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# A NOVEL MARCHING ALGORITHM FOR RADIO WAVE PROPAGATION MODELING OVER ROUGH SURFACES

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**Abstract**—In this paper, the parabolic approximation of wave equation will be solved by the method of least squares. At first, the radio wave propagation in homogeneous media will be considered. The electromagnetic field will be expanded by proper expansion functions, which satisfy the parabolic equation in homogeneous media. The expansion coefficients will be derived by the least square method through enforcing initial and boundary conditions. The least square functionals satisfy the initial and boundary conditions. Similar to the split step method, the field in the inhomogeneous media with known profile of refractive index can be obtained by proper phase shifting of the field in homogeneous media. The proposed method is more reliable than the split step method and can be applied over rough boundary without any excess computations. In comparison with the finite difference method, the proposed method is very fast.

## 1. INTRODUCTION

The parabolic equation method (PEM) as an effective algorithm for modeling radio wave propagation in the troposphere, has been used for many years. A proper solution for the wave equation is chosen and substituted in it and by neglecting the backward propagating term of the field, a parabolic approximation of the wave equation may be derived.

The finite difference (FD) and split step (SS) methods, usually have been used for the solution of PEM in the troposphere [1]. In the finite difference method, the derivatives of parabolic equation are replaced by appropriate difference equations and the partial differential equation is changed into a matrix equation. In the split step method, the effect of refractive index is separated from the effect of surfaces. The effect of refractive index is implemented by proper phase shift of the field in homogeneous medium.

Recently, the finite element method (FEM) has been employed for the solution of parabolic equations [2]. A linear dependence for the range variation of electromagnetic field is assumed. With such assumptions, the parabolic partial differential equation(PDE) is reduced to an ordinary differential equation(ODE). FEM has been used for the solution of derived ODE.

Numerical investigations of several authors show that,in comparison with FD method, the SS method requires larger height and range for treating the wave propagation in the troposphere [3]. But FD is flexible and can evaluate the field over boundaries with any roughness profile [3, 4]. Linear function assumption for range variation of electromagnetic fields in FEM, is far from the exact field variation, which is an exponential function for homogeneous media. Consequently, accurate numerical field calculation requires small range steps. This process increases the required memory resources, computation tasks and slows down the algorithm.

In this paper, a least square method (LSM) using the marching algorithm will be presented for modeling radio wave propagation in the troposphere. This algorithm is quite reliable for the radio wave propagation problem in the troposphere with any profile of refractive index. The basis functions used in this algorithm are derived by the solution of parabolic equation in homogeneous media. With due regard to small variations of refractivity profile of troposphere, in comparison with homogeneous media, it is expected that longer range steps may be used for field computations.

In the second section of the paper, the method of least squares for the solution of PE problem is formulated.The proposed algorithm tries, using the advantages of the aforementioned algorithms simultaneously. Similar to the split step method, exponential basis functions have been used for the field expansion. Consequently, in comparison with the finite difference and finite element methods, less range and height resolutions for the field sampling are required. Also the FFT algorithm can be used for the acceleration of calculations. Boundary conditions can be exerted by a simple weighting summation of boundary condition dependent terms with the initial term dependent functional. Only a diogonal matrix, i.e., $\underline{\underline{S}}$, must be changed, of which the elements can readily be calculated by the Neumann expansion. Unlike the split step method it is not mandatory to apply the Fresnel-Kirchhoff approximation of initial field and similar to the finite difference method it can be treated directly and without any approximation. At first the

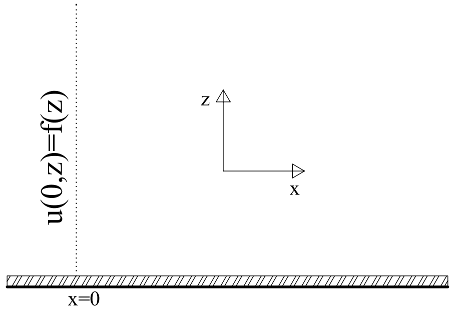


**Figure 1.** Geometry of problem.

radio wave propagation over the flat earth is considered (as shown in Fig. 1) and LSM algorithm for this geometry will be developed. Then, the algorithm will be extended for rough surfaces.

In the third section, the least square functionals will be constructed. By minimization of these functionals, the field in troposphere will be calculated. In the fourth section, the algorithm is extended for calculation of the field over rough surfaces. Finally, the results of LSM are compared with those of other methods.

## 2. PARABOLIC EQUATION METHOD

The radio wave propagation modeling requires the solution of wave equation with proper boundary conditions.

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2} + k^2 n^2 \varphi = 0 \tag{1}$$

where the field is considered $y$ independent, $k$ and $n$ are the propagation constant and refractive index, respectively, $\varphi$ is either $E_y$ or $H_y$ depending on the polarization of wave. We define $u$ as follows [4]:

$$u(x, z) = \varphi(x, z) e^{-jkx} \tag{2}$$

By substituting (2) in (1) and some approximations, a partial differential equation for u can be extracted.

$$\frac{\partial^2 u}{\partial z^2} + 2jk \frac{\partial u}{\partial x} + k^2 \left(n^2 - 1\right) u = 0 \tag{3}$$

In this approximation we follow reference [4]. It should be noted that in the parabolic equation method, it is assumed that EM fields are concentrated around the horizon. Hence, in Eq. (2), the function $u$ as the amplitude of $\varphi$ is defined in a such a way as to have less variations with respect to horizon (namely, $x$). Consequently, its second derivative with respect to $x$ may be neglected in comparison with its first derivative.Obviousely, it causes some errors in the parabolic equation method for angles far from the horizon. Several methods have been proposed in order to overcome this problem. A novel algorithm for radio wave propagation far from the horizon is introduced in [5].

In this paper, similar to the split step method, the effect of refractive index and boundary conditions are considered separately. The field variation with respect to height ($z$) at $x = 0$ is assumed as $f(z)$ (see Fig. 1).

At first, the problem for the homogeneous medium ($n = 1$) is considered.

$$\begin{aligned} \frac{\partial^2 u}{\partial z^2} + 2jk\frac{\partial u}{\partial x} &= 0 \\ u(0,z) &= f(z) \\ \frac{\partial u(x,0)}{\partial z} + \beta u(x,0) &= 0 \end{aligned} \tag{4}$$

where the latter two equations are the initial and boundary conditions, and $\beta$ depends on the electrical properties of ground (i.e., conductivity and dielectric constant) and frequency. For radio wave propagation beyond VHF band,the skin depth of radio wave is small in comparison with the wavelength. Therefore, considering the earth as a perfect conductor does not cause any substantial error on the radio wave propagation modeling. In these frequency bands and for small grazing angles, it is good approximation to assume that impedance of earth is infinite for both horizontal and vertical polarization [4, p. 24]. Consequently only Dirichlet boundary condition (namely $u(x,0) = 0$ ) must be applied for field calculation. Our simulation experiments show that this algorithm can only consider the Dirichlet boundary condition. Therefore, the algorithm can be applied for both polarizations beyond the VHF band. This is due to the choice of basis functions. Our investigations with another basis function (i.e., sub-domain $2^{nd}$ order spline basis function), shows that LSM can consider the Neumann and hybrid boundary conditions [6].

Then, by multiplying the computed field by $e^{-jk\frac{n^2-1}{2}}$, the effect of troposphere is applied [3].

We truncate the upper height of problem domain at a fixed finite height $(z = h)$. For $u$ we consider the following solution:

$$u(x,z) = \sum_{m=-M}^{M-1} a_m^* e^{-\frac{1}{j2k}\left(\frac{2m\pi}{h}\right)^2 x} e^{j\frac{2m\pi}{h}z} \tag{5}$$

This equation satisfies the parabolic equation in homogeneous media. In order to satisfy the initial and boundary conditions the coefficients $a_m^*$ must be determined appropriately.

$$\sum_{m=-M}^{M-1} a_m^* e^{\frac{j2m\pi}{h}z} = f(z) \tag{6}$$

$$\sum_{m=-M}^{M-1} a_m^* e^{-\frac{1}{j2k}\left(\frac{2m\pi}{h}\right)^2 x} = 0 \tag{7}$$

## 3. THE LEAST SQUARE FUNCTIONAL

Assuming that the expansion functions satisfy the parabolic equations, the least square functional must be constructed in such a manner as to satisfy the initial and boundary conditions. Similar to the other solution methods of parabolic equation , its solution in a finite interval is considered. The least square functional is constructed directly for both the initial conditions and boundary conditions.

By defining the inner product and norm as

$$\begin{aligned} \langle v, w \rangle &= \frac{1}{h}\int_0^h v w^* dz \\ \|v\|^2 &= \langle v, v \rangle \end{aligned} \tag{8}$$

the least square functional can be written as follows

$$F = \|u(0,z) - f(z)\|^2 + \alpha \int_0^x |u(x,0)|^2 \, dx \tag{9}$$

where the $\alpha$ is a constant, which is chosen to equalize the effect of boundary and initial condition terms in the functional and $x$ is the known marching range. Equations (6) and (7) are substituted into Eq. (9) and by some mathematical manipulations, the functional can be written as

$$\begin{aligned} F &= \sum_{m=-M}^{M-1} |a_m|^2 + \|f(z)\|^2 - \sum_{m=-M}^{M-1} a_m^* f_m \\ &\quad - \sum_{m=-M}^{+M-1} f_m^* a_m + \alpha \sum_{m=-M}^{M-1} \sum_{n=-M}^{M-1} s_{m,n} a_m^* a_n \end{aligned} \tag{10}$$

where

$$f_m = \left\langle e^{j\frac{2m\pi}{h}z}, f(z) \right\rangle \tag{11}$$

and

$$s_{m,n} = \begin{cases} x & m = n \\ \frac{e^{vx}-1}{v} & m \neq n. \end{cases} \tag{12}$$

$$v = -\frac{1}{j2k}\left(\frac{2\pi}{h}\right)^2 \left(m^2 - n^2\right)$$

Matrix formulation of the functional facilitates its mathematical manipulation. We define the following matrices

$$\underline{a} = [a_{-M}, a_{-M+1}, ..., a_{M-1}]^T \tag{13}$$

$$\underline{f} = [f_{-M}, f_{-M+1}, ..., f_{M-1}]^T \tag{14}$$

$$\underline{\underline{S}} = [s_{n,m}] \tag{15}$$

where $-M \leq m, n \leq M - 1$ and $s_{n,m}$ may be obtained from(12). Finally, the functional(10) may be expressed in a closed form

$$F = \underline{a}^H \underline{a} - \underline{f}^H \underline{a} - \underline{a}^H \underline{f} + \alpha \underline{a}^H \underline{\underline{S}}\, \underline{a} + \|f(z)\|^2 \tag{16}$$

## 4. MINIMIZATION OF THE LS FUNCTIONAL

Minimization of the functional requires that its gradient become zero [7].

$$\nabla F = \frac{\partial F}{\partial \underline{a}^H} = \underline{a} - \underline{f} + \alpha \underline{\underline{S}}\, \underline{a} = 0 \tag{17}$$

Therefore, $\underline{a}$ can be evaluated as follows

$$\underline{a} = \left(\underline{\underline{I}} + \alpha \underline{\underline{S}}\right)^{-1} \underline{f} \tag{18}$$

where$\underline{\underline{I}}$ is the unit matrix. In the rest of the paper, we denote the expression $\left(\underline{\underline{I}} + \alpha \underline{\underline{S}}\right)^{-1}$ by a propagator matrix.

For small $\alpha$, Neumann expansion of $\left(\underline{\underline{I}} + \alpha \underline{\underline{S}}\right)^{-1}$ may be used for the calculation of the field [8].

$$\left(\underline{\underline{I}} + \alpha \underline{\underline{S}}\right)^{-1} = \underline{\underline{I}} + \sum_{n=1}^{\infty} \left(-\alpha \underline{\underline{S}}\right)^n \tag{19}$$

This approximation can reduce the order of computations from $O(M^3)$ to $O(M^2)$.

## 5. EXTENSION OF THE LSM ALGORITHM FOR RADIO WAVE PROPAGATION OVER ROUGH SURFACES

Rough surfaces with arbitrary profile of roughness may be approximated by a surface with staircase profile. Propagation modeling requires the satisfaction of boundary conditions over a terrain with non-zero height. Therefore, the expression for boundary conditions in Eq. (7) changes into

$$\sum_{m=-M}^{M-1} a_m^* e^{\frac{2m\pi}{h}h_t} e^{-\frac{1}{j2k}\left(\frac{2m\pi}{h}\right)^2 x} = 0 \tag{20}$$

where $h_t$ is the local terrain height with respect to a specified reference ground plane. The least square functional can be written as follows:

$$F = \underline{a}^H \underline{a} - \underline{f}^H \underline{a} - \underline{a}^H \underline{f} + \alpha \underline{a}^H \underline{\underline{S_t}}^* \underline{\underline{S}}\,\underline{\underline{S_t}}\,\underline{a} + \|f(z)\|^2 \tag{21}$$

where $\underline{\underline{S_t}}$ is a diagonal matrix with $e^{\frac{2m\pi}{h}h_t}(-M \leq m \leq M-1)$ as the diagonal entries.

Therefore, over rough terrains, $\underline{a}$ can be obtained as follows:

$$\underline{a} = \left(\underline{\underline{I}} + \alpha \underline{\underline{S_t}}^* \underline{\underline{S}}\,\underline{\underline{S_t}}\right)^{-1} \underline{f} \tag{22}$$

Furthermore, the Neumann expansion for calculation of propagator matrix changes into

$$\left(\underline{\underline{I}} + \alpha \underline{\underline{S_t}}^* \underline{\underline{S}}\,\underline{\underline{S_t}}\right)^{-1} = \underline{\underline{I}} + \underline{\underline{S_t}}^* \sum_{n=1}^{\infty} \left(-\alpha \underline{\underline{S}}\right)^n \underline{\underline{S_t}} \tag{23}$$

## 6. NUMERICAL RESULTS AND DISCUSSIONS

For the investigation of efficacy of LSM algorithm for radio wave propagation modeling, its application to and results for several surfaces with different roughness profiles are compared with those of the split step parabolic equation method.

Figures 2 to 4 compare the LSM computation of propagation loss over flat earth with those of SSPE for different range steps. All figures are plotted for Guassian initial field with 5° beam width. However, the effect of troposphere is neglected in these figures, although the refractive index can be readily considered by a proper phase shift in the field of homogeneous medium in each marching step. This phase

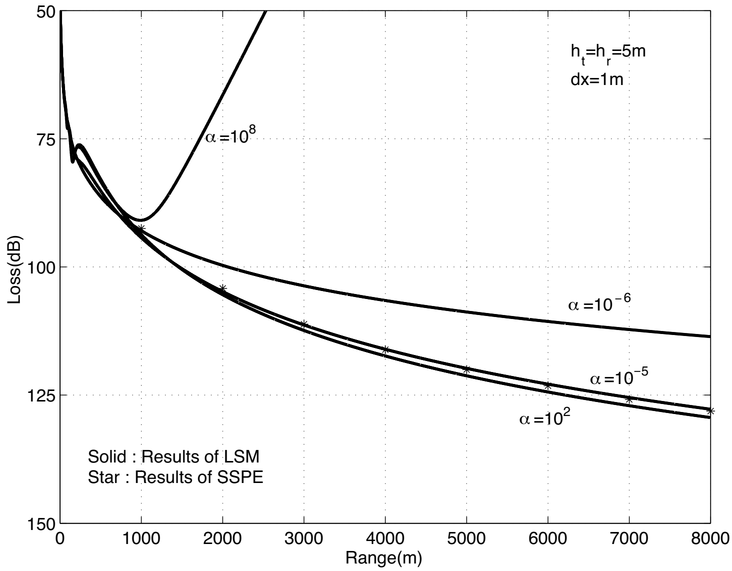


**Figure 2.** Computation of propagation loss over flat earth for several $\alpha$ and 1 m range step.

shift has been derived in references (for example in [4]) and is also shown in the second section. In each figure calculations are carried out for several values of $\alpha$.

It is seen that small values of $\alpha$ reduce the effect of boundary conditions on the LSM functional, which lead to their improper consideration.

For large range steps, deviation from the optimum values of $\alpha$, causes large errors in field calculation. For example, for $dx = 1\,\text{m}$ and $\alpha = 5 \times 10^{-4}$, the propagation loss coincides with the results of optimum value of $\alpha(\alpha = 10^{-4})$, while for $dx = 100\,\text{m}$, the propagation loss obtained for $\alpha = 5 \times 10^{-4}$ shows large errors.

Fig. 5 shows the minimum value of functional versus range for several range steps and the same weighting factor for boundary conditions ($\alpha = 5 \times 10^{-4}$). It is observed that for all range steps, the LSM functional curves show similar behaviour with different peak values. Similar behavior is observed for other values of $\alpha$, as it is seen in Fig. 6 for $\alpha = 100$. More investigations show that for values of $\alpha$, for which the algorithm becomes unstable, the minimum value of functional is large and the instability of the algorithm can be predicted from the functional curve. For example, see Fig. 7 for $\alpha = 10^9$ and range step of 1 m.

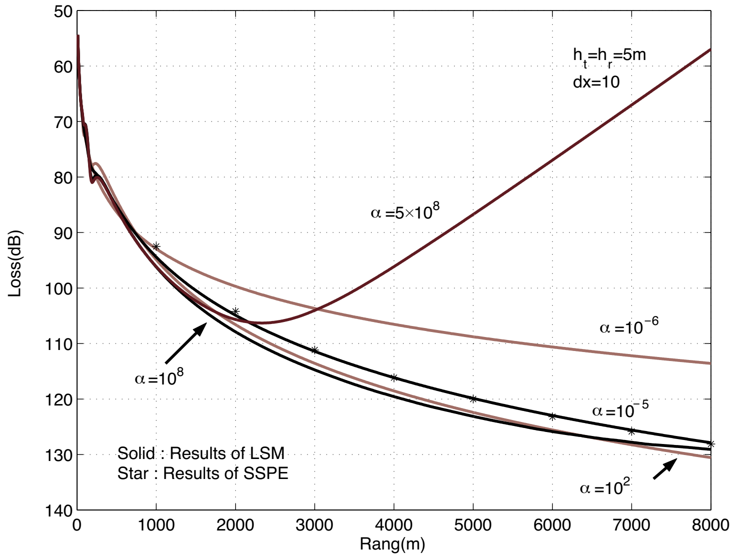


**Figure 3.** Computation of propagation loss over flat earth for several $\alpha$ and 10 m range step.

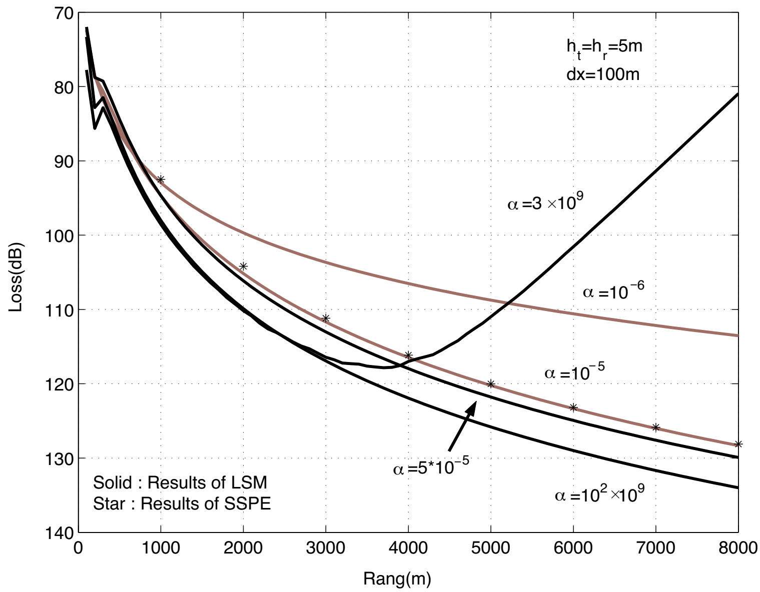


**Figure 4.** Computation of propagation loss over flat earth for several $\alpha$ and 100 m range step.

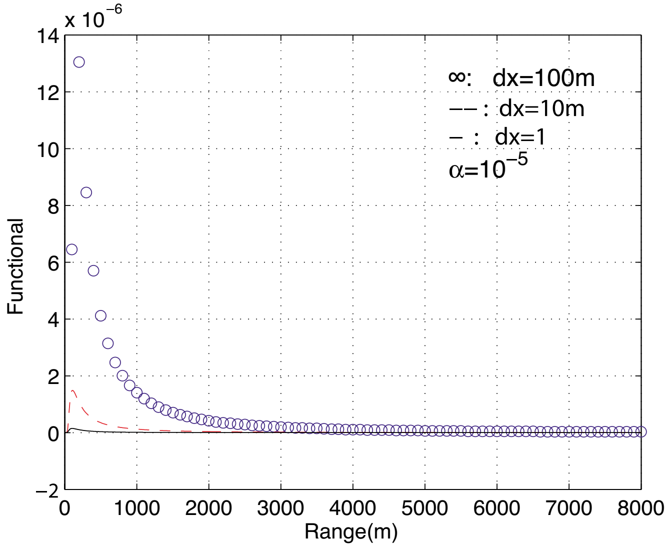


**Figure 5.** Minimum of functional versus range for $\alpha = 10^{-5}$ and several range steps.

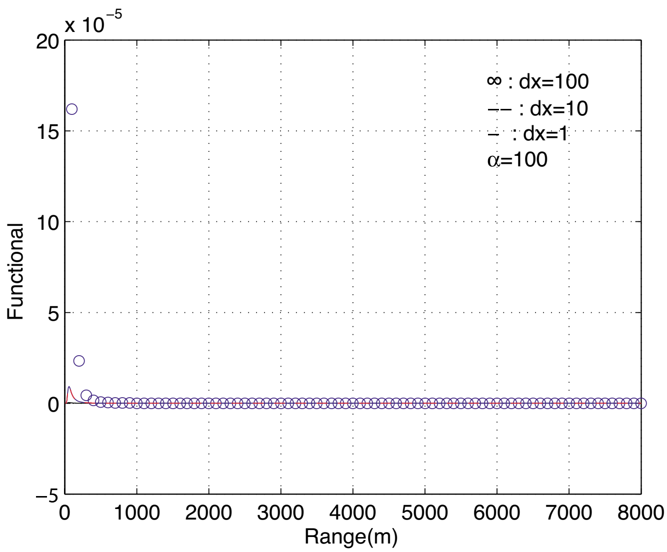


**Figure 6.** Minimum of Functional Versus range for $\alpha = 100$ and several range steps.

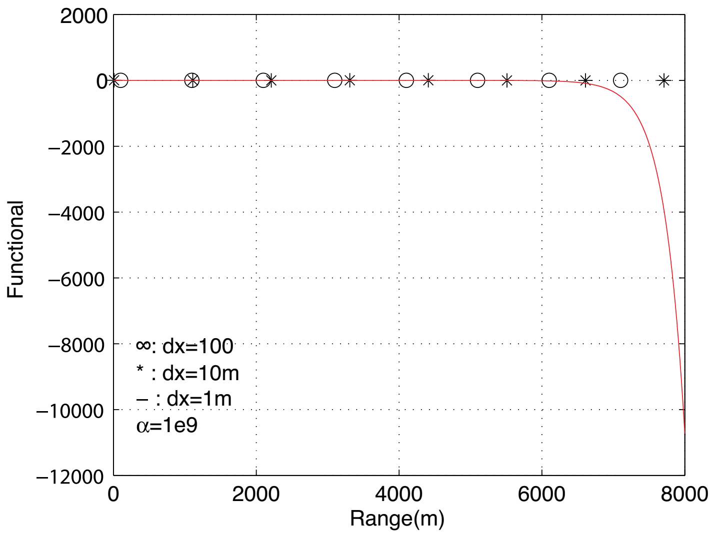


**Figure 7.** Minimum of functional versus range for $\alpha = 10^9$ and several range steps.

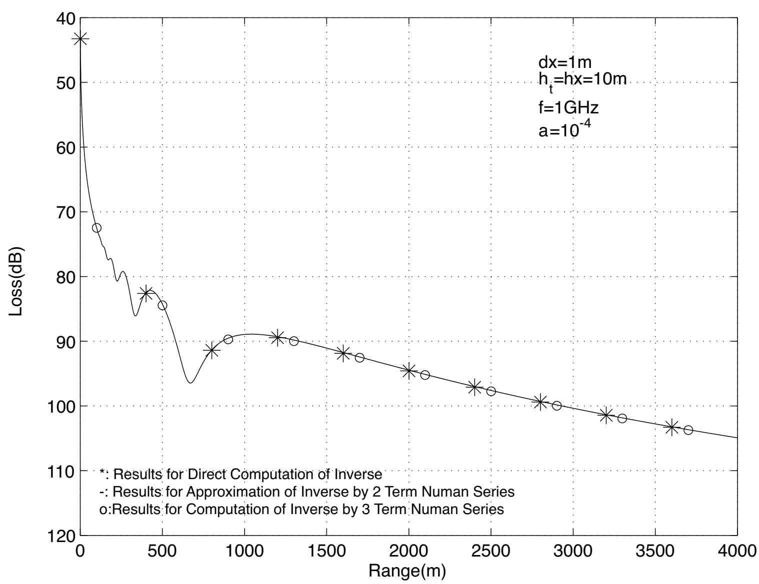


**Figure 8.** The effect of propagator matrix inversion method on propagation loss.

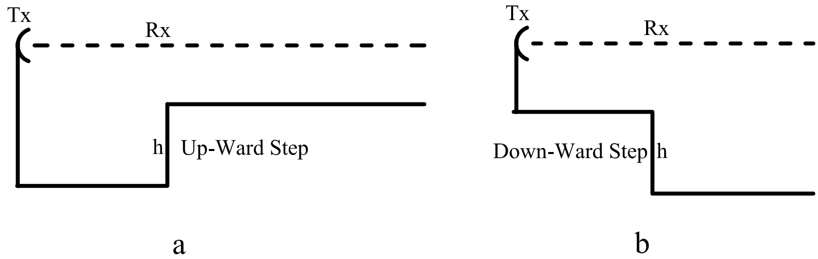


**Figure 9.** Geometry of (a) Up-ward and (b) Down-ward step.

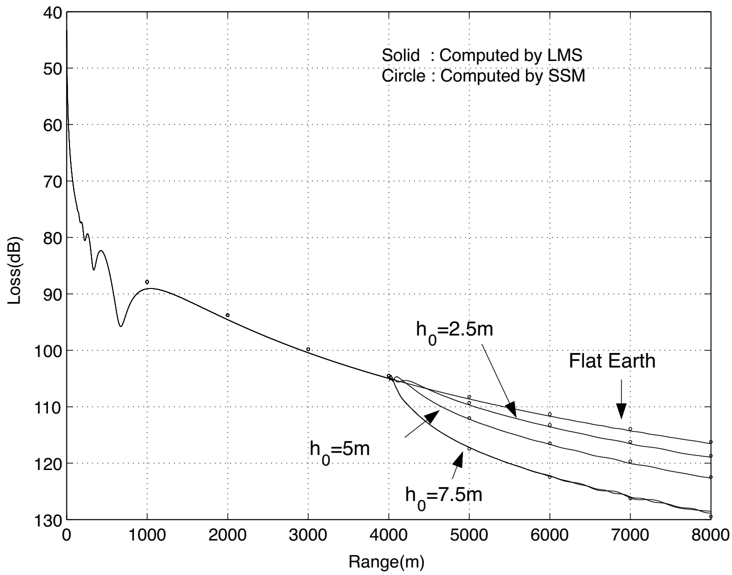


**Figure 10.** Propagation loss in the presence of up-ward step.

Large values of $\alpha$ cause ill-conditioning of the algorithm. For large $\alpha$, in comparison with $\alpha\underline{\underline{S}}$, the effect of unit matrix in Eq. (18) may be neglected. The condition number of $\alpha\underline{\underline{S}}$ is a monotonic increasing function of $\alpha$. For large values of $\alpha$, this condition number increases and causes ill conditioning in the computation of propagator matrix in Eq. (18). Therefore, the algorithm becomes ill-conditioned. Experiments show that, there is a finite range for which ill-conditioning of the algorithm shows its effect. This range depends on the number of marching steps and/or the step size. By increasing the step size, the range of ill-conditioning of the algorithm increases. Similar dependence

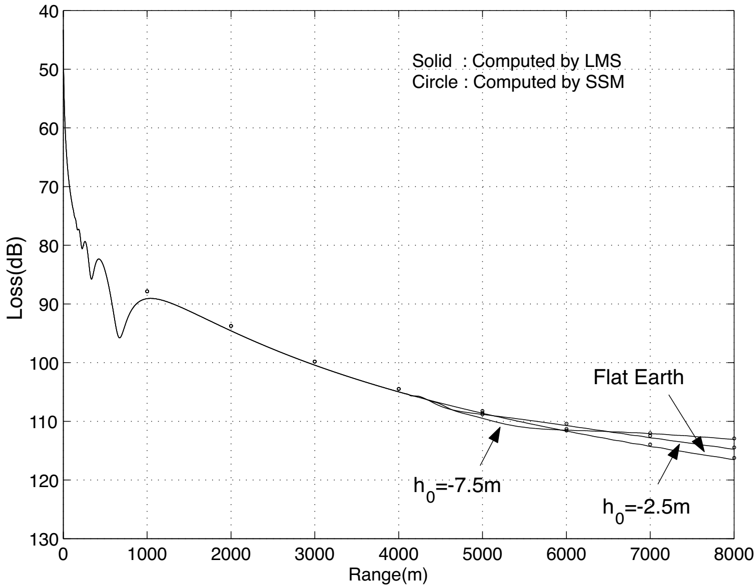


**Figure 11.** Propagation loss in the presence of down-ward step.

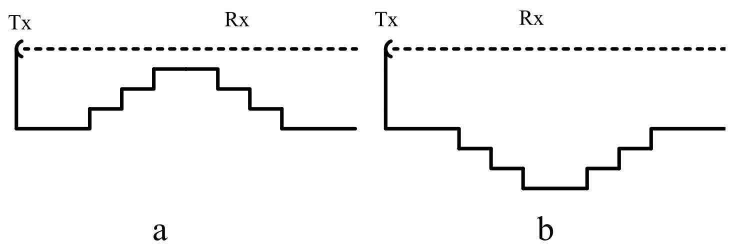


**Figure 12.** Complex profiles between transmitter and receiver. a) Variable height profile and b) Valley with variable depth profile.

for maximum value of $\alpha$ is observed(as shown in Figs. 2, 3 and 4).

It is seen that, in comparison with the diagonal entries of unit matrix ($\underline{\underline{I}}$), the maximum values of elements of $\alpha\underline{\underline{S}}$ (i.e., $\alpha x$) are very small. Therefore, it is expected that the Neumann series can be used for the approximate computation of the propagator matrix. Fig. 8 compares the propagation loss for direct and indirect computation of the propagator matrix. It is seen that only 2 terms of the Neumann series for propagator computation is sufficient.

Figures 10 and 11 show the propagation loss in the presence of

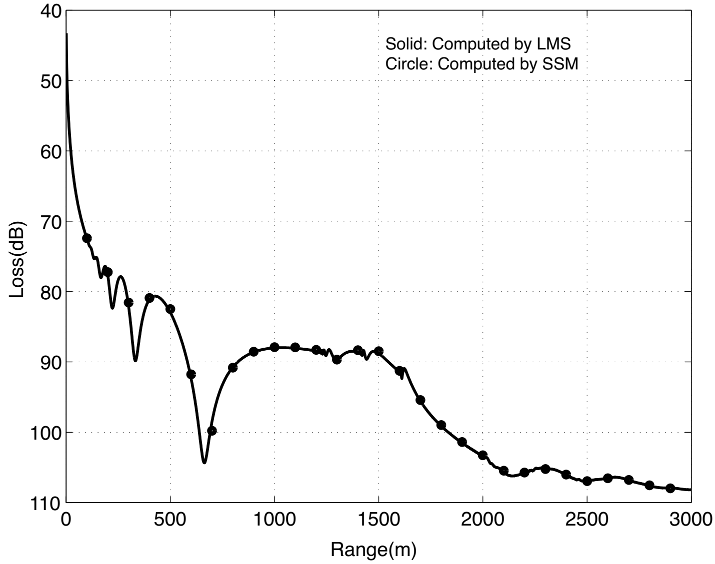


**Figure 13.** Propagation loss in the presence of complex obstacle.

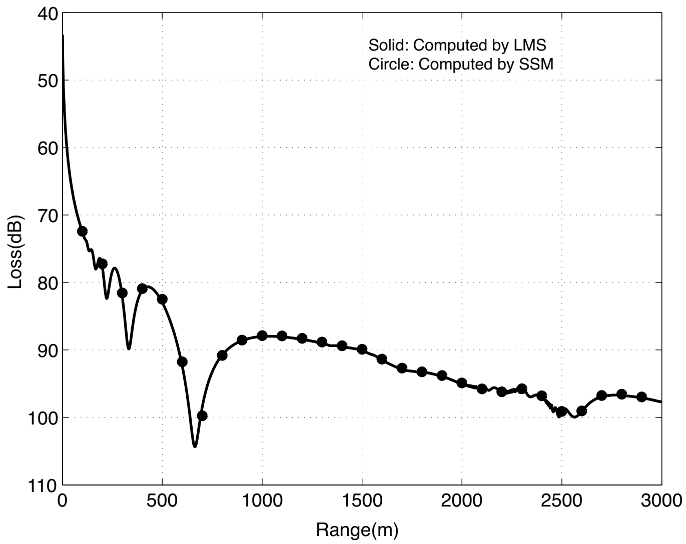


**Figure 14.** Propagation loss in the presence of valley with complex height profile.

up-ward and down-ward steps for several values of terrain heights. (The geometry of problem is shown at Fig. 9). It is seen that the propagation losses computed by the LSM algorithm exactly agree with those of SSPE using the Fresnel-Kirchhoff approximation for the initial field.

Figures 13 and 14 show the propagation loss in the presence of terrains with complex height profile. (The geometry of problem is shown in Fig. 12). Exact agreement between LSM and SSPE is seen for the propagation losses computed by these methods over complex terrains. It is noted that in the proposed algorithm, only the propagator matrix changes and no approximation is used for the initial field.

## 7. CONCLUSION

In this paper a new least square based marching algorithm for the solution of parabolic equations is developed. It is seen that the results of Least Square Method coincide with those of other methods for the solution of parabolic equations. Also FFT algorithm and Neumann expansion can be used to reduce the computation time. The least square algorithm can readily be extended for treating radio wave propagation over stepped boundaries. Therefore, it can be used for radio wave propagation over any step-wise approximation of rough terrains. However, to prevent ill-conditioning of the algorithm and to increase its accuracy, proper equalizing weighting constants must be multiplied by the term due to boundary conditions. Small equalizing constant can not consider the effect of boundary conditions properly, whereas large equalization constants cause ill conditioning of the algorithm.Experiments show that there is a suitable range of weighting constants, whereby stable and fast algorithm (through applying Neumann expansion for propagator) may be implemented. It is seen that the proposed algorithm can model radio wave propagation over rough terrains with complex height profiles. For propagation modeling over rough terrains, only the propagator matrix changes and no approximation is used for the initial field.

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