms of an inverse-distance power series and a differential operator acting on α in $\bar{E}_{\alpha n \nu}$, and after the derivatives have been effected replacing α with φ . The details of this technique, based on Twersky's inverse-distance differential operators [17, 18], have been discussed before, see [4, 19] and need not be reiterated here.

Finally, the spatiotemporal transformation (2), (3), in the form (16) relevant to the present problem can be substituted, in order to derive the scattered wave in terms of the coordinates of an observer at-rest with respect to the initial medium $\{1\}$ at-rest.

5. CONCLUDING REMARKS

The present study is centered on the analysis of scattering of a plane electromagnetic wave by harmonically pulsating objects. The present model is based on a first-order in v/c *quasi*-Lorentzian transformation of coordinates, which also allows to use Doppler effect and Fresnel drag effect formulas for non-uniform velocity fields. As expected, the motion modulates the waves and a spectrum peaking at sideband frequencies $\omega_{ex} \pm n\Omega$ is created.

Inasmuch as the analysis is very complicated, the discussion is limited to two examples of harmonically pulsating interfaces moving through the ambient undisturbed medium: a plane interface with normally excitation plane wave, and a circular cylinder excited by a plane wave normal to the cylindrical axis.

The analysis deals with the boundary-value problems and with the representation of the scattered waves in the initial medium. It is typical of such problems that they lead to infinite sets of equations on the coefficients, where coefficients of various indices are involved in each equation. therefore. The details are quite complicated, leading to infinite series that will have to be appropriately truncated in order to derive appropriate expressions for numerical simulations.

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