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RADIATION FROM DIELECTRIC COATED ELLIPTIC CONDUCTING CYLINDER BY ASSIGNED ELECTRIC CURRENT DISTRIBUTION

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Abstract—In this paper the analysis of radiation from known current distributions placed on a dielectric coated elliptic conducting cylinder has been effected. The analysis is carried out considering the expansion of the field as a series of Mathieu functions. The presence of the current is taken into account in the boundary conditions at the air-dielectric interface. The behavior of the radiated field as a function of the minor over major axis ratio of the elliptic conducting cylinder has been investigated and discussed.

1. INTRODUCTION

The analysis of antennas conformal to cylindrical surfaces is a challenging problem and it is particularly difficult when the structure is non-homogeneous, as it is for patch antennas printed on curved substrates. In this case the analysis is usually carried out by solving the electric field integral equation with the moments method [1, 2]. This technique has been proved to be very effective but it is conditioned by the knowledge of the appropriate Green's function which is expressed as a series of eigenfunctions. A limitation of the method stems from the fact that complete sets of eigenfunctions are available only for a few coordinate systems. Furthermore, the functions involved are very often difficult to treat. For this reason, only antennas printed on layered circular cylinders have been fully investigated.

As far as the most general case of cylindrical surfaces with non constant radius of curvature is concerned, results based on asymptotic techniques are available for radiation by sources placed on metallic surfaces [3–5]. Convex metallic surfaces were also modelled by choosing an appropriate coordinate system like in [6, 7] where radiating elements recessed in an elliptic metallic cylinder were analyzed with an hybrid full wave technique (BI-FEM). Conducting convex surfaces with a dielectric coating were also analyzed extending the results obtained for PEC surfaces with asymptotic techniques [8].

This paper proposes the study of the field radiated by a rectangular patch placed on a dielectric coated elliptic cylinder employing a modal expansion. Results on layered elliptic cylinder are already available but they are mainly related to the scattering by TM or TE plane waves or the radiation by 2D structures (see [10–12] and references therein). In the following, the fields in the Fourier space within and outside the dielectric layer will be expressed as a series of Mathieu functions with unknown coefficients. The presence of the electric current will be considered in the boundary conditions at the air-dielectric interface and the Galerkin method will be applied to obtain the unknown coefficients. The main difficulty arises from the properties of the Mathieu functions. In fact, as it is well known, Mathieu functions depend on the wave number which assumes different values in layers which have different dielectric constants. For this reason orthogonality between angular functions cannot be invoked when boundary conditions at the separation between two layers are imposed. As a result, the matrix relating electric field and current sources is no more block diagonal as it was in the circular case and it has to be numerically inverted. The radiation from a current flowing on patch antenna will be then considered. A guess for the current distribution will be obtained with the help of the cavity model, the approximation being valid, as the stored energy under the patch is great with respect to the radiated one [13]. Results of the radiated far fields relevant to the TM_{10} and TM_{01} modes will be presented and discussed.

2. THEORY

The geometry of the coated elliptic conducting cylinder is shown in Fig. 1. Here u and v are the radial and the angular coordinates, respectively, and d is the focal length. The contour of the conducting cylinder (placed in $u = u_0$) and the one of the dielectric layer (placed in $u = u_1$) will be considered to have the same focal length. This position can be considered as valid for thin dielectric layers and simplifies the

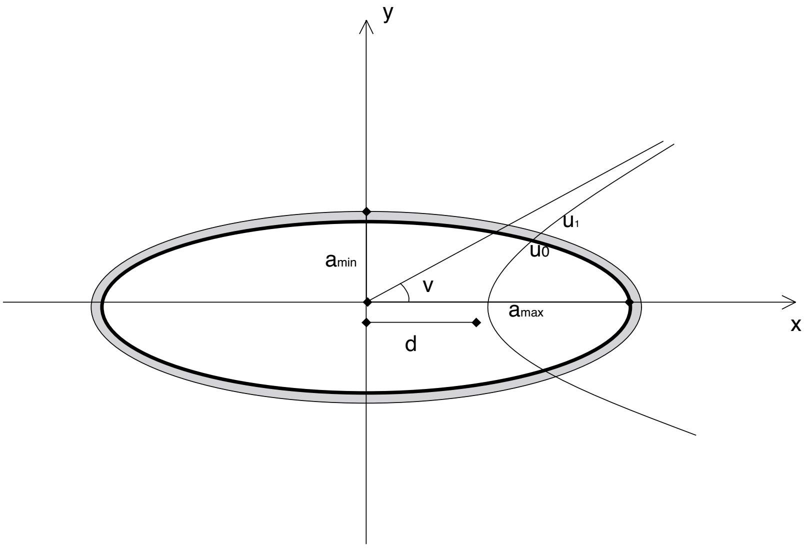

Figure 1. Elliptic coordinate system.

analytical treatment. In fact, a more general theory considering non confocal ellipses would require the use of the addition theorem for the Mathieu functions increasing the complexity of the mathematical developments [14]. In the following time-harmonic fields are assumed and the time dependance $e^{j\omega t}$ is suppressed. In order to analyze the radiation we have used electric and magnetic potentials Fourier transformed along z [15] and then expressed as a series of Mathieu functions. In the dielectric layer (Region I) one has

$$\begin{aligned}\tilde{\psi}_z^{M,1} &= \sum_{m=0}^{\infty} [e_{m,3}^{M,1} Mc_m^3(q_1,u) + e_{m,4}^{M,1} Mc_m^4(q_1,u)] ce_m(q_1,v) \\ &\quad + \sum_{m=1}^{\infty} [o_{m,3}^{M,1} Ms_m^3(q_1,u) + o_{m,4}^{M,1} Ms_m^4(q_1,u)] se_m(q_1,v)\end{aligned} \tag{1}$$

$$\begin{aligned}\tilde{\psi}_z^{E,1} &= \sum_{m=0}^{\infty} [e_{m,3}^{E,1} Mc_m^3(q_1,u) + e_{m,4}^{E,1} Mc_m^4(q_1,u)] ce_m(q_1,v) \\ &\quad + \sum_{m=1}^{\infty} [o_{m,3}^{E,1} Ms_m^3(q_1,u) + o_{m,4}^{E,1} Ms_m^4(q_1,u)] se_m(q_1,v)\end{aligned} \tag{2}$$

and in the exterior region (Region II)

$$\begin{aligned}\tilde{\psi}_z^{M,2} &= \sum_{m=0}^{\infty} e_{m,4}^{M,2} Mc_m^4(q_2,u)ce_m(q_2,v) \\ &\quad + \sum_{m=1}^{\infty} o_{m,4}^{M,2} Ms_m^4(q_2,u)se_m(q_2,v)\end{aligned} \tag{3}$$

$$\begin{aligned}\tilde{\psi}_z^{E,2} &= \sum_{m=0}^{\infty} e_{m,4}^{E,2} Mc_m^4(q_2,u)ce_m(q_2,v) \\ &\quad + \sum_{m=1}^{\infty} o_{m,4}^{E,2} Ms_m^4(q_2,u)se_m(q_2,v)\end{aligned} \tag{4}$$

Coefficients e and o are relevant to even and odd functions, the apexes M and E indicate coefficients related to the magnetic and electric potentials, apexes 1 and 2 are for quantities in the dielectric layer (Region I) and in the vacuum (Region II) and indexes indicate the order and the kind of the radial eigenfunctions. The dot on Mathieu functions indicates derivatives with respect to v or u variable. The parameters q_1 and q_2 are related to the transverse propagation constant k_z as follows

$$q_i = \frac{(k_i^2 - k_z^2)d^2}{4} \quad i \in \{1,2\} \tag{5}$$

where k_i is $k_1 = \omega\sqrt{\epsilon_0\epsilon_r\mu_0}$ in Region I and $k_2 = \omega\sqrt{\epsilon_0\mu_0}$ in Region II. The functions $Mc_m^3(q_i,u)$, $Mc_m^4(q_i,u)$, $Ms_m^3(q_i,u)$, $Ms_m^4(q_i,u)$ are even (c) and odd (s) Mathieu functions of the third and fourth kind expressing ingoing and outgoing elliptical waves, while $ce_m(q_i,v)$ and $se_m(q_i,v)$ are even and odd angular Mathieu functions [14]. The electric and magnetic field components in the spectral domain are derived from the potentials as follows [15]

$$\tilde{E}_u = \frac{-k_z}{\omega\epsilon h}\frac{\partial\tilde{\psi}_z^M}{\partial u} - \frac{1}{h}\frac{\partial\tilde{\psi}_z^E}{\partial v} \quad \tilde{E}_v = \frac{-k_z}{\omega\epsilon h}\frac{\partial\tilde{\psi}_z^M}{\partial v} + \frac{1}{h}\frac{\partial\tilde{\psi}_z^E}{\partial u} \tag{6}$$

$$\tilde{H}_u = \frac{1}{h}\frac{\partial\tilde{\psi}_z^M}{\partial v} - \frac{k_z}{\omega\mu h}\frac{\partial\tilde{\psi}_z^E}{\partial u} \quad \tilde{H}_v = \frac{-1}{h}\frac{\partial\tilde{\psi}_z^M}{\partial u} - \frac{k_z}{\omega\mu h}\frac{\partial\tilde{\psi}_z^E}{\partial v} \tag{7}$$

$$\tilde{E}_z = \frac{k^2 - k_z^2}{j\omega\epsilon}\tilde{\psi}_z^M \quad \tilde{H}_z = \frac{k^2 - k_z^2}{j\omega\mu}\tilde{\psi}_z^E \tag{8}$$

where $h = d\sqrt{\cosh^2 u - \cos^2 v}$. Inserting equations (1)–(4) into (6)–(8) one obtains the expressions of the fields in regions I and II reported below

Region I (dielectric layer)

$$\tilde{E}_v^1 = \frac{-k_z}{\omega\epsilon_1 h}\left[\sum_{m=0}^{\infty} T_m^{M,e}\dot{ce}_m(q_1,v) + \sum_{m=1}^{\infty} T_m^{M,o}\dot{se}_m(q_1,v)\right] + \frac{1}{h}\left[\sum_{m=0}^{\infty} \dot{T}_m^{E,e} ce_m(q_1,v) + \sum_{m=1}^{\infty} \dot{T}_m^{E,o} se_m(q_1,v)\right] \quad (9)$$

$$\tilde{E}_z^1 = \frac{k_1^2 - k_z^2}{j\omega\epsilon_1}\left[\sum_{m=0}^{\infty} T_m^{M,e} ce_m(q_1,v) + \sum_{m=1}^{\infty} T_m^{M,o} se_m(q_1,v)\right] \quad (10)$$

$$\tilde{H}_v^1 = \frac{-1}{h}\left[\sum_{m=0}^{\infty} \dot{T}_m^{M,e} ce_m(q_1,v) + \sum_{m=1}^{\infty} \dot{T}_m^{M,o} se_m(q_1,v)\right] - \frac{k_z}{\omega\mu_1 h}\left[\sum_{m=0}^{\infty} T_m^{E,e}\dot{ce}_m(q_1,v) + \sum_{m=1}^{\infty} T_m^{E,o}\dot{se}_m(q_1,v)\right] \quad (11)$$

$$\tilde{H}_z^1 = \frac{k_1^2 - k_z^2}{j\omega\mu_1}\left[\sum_{m=0}^{\infty} T_m^{E,e} ce_m(q_1,v) + \sum_{m=1}^{\infty} T_m^{E,o} se_m(q_1,v)\right] \quad (12)$$

with

$$T_m^{M,e} = e_{m,3}^{M,1} Mc_m^3(q_1,u) + e_{m,4}^{M,1} Mc_m^4(q_1,u) \quad (13)$$

$$T_m^{M,o} = o_{m,3}^{M,1} Ms_m^3(q_1,u) + o_{m,4}^{M,1} Ms_m^4(q_1,u) \quad (14)$$

$$T_m^{E,e} = e_{m,3}^{E,1} Mc_m^3(q_1,u) + e_{m,4}^{E,1} Mc_m^4(q_1,u) \quad (15)$$

$$T_m^{E,o} = o_{m,3}^{E,1} Ms_m^3(q_1,u) + o_{m,4}^{E,1} Ms_m^4(q_1,u) \quad (16)$$

$$\dot{T}_m^{E,e} = e_{m,3}^{E,1} \dot{M}c_m^3(q_1,u) + e_{m,4}^{E,1} \dot{M}c_m^4(q_1,u) \quad (17)$$

$$\dot{T}_m^{E,o} = o_{m,3}^{E,1} \dot{M}s_m^3(q_1,u) + o_{m,4}^{E,1} \dot{M}s_m^4(q_1,u) \quad (18)$$

$$\dot{T}_m^{M,e} = e_{m,3}^{M,1} \dot{M}c_m^3(q_1,u) + e_{m,4}^{M,1} \dot{M}c_m^4(q_1,u) \quad (19)$$

$$\dot{T}_m^{M,o} = o_{m,3}^{M,1} \dot{M}s_m^3(q_1,u) + o_{m,4}^{M,1} \dot{M}s_m^4(q_1,u) \quad (20)$$

Region II (vacuum)

$$\tilde{E}_v^2 = \frac{-k_z}{\omega\epsilon_2 h}\left[\sum_{m=0}^{\infty} e_{m,4}^{M,2} Mc_m^4(q_2,u)\dot{ce}_m(q_2,v) + \sum_{m=1}^{\infty} o_{m,4}^{M,2} Ms_m^4(q_2,u)\dot{se}_m(q_2,v)\right] + \frac{1}{h}\left[\sum_{m=0}^{\infty} e_{m,4}^{E,2} \dot{M}c_m^4(q_2,u) ce_m(q_2,v)\right.$$

$$+\sum_{m=1}^{\infty} o_{m,4}^{E,2}\dot{M}s_m^4(q_2,u)se_m(q_2,v)\Bigg] \tag{21}$$

$$\tilde{E}_z^2 = \frac{k_2^2-k_z^2}{j\omega\epsilon_2}\left[\sum_{m=0}^{\infty} e_{m,4}^{M,2}Mc_m^4(q_2,u)ce_m(q_2,v) + \sum_{m=1}^{\infty} o_{m,4}^{M,2}Ms_m^4(q_2,u)se_m(q_2,v)\right] \tag{22}$$

$$\tilde{H}_v^2 = \frac{-1}{h}\left[\sum_{m=0}^{\infty} e_{m,4}^{M,2}\dot{M}c_m^4(q_2,u)ce_m(q_2,v) + \sum_{m=1}^{\infty} o_{m,4}^{M,2}\dot{M}s_m^4(q_2,u)se_m(q_2,v)\right] - \frac{k_z}{\omega\mu_2 h}\left[\sum_{m=0}^{\infty} e_{m,4}^{E,2}Mc_m^4(q_2,u)\dot{ce}_m(q_2,v) + \sum_{m=1}^{\infty} o_{m,4}^{E,2}Ms_m^4(q_2,u)\dot{se}_m(q_2,v)\right] \tag{23}$$

$$\tilde{H}_z^2 = \frac{k_2^2-k_z^2}{j\omega\mu_2}\left[\sum_{m=0}^{\infty} e_{m,4}^{E,2}Mc_m^4(q_2,u)ce_m(q_2,v) + \sum_{m=1}^{\infty} o_{m,4}^{E,2}Ms_m^4(q_2,u)se_m(q_2,v)\right] \tag{24}$$

Applying the boundary condition on the surface of the inner elliptic conductor

$$\tilde{E}_z^1 = \tilde{E}_v^1 = 0 \quad in\ u = u_0 \tag{25}$$

and testing the field expressions with the functions $ce_m(q_1, v)$ and $se_m(q_1, v)$ one obtains

$$\tilde{E}_v^1 = \frac{-k_z}{\omega\epsilon_1 h}\left[\sum_{m=0}^{\infty} e_{m,3}^{M,1}\Omega_m^{M,e}\dot{ce}_m(q_1,v) + \sum_{m=1}^{\infty} o_{m,3}^{M,1}\Omega_m^{M,o}\dot{se}_m(q_1,v)\right] + \frac{1}{h}\left[\sum_{m=0}^{\infty} e_{m,3}^{E,1}\Psi_m^{E,e}ce_m(q_1,v) + \sum_{m=1}^{\infty} o_{m,3}^{E,1}\Psi_m^{E,o}se_m(q_1,v)\right] \tag{26}$$

$$\tilde{E}_z^1 = \frac{k_1^2-k_z^2}{j\omega\epsilon_1}\left[\sum_{m=0}^{\infty} e_{m,3}^{M,1}\Omega_m^{M,e}ce_m(q_1,v) + \sum_{m=1}^{\infty} o_{m,3}^{M,1}\Omega_m^{M,o}se_m(q_1,v)\right] \tag{27}$$

$$\tilde{H}_v^1 = \frac{-1}{h}\left[\sum_{m=0}^{\infty} e_{m,3}^{M,1}\Psi_m^{M,e} ce_m(q_1,v) + \sum_{m=1}^{\infty} o_{m,3}^{M,1}\Psi_m^{M,o} se_m(q_1,v)\right]$$
$$-\frac{k_z}{\omega\mu_1 h}\left[\sum_{m=0}^{\infty} e_{m,3}^{E,1}\Omega_m^{E,e} \dot{ce}_m(q_1,v) + \sum_{m=1}^{\infty} o_{m,3}^{E,1}\Omega_m^{E,o} \dot{se}_m(q_1,v)\right] \quad (28)$$

$$\tilde{H}_z^1 = \frac{k_1^2 - k_z^2}{j\omega\mu_1}\left[\sum_{m=0}^{\infty} e_{m,3}^{E,1}\Omega_m^{E,e} ce_m(q_1,v) + \sum_{m=1}^{\infty} o_{m,3}^{E,1}\Omega_m^{E,o} se_m(q_1,v)\right] \quad (29)$$

where

$$\Omega_m^{M,e} = Mc_m^3(q_1,u) - \alpha_m^{M,e} Mc_m^4(q_1,u) \quad (30)$$
$$\Omega_m^{M,o} = Ms_m^3(q_1,u) - \alpha_m^{M,o} Ms_m^4(q_1,u) \quad (31)$$
$$\Omega_m^{E,e} = Mc_m^3(q_1,u) - \alpha_m^{E,e} Mc_m^4(q_1,u) \quad (32)$$
$$\Omega_m^{E,o} = Ms_m^3(q_1,u) - \alpha_m^{E,o} Ms_m^4(q_1,u) \quad (33)$$
$$\Psi_m^{E,e} = \dot{M}c_m^3(q_1,u) - \alpha_m^{E,e} \dot{M}c_m^4(q_1,u) \quad (34)$$
$$\Psi_m^{E,o} = \dot{M}s_m^3(q_1,u) - \alpha_m^{E,o} \dot{M}s_m^4(q_1,u) \quad (35)$$
$$\Psi_m^{M,e} = \dot{M}c_m^3(q_1,u) - \alpha_m^{M,e} \dot{M}c_m^4(q_1,u) \quad (36)$$
$$\Psi_m^{M,o} = \dot{M}s_m^3(q_1,u) - \alpha_m^{M,o} \dot{M}s_m^4(q_1,u) \quad (37)$$

and

$$\alpha_m^{M,e} = \frac{Mc_m^3(q_1,u_0)}{Mc_m^4(q_1,u_0)} \quad \alpha_m^{M,o} = \frac{Ms_m^3(q_1,u_0)}{Ms_m^4(q_1,u_0)} \quad (38)$$
$$\alpha_m^{E,e} = \frac{\dot{M}c_m^3(q_1,u_0)}{\dot{M}c_m^4(q_1,u_0)} \quad \alpha_m^{E,o} = \frac{\dot{M}s_m^3(q_1,u_0)}{\dot{M}s_m^4(q_1,u_0)} \quad (39)$$

At the interface between the dielectric and the air the following continuity conditions are imposed

$$\left.\begin{array}{c} \tilde{E}_v^2 - \tilde{E}_v^1 = 0 \\ \tilde{E}_z^2 - \tilde{E}_z^1 = 0 \\ \tilde{H}_v^2 - \tilde{H}_v^1 = \tilde{J}_z \\ \tilde{H}_z^2 - \tilde{H}_z^1 = -\tilde{J}_v \end{array}\right\} \quad (40)$$

where the currents are expanded is terms of Mathieu functions as follows

$$\begin{aligned}\tilde{J}_z &= \sum_{m=0}^{\infty} \tilde{J}_{zm}^{e} ce_m(q_2, v) + \sum_{m=1}^{\infty} \tilde{J}_{zm}^{o} se_m(q_2, v) \\ \tilde{J}_v &= \sum_{m=0}^{\infty} \tilde{J}_{vm}^{e} ce_m(q_2, v) + \sum_{m=1}^{\infty} \tilde{J}_{vm}^{o} se_m(q_2, v)\end{aligned} \tag{41}$$

Galerkin's method is applied to conditions (40). Testing (40) with angular functions $ce_m(q_2, v)$ and $se_m(q_2, v)$, we obtain a relation between currents and expansion coefficients in a matrix form. The matrix equation is organized as follows

$$\tilde{\mathbf{M}}\mathbf{c}^2 = \tilde{\mathbf{T}}\mathbf{c}^1 + \tilde{\mathbf{J}} \tag{42}$$

where $\tilde{\mathbf{M}}$ is a matrix that accounts for the external field, $\tilde{\mathbf{T}}$ is a matrix accounting for the inner fields, $\mathbf{c}^2$ and $\mathbf{c}^1$ denote the vectors or column matrices of the expansion coefficients of the external and internal fields and, finally, $\tilde{\mathbf{J}}$ stands for the vector of the coefficients of the expansion of the electric current. Notice that, in the case of circular cylinders, Galerkin's method leads to a block diagonal matrix system in which the elements of the inverse matrix can be computed in an analytical form. For the present case, because of the well-known lack of orthogonality of the angular Mathieu functions with different eigen-numbers [14], the matrix system (42) is no more block diagonal as it was in the circular case and it must to be inverted numerically. In Table 1, the sizes of the matrices considering $2N+1$ expansion functions ($N+1$ even and N odd) are reported. The structure of the matrices and of the vectors in (42) can be organized in blocks as follows

$$\tilde{\mathbf{M}} = \begin{bmatrix} ME_{0,0}, ME_{0,1}, \ldots, ME_{0,N} \\ ME_{1,0}, ME_{1,1}, \ldots, ME_{1,N} \\ MO_{1,0}, MO_{1,1}, \ldots, MO_{1,N} \\ \ldots\ \ldots\ \ldots\ \ldots \\ ME_{N,0}, ME_{N,1}, \ldots, ME_{N,N} \\ MO_{N,0}, MO_{N,1}, \ldots, MO_{N,N} \end{bmatrix} \tag{43}$$

$$\tilde{\mathbf{T}} = \begin{bmatrix} TE_{0,0}, TE_{0,1}, \ldots, TE_{0,N} \\ TE_{1,0}, TE_{1,1}, \ldots, TE_{1,N} \\ TO_{1,0}, TO_{1,1}, \ldots, TO_{1,N} \\ \ldots\ \ldots\ \ldots\ \ldots \\ TE_{N,0}, TE_{N,1}, \ldots, TE_{N,N} \\ TO_{N,0}, TO_{N,1}, \ldots, TO_{N,N} \end{bmatrix} \tag{44}$$

Table 1. Matrices sizes of eq. (42).

Term	*Dimension*
$\tilde{\mathbf{T}}$	$(8N+4)\times(4N+2)$
$\tilde{\mathbf{M}}$	$(8N+4)\times(4N+2)$
$\tilde{\mathbf{J}}$	$(8N+4)\times 1$
$\mathbf{c}^1$	$(4N+2)\times 1$
$\mathbf{c}^2$	$(4N+2)\times 1$

$$\tilde{\mathbf{J}} = \begin{bmatrix} JE_0 \\ JO_0 \\ \dots \\ JE_N \\ JO_N \end{bmatrix} \mathbf{c}^1 = \begin{bmatrix} C_0^1 \\ C_1^1 \\ \dots \\ C_{N-1}^1 \\ C_N^1 \end{bmatrix} \quad \mathbf{c}^2 = \begin{bmatrix} C_0^2 \\ C_1^2 \\ \dots \\ C_{N-1}^2 \\ C_N^2 \end{bmatrix} \tag{45}$$

A detailed description of these matrix elements and of the coefficients is given in the appendix A. Since the odd radial eigenfunctions of zero order do not exists, the blocks belonging to the first column of the matrices $\tilde{\mathbf{M}}$ and $\tilde{\mathbf{T}}$ are of size (4×2) while all the others are (4×4). A similar reasoning is applied to the vectors $\mathbf{c}^1$, $\mathbf{c}^2$ and $\tilde{\mathbf{J}}$ for which the first block is (2×1) and the remaining are (4×1). To obtain a more direct relation between currents and radiated fields equation (42) is reconsidered in the following form

$$[\tilde{\mathbf{M}}(k_z), -\tilde{\mathbf{T}}(k_z)] \begin{bmatrix} \mathbf{c}^2 \\ \mathbf{c}^1 \end{bmatrix} = [\tilde{\mathbf{Q}}(k_z)] \begin{bmatrix} \mathbf{c}^2 \\ \mathbf{c}^1 \end{bmatrix} \tilde{\mathbf{J}}(k_z) \tag{46}$$

which can be conveniently recast as follows

$$\begin{bmatrix} \tilde{\mathbf{Q}}_{11} & \tilde{\mathbf{Q}}_{12} \\ \tilde{\mathbf{Q}}_{21} & \tilde{\mathbf{Q}}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{c}^2 \\ \mathbf{c}^1 \end{bmatrix} = \tilde{\mathbf{J}}(k_z) \tag{47}$$

From the block matrix equation (47) the coefficients of the field radiated in the free space (Region II) can be computed as follows

$$\mathbf{c}^2 = \left[\tilde{\mathbf{H}}_1 \ \tilde{\mathbf{H}}_2\right] \tilde{\mathbf{J}}(k_z) \tag{48}$$

where the matrices $\tilde{\mathbf{H}}_1$ and $\tilde{\mathbf{H}}_2$ have the following structures

$$\begin{aligned} \tilde{\mathbf{H}}_1 &= \tilde{\mathbf{Q}}_{11}^{-1} + \tilde{\mathbf{Q}}_{11}^{-1}\tilde{\mathbf{Q}}_{12}\tilde{\mathbf{Q}}_{22}^{-1}\tilde{\mathbf{Q}}_{21}\tilde{\mathbf{Q}}_{11}^{-1} \\ \tilde{\mathbf{H}}_2 &= -\tilde{\mathbf{Q}}_{11}^{-1}\tilde{\mathbf{Q}}_{12}\tilde{\mathbf{Q}}_{22}^{-1} \end{aligned} \tag{49}$$

Table 2. Matrices sizes of eqs. (47)–(50).

Term	*Dimension*
$\tilde{\mathbf{Q}}$	$(8N+4)\times(8N+4)$
$\tilde{\mathbf{Q}}_{ij}$	$(4N+2)\times(4N+2)$
$\tilde{\mathbf{H}}_i$	$(4N+2)\times(4N+2)$
$\tilde{\mathbf{P}}_{ij}$	$1\times(2N+1)$

Inserting these coefficients into eqs. (21)–(24) (truncated at the order N), we obtain the field components in Region II. This can be written in matrix form as

$$\tilde{\mathbf{E}}(k_z) = \begin{bmatrix} \tilde{E}_v(k_z) \\ \tilde{E}_z(k_z) \end{bmatrix} = \begin{bmatrix} \left[\tilde{\mathbf{P}}_{11}\ \tilde{\mathbf{P}}_{12}\right] \\ \left[\tilde{\mathbf{P}}_{21}\quad \mathbf{0}\right] \end{bmatrix} \mathbf{c}^2 \tag{50}$$

where

$$\begin{aligned} \tilde{\mathbf{P}}_{11} &= \frac{-k_z}{\omega\epsilon_2 h}\Big[Mc_0^4(q_2,u)\dot{ce}_0(q_2,v), Ms_1^4(q_2,u)\dot{se}_1(q_2,v), \\ &\quad \ldots, Mc_N^4(q_2,u)\dot{ce}_N(q_2,v), Ms_N^4\dot{se}_N(q_2,v)\Big] \end{aligned} \tag{51}$$

$$\begin{aligned} \tilde{\mathbf{P}}_{12} &= \frac{-k_z}{\omega\epsilon_2 h}\Big[\dot{M}c_0^4(q_2,u)ce_0(q_2,v), \dot{M}s_1^4(q_2,u)se_1(q_2,v), \\ &\quad \ldots, \dot{M}c_N^4(q_2,u)ce_N(q_2,v), \dot{M}s_N^4(q_2,u)se_N(q_2,v)\Big] \end{aligned} \tag{52}$$

$$\begin{aligned} \tilde{\mathbf{P}}_{21} &= \frac{k_2^2-k_z^2}{j\omega\epsilon_2}\Big[Mc_0^4(q_2,u)ce_0(q_2,v), Ms_1^4(q_2,u)se_1(q_2,v), \\ &\quad \ldots, Mc_{N-1}^4(q_2,u)ce_N(q_2,v), Ms_N^4(q_2,u)se_N(q_2,v)\Big] \end{aligned} \tag{53}$$

The dimensions of the terms $\tilde{\mathbf{Q}}$, $\tilde{\mathbf{H}}_1$, $\tilde{\mathbf{H}}_2$ and $\tilde{\mathbf{P}}$ in eqs. (47)–(50) are reported in Table 2. Combining equations (47), (50) we can write

$$\tilde{\mathbf{E}}(k_z) = \tilde{\mathbf{G}}(k_z)\cdot\tilde{\mathbf{J}}(k_z) \tag{54}$$

where $\tilde{\mathbf{G}}(k_z)$ can be regarded as the electric-type dyadic Green's function in the spectral domain written in matrix form (the size is $2\times(8N+4)$) relating the electric field in the exterior region to the current sources.

3. NUMERICAL RESULTS

The presented previously theory was implemented in a computer code. The code was written considering the steepest descent solution of the Fourier integrals and using the following approximation valid in the far field

$$M^4_{(c,s),p}(q,u) \rightarrow H^2_p(2\sqrt{q}\cosh(u)) \tag{55}$$

where

$$q = \frac{1}{4}k_2^2 \sin^2(\theta) d^2 \tag{56}$$

and θ is the elevation angle of a spherical system centered in the origin of the elliptic-cylinder coordinates. A 4×3 cm rectangular patch printed on a cylinder with major axis $a_{max} = 15$ cm directed along the x axis and covered with a dielectric layer with $\epsilon_r = 2.32$ and thickness $h = 0.78$ mm, is considered. The patch is placed in a symmetrical position with respect to the x axis. Radiation patterns were calculated varying the minor over major axis ratio e of the cylinder and keeping its major axis a_{max} unchanged. Patterns in Fig. 2 and Fig. 3 are relevant to an electric current flowing on the patch in the azimuthal direction and corresponding to a patch resonating on TM_{10} mode. The following current distribution is assumed

$$J_{10} = \sin\left(\frac{\pi s}{S}\right) \tag{57}$$

where S is the length of the elliptical arc covered by the current distribution defined as

$$S = \int_{-v_0}^{v_0} \sqrt{\cosh^2 u_1 - \cos^2 v dv} \tag{58}$$

with $(-v_0, v_0)$ the angular extension of the patch, and s is the curvilinear coordinate that follows the arc. In Fig. 4 and Fig. 5 an axial current flow, corresponding to the TM_{01} mode, with no azimuthal variation and with the following longitudinal law

$$J_{01} = \sin\left(\frac{\pi z}{L}\right) \tag{59}$$

with L the axial length of the antenna, is considered. Radiation patterns were normalized with respect to the maximum of the radiated field for the less eccentric cylinder. In all cases the effects of the eccentricity of the cylinder are evident. In particular, in the case of the azimuthal mode TM_{10}, the current in the half of the patch above the x axis and the current in the half below the x axis flow in opposite

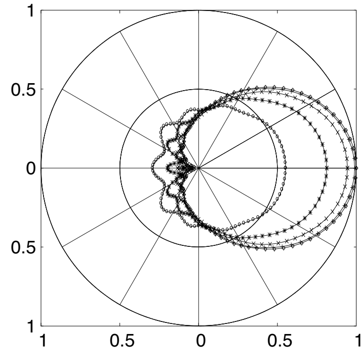

Figure 2. Normalized radiation pattern in E-plane of the patch operating on TM$_{10}$ mode. Patterns correspond to cylinders with $(-\diamond-)$ $a_{\min}/a_{\max} = 0.999$, $(-+-)$ $a_{\min}/a_{\max} = 0.8$, $(-\times-)$ $a_{\min}/a_{\max} = 0.6$, $(-*-)$ $a_{\min}/a_{\max} = 0.4$, $(-o-)$ $a_{\min}/a_{\max} = 0.1$.

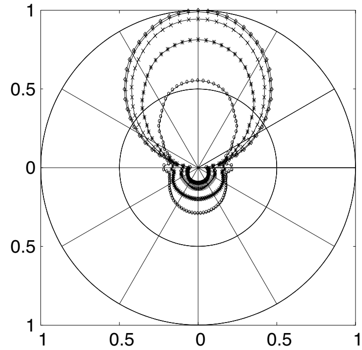

Figure 3. Normalized radiation pattern in H-plane of the patch operating on TM$_{10}$ mode. Patterns correspond to cylinders with $(-\diamond-)$ $a_{\min}/a_{\max} = 0.999$, $(-+-)$ $a_{\min}/a_{\max} = 0.8$, $(-\times-)$ $a_{\min}/a_{\max} = 0.6$, $(-*-)$ $a_{\min}/a_{\max} = 0.4$, $(-o-)$ $a_{\min}/a_{\max} = 0.1$.

direction in the limit of the minor over major axis ratio approaching to 1. This therefore strongly affects the antenna radiation patterns. As it is well known, the number of eigenfunctions needed to express the fields depends on the size in wavelengths of the structure under analysis. In the present case it has been experimentally observed that a number of functions equal to $4 * \lambda_0/\pi$ times the ellipse major axis is enough for a satisfactory accuracy.

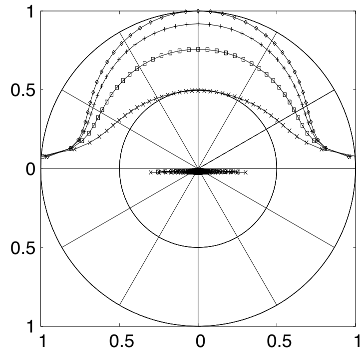

Figure 4. Normalized radiation pattern in E-plane of the patch operating on TM_{01} mode. Patterns correspond to cylinders with $(-\diamond-)$ $a_{\min}/a_{\max} = 0.999$, $(-+-)$ $a_{\min}/a_{\max} = 0.8$, $(-\times-)$ $a_{\min}/a_{\max} = 0.6$, $(-*-)$ $a_{\min}/a_{\max} = 0.4$, $(-o-)$ $a_{\min}/a_{\max} = 0.1$.

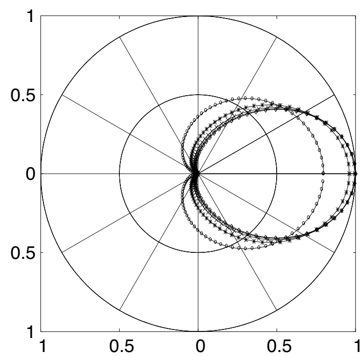

Figure 5. Normalized radiation pattern on H-plane of the patch operating on TM_{01} mode. Patterns correspond to cylinders with $(-\diamond-)$ $a_{\min}/a_{\max} = 0.999$, $(-+-)$ $a_{\min}/a_{\max} = 0.8$, $(-\times-)$ $a_{\min}/a_{\max} = 0.6$, $(-*-)$ $a_{\min}/a_{\max} = 0.4$, $(-o-)$ $a_{\min}/a_{\max} = 0.1$.

4. CONCLUSIONS

In this paper, an analysis of the radiation from assigned current distributions placed on a coated elliptic conducting cylinder has been effected. The fields were expanded as a series of elliptical wave-functions with unknown coefficients. The presence of the current were included in the boundary conditions at the air-dielectric interface and Galerkin's method was used to obtain the coefficients. Because of the well known properties of the Mathieu functions the analysis result more elaborate than the circular case and the matrix relating sources with the electric field had to be numerically inverted. However it has to be noticed that the major computational burden is due to the computation of the Mathieu functions rather than the numerical solution of the linear system. As results, the radiated far field from rectangular patches were produced and the effects of eccentricity of the elliptical cylinder on the radiation patterns were investigated. Even if in this work only known current distributions have been taken into account the development presented can be considered as the starting point for the full wave analysis of printed structures conformal to elliptic cylindrical surfaces.

APPENDIX A.

The elements of the $\tilde{\mathbf{M}}$ and $\tilde{\mathbf{T}}$ matrices are results of the testing operation on the boundary conditions at the interface between the layers. In particular, conditions (40) when tested with angular functions $ce_m(q_2, v)$ and $se_m(q_2, v)$ give rise to integrals that can be expressed as series of the coefficient of the expansions of the angular Mathieu functions [14]. The integrals involve the angular functions and their derivatives ,computed in both regions, and they have been named as follows $I_{m,n}^{p,(e/o),(e/o)'}$. The indexes n and m are the order of the test and expansion functions, respectively, p assumes values 1 or 2 according to the region, the couple $(e/o, e/o)$ in the apexes indicates even/odd test and expansions functions, respectively, and the dot indicates derivative of the expansion functions with respect to v variable. As an example the integrals for an even test function of order m, involving expansion functions in Region I, are

$$\begin{cases} I_{m,n}^{1,e,e} \\ I_{m,n}^{1,e,o} \\ I_{m,n}^{1,e,\dot{e}} \\ I_{m,n}^{1,e,\dot{o}} \end{cases} = \int_0^{2\pi} ce_m(q_2, v) \begin{cases} ce_n(q_1, v) \\ se_n(q_1, v) \\ \dot{ce}_n(q_1, v) \\ \dot{se}_n(q_1, v) \end{cases} dv \tag{A1}$$

The previous integrals can be then expressed as

$$I_{m,n}^{1,e,e} = \begin{cases} \displaystyle\sum_{r=0}^{+\infty} A_{2r}^{(n)}(q_2) A_{2r}^{(m)}(q_1) & n,m \text{ even} \\ \displaystyle\sum_{r=0}^{+\infty} A_{2r+1}^{(n)}(q_2) A_{2r+1}^{(m)}(q_1) & n,m \text{ odd} \\ 0 & \text{otherwise} \end{cases} \tag{A2}$$

$$I_{m,n}^{1,e,o} = I_{m,n}^{1,e,\dot{e}} = 0 \quad \forall n,m \tag{A3}$$

$$I_{m,n}^{1,e,o'} = \begin{cases} \displaystyle\sum_{r=1}^{+\infty}(2r)A_{2r}^{(n)}(q_2)B_{2r}^{(m)}(q_1) & n,m \text{ even} \\ \displaystyle\sum_{r=0}^{+\infty}(2r+1)A_{2r+1}^{(n)}(q_2)B_{2r+1}^{(m)}(q_1) & n,m \text{ odd} \\ 0 & \text{otherwise} \end{cases} \tag{A4}$$

where A_r^n and B_r^n are the coefficients of the expansion in angular circular functions of the Mathieu functions of order n [14]. The blocks of the $\tilde{T}$ and $\tilde{M}$ matrices have the following structure $TE(O)_{m,n} = \{te(o)^{m,n}\}_{i,j}$ and $ME(O)_{m,n} = \{me(o)^{m,n}\}_{i,j}$ with $i = \{1,4\}$, $j = \{1,2\}$ if the block belongs to the first column and $j = \{1,4\}$ otherwise. The elements in the first column of (44) are

$$\left\{ \begin{array}{c} te_{1,1}^{m,n} \\ to_{1,1}^{m,n} \end{array} \right\} = \frac{k_z}{\omega\epsilon_1}\Omega_m^{M,e} \left\{ \begin{array}{c} I_{m,n}^{1,e,\dot{e}} \\ I_{m,n}^{1,o,\dot{e}} \end{array} \right\} \tag{A5}$$

$$\left\{ \begin{array}{c} te_{1,2}^{m,n} \\ to_{1,2}^{m,n} \end{array} \right\} = -\Psi_m^{E,e} \left\{ \begin{array}{c} I_{m,n}^{1,e,e} \\ I_{m,n}^{1,o,e} \end{array} \right\} \tag{A6}$$

$$\left\{ \begin{array}{c} te_{2,1}^{m,n} \\ to_{2,1}^{m,n} \end{array} \right\} = -\frac{k_1^2-k_z^2}{j\omega\epsilon_1}\Omega_m^{M,e} \left\{ \begin{array}{c} I_{m,n}^{1,e,e} \\ I_{m,n}^{1,o,e} \end{array} \right\} \tag{A7}$$

$$\left\{ \begin{array}{c} te_{2,2}^{m,n} \\ to_{2,2}^{m,n} \end{array} \right\} = 0 \tag{A8}$$

$$\left\{ \begin{array}{c} te_{3,1}^{m,n} \\ to_{3,1}^{m,n} \end{array} \right\} = \Psi_m^{M,e} \left\{ \begin{array}{c} I_{m,n}^{1,e,e} \\ I_{m,n}^{1,o,e} \end{array} \right\} \tag{A9}$$

$$\left\{ \begin{array}{c} te_{3,2}^{m,n} \\ to_{3,2}^{m,n} \end{array} \right\} = \frac{k_z}{\omega\mu_1}\Omega_m^{E,e} \left\{ \begin{array}{c} I_{m,n}^{1,e,e} \\ I_{m,n}^{1,o,e} \end{array} \right\} \tag{A10}$$

$$\left\{ \begin{array}{c} te_{4,1}^{m,n} \\ to_{4,1}^{m,n} \end{array} \right\} = 0 \tag{A11}$$

$$\left\{ \begin{array}{c} te_{4,2}^{m,n} \\ to_{4,2}^{m,n} \end{array} \right\} = -\frac{k_1^2-k_z^2}{j\omega\mu_1}\Omega_m^{E,e} \left\{ \begin{array}{c} I_{m,n}^{1,o,\dot{e}} \\ I_{m,n}^{1,o,\dot{e}} \end{array} \right\} \tag{A12}$$

For the elements in the first column of (43) one has

$$\left\{ \begin{array}{c} me_{1,1}^{m,n} \\ mo_{1,1}^{m,n} \end{array} \right\} = \frac{-k_z}{\omega\epsilon_2}Mc_m^4(q_2,u) \left\{ \begin{array}{c} 0 \\ I_{m,n}^{2,o,\dot{e}} \end{array} \right\} \tag{A13}$$

$$\left\{\begin{array}{c} me_{1,2}^{m,n} \\ mo_{1,2}^{m,n} \end{array}\right\} = \dot{M}c_m^4(q_2,u)\left\{\begin{array}{c} I_{m,n}^{2,e,e} \\ 0 \end{array}\right\} \tag{A14}$$

$$\left\{\begin{array}{c} me_{2,1}^{m,n} \\ mo_{2,1}^{m,n} \end{array}\right\} = \frac{k_2^2-k_z^2}{j\omega\epsilon_2}Mc_m^4(q_2,u)\left\{\begin{array}{c} I_{m,n}^{2,e,e} \\ 0 \end{array}\right\} \tag{A15}$$

$$\left\{\begin{array}{c} me_{2,2}^{m,n} \\ mo_{2,2}^{m,n} \end{array}\right\} = 0 \tag{A16}$$

$$\left\{\begin{array}{c} me_{3,1}^{m,n} \\ mo_{3,1}^{m,n} \end{array}\right\} = -\dot{M}c_m^4(q_2,u)\left\{\begin{array}{c} I_{m,n}^{2,e,e} \\ 0 \end{array}\right\} \tag{A17}$$

$$\left\{\begin{array}{c} me_{3,2}^{m,n} \\ mo_{3,2}^{m,n} \end{array}\right\} = \frac{-k_z}{\omega\mu_2}Mc_m^4(q_2,u)\left\{\begin{array}{c} I_{m,n}^{2,e,e} \\ 0 \end{array}\right\} \tag{A18}$$

$$\left\{\begin{array}{c} me_{4,1}^{m,n} \\ mo_{4,1}^{m,n} \end{array}\right\} = 0 \tag{A19}$$

$$\left\{\begin{array}{c} me_{4,2}^{m,n} \\ mo_{4,2}^{m,n} \end{array}\right\} = \frac{k_2^2-k_z^2}{j\omega\mu_2}Mc_m^4(q_2,u)\left\{\begin{array}{c} I_{m,n}^{2,e,\dot{e}} \\ 0 \end{array}\right\} \tag{A20}$$

The remaining terms of the $TE(O)_{m,n}$ and $ME(O)_{m,n}$ blocks are

$$\left\{\begin{array}{c} te_{1,1}^{m,n} \\ to_{1,1}^{m,n} \end{array}\right\} = \frac{k_z}{\omega\epsilon_1}\Omega_m^{M,e}\left\{\begin{array}{c} 0 \\ I_{m,n}^{1,o,\dot{e}} \end{array}\right\} \tag{A21}$$

$$\left\{\begin{array}{c} te_{1,2}^{m,n} \\ to_{1,2}^{m,n} \end{array}\right\} = \frac{k_z}{\omega\epsilon_1}\Omega_m^{M,o}\left\{\begin{array}{c} I_{m,n}^{1,e,\dot{o}} \\ 0 \end{array}\right\} \tag{A22}$$

$$\left\{\begin{array}{c} te_{1,3}^{m,n} \\ to_{1,3}^{m,n} \end{array}\right\} = -\Psi_m^{E,e}\left\{\begin{array}{c} I_{m,n}^{1,e,e} \\ I_{m,n}^{1,o,e} \end{array}\right\} \tag{A23}$$

$$\left\{\begin{array}{c} te_{1,4}^{m,n} \\ to_{1,4}^{m,n} \end{array}\right\} = -\Psi_m^{E,o}\left\{\begin{array}{c} I_{m,n}^{1,e,o} \\ I_{m,n}^{1,o,o} \end{array}\right\} \tag{A24}$$

$$\left\{\begin{array}{c} te_{2,1}^{m,n} \\ to_{2,1}^{m,n} \end{array}\right\} = -\frac{k_1^2-k_z^2}{j\omega\epsilon_1}\Omega_m^{M,e}\left\{\begin{array}{c} I_{m,n}^{1,e,e} \\ I_{m,n}^{1,o,e} \end{array}\right\} \tag{A25}$$

$$\left\{\begin{array}{c} te_{2,2}^{m,n} \\ to_{2,2}^{m,n} \end{array}\right\} = -\frac{k_1^2-k_z^2}{j\omega\epsilon_1}\Omega_m^{M,o}\left\{\begin{array}{c} 0 \\ I_{m,n}^{1,e,o} \end{array}\right\} \tag{A26}$$

$$\left\{\begin{array}{c} te_{2,3}^{m,n} \\ to_{2,3}^{m,n} \end{array}\right\} = \left\{\begin{array}{c} te_{2,4}^{m,n} \\ to_{2,4}^{m,n} \end{array}\right\} = 0 \tag{A27}$$

$$\left\{\begin{array}{c} te_{3,1}^{m,n} \\ to_{3,1}^{m,n} \end{array}\right\} = \Psi_m^{M,e}\left\{\begin{array}{c} I_{m,n}^{1,e,e} \\ I_{m,n}^{1,o,e} \end{array}\right\} \tag{A28}$$

$$\left\{ \begin{array}{c} te_{3,2}^{m,n} \\ to_{3,2}^{m,n} \end{array} \right\} = \Psi_m^{M,o} \left\{ \begin{array}{c} I_{m,n}^{1,e,o} \\ I_{m,n}^{,o,e} \end{array} \right\} \tag{A29}$$

$$\left\{ \begin{array}{c} te_{3,3}^{m,n} \\ to_{3,3}^{m,n} \end{array} \right\} = \frac{k_z}{\omega\mu_1} \Omega_m^{E,e} \left\{ \begin{array}{c} I_{m,n}^{1,e,\dot{e}} \\ I_{m,n}^{1,o,\dot{e}} \end{array} \right\} \tag{A30}$$

$$\left\{ \begin{array}{c} te_{3,4}^{m,n} \\ to_{3,4}^{m,n} \end{array} \right\} = \frac{k_z}{\omega\mu_1} \Omega_m^{E,o} \left\{ \begin{array}{c} I_{m,n}^{1,e,\dot{o}} \\ I_{m,n}^{1,o,\dot{o}} \end{array} \right\} \tag{A31}$$

$$\left\{ \begin{array}{c} te_{4,1}^{m,n} \\ to_{4,1}^{m,n} \end{array} \right\} = \left\{ \begin{array}{c} te_{4,2}^{m,n} \\ to_{4,2}^{m,n} \end{array} \right\} = 0 \tag{A32}$$

$$\left\{ \begin{array}{c} te_{4,3}^{m,n} \\ to_{4,3}^{m,n} \end{array} \right\} = -\frac{k_1^2 - k_z^2}{j\omega\mu_1} \Omega_m^{E,e} \left\{ \begin{array}{c} I_{m,n}^{1,e,e} \\ I_{m,n}^{1,o,e} \end{array} \right\} \tag{A33}$$

$$\left\{ \begin{array}{c} te_{4,4}^{m,n} \\ to_{4,4}^{m,n} \end{array} \right\} = -\frac{k_1^2 - k_z^2}{j\omega\mu_1} \Omega_m^{E,o} \left\{ \begin{array}{c} I_{m,n}^{1,e,o} \\ I_{m,n}^{1,o,o} \end{array} \right\} \tag{A34}$$

$$\left\{ \begin{array}{c} me_{1,1}^{m,n} \\ mo_{1,1}^{m,n} \end{array} \right\} = \frac{-k_z}{\omega\epsilon_2} Mc_m^4(q_2, u) \left\{ \begin{array}{c} I_{m,n}^{2,e,\dot{e}} \\ I_{m,n}^{2,o,\dot{e}} \end{array} \right\} \tag{A35}$$

$$\left\{ \begin{array}{c} me_{1,2}^{m,n} \\ mo_{1,2}^{m,n} \end{array} \right\} = \frac{-k_z}{\omega\epsilon_2} Ms_m^4(q_2, u) \left\{ \begin{array}{c} I_{m,n}^{2,e,\dot{o}} \\ 0 \end{array} \right\} \tag{A36}$$

$$\left\{ \begin{array}{c} me_{1,3}^{m,n} \\ mo_{1,3}^{m,n} \end{array} \right\} = \dot{M}c_m^4(q_2, u) \left\{ \begin{array}{c} I_{m,n}^{2e,e} \\ 0 \end{array} \right\} \tag{A37}$$

$$\left\{ \begin{array}{c} me_{1,4}^{m,n} \\ mo_{1,4}^{m,n} \end{array} \right\} = \dot{M}s_m^4(q_2, u) \left\{ \begin{array}{c} 0 \\ I_{m,n}^{2,o,o} \end{array} \right\} \tag{A38}$$

$$\left\{ \begin{array}{c} me_{2,1}^{m,n} \\ mo_{2,1}^{m,n} \end{array} \right\} = \frac{k_2^2 - k_z^2}{j\omega\epsilon_2} Mc_m^4(q_2, u) \left\{ \begin{array}{c} I_{m,n}^{2,e,e} \\ 0 \end{array} \right\} \tag{A39}$$

$$\left\{ \begin{array}{c} me_{2,2}^{m,n} \\ mo_{2,2}^{m,n} \end{array} \right\} = \frac{k_2^2 - k_z^2}{j\omega\epsilon_2} Ms_m^4(q_2, u) \left\{ \begin{array}{c} 0 \\ I_{m,n}^{2,o,o} \end{array} \right\} \tag{A40}$$

$$\left\{ \begin{array}{c} me_{2,3}^{m,n} \\ mo_{2,3}^{m,n} \end{array} \right\} = \left\{ \begin{array}{c} me_{2,4}^{m,n} \\ mo_{2,4}^{m,n} \end{array} \right\} = 0 \tag{A41}$$

$$\left\{ \begin{array}{c} me_{3,1}^{m,n} \\ mo_{3,1}^{m,n} \end{array} \right\} = -\dot{M}c_m^4(q_2, u) \left\{ \begin{array}{c} I_{m,n}^{2,e,e} \\ 0 \end{array} \right\} \tag{A42}$$

$$\left\{ \begin{array}{c} me_{3,2}^{m,n} \\ mo_{3,2}^{m,n} \end{array} \right\} = -\dot{M}s_m^4(q_2, u) \left\{ \begin{array}{c} 0 \\ I_{m,n}^{2,o,o} \end{array} \right\} \tag{A43}$$

$$\left\{ \begin{array}{c} me_{3,3}^{m,n} \\ mo_{3,3}^{m,n} \end{array} \right\} = \frac{-k_z}{\omega\mu_2 h} Mc_m^4(q_2,u) \left\{ \begin{array}{c} 0 \\ I_{m,n}^{2,o,\dot{e}} \end{array} \right\} \tag{A44}$$

$$\left\{ \begin{array}{c} me_{3,4}^{m,n} \\ mo_{3,4}^{m,n} \end{array} \right\} = \frac{-k_z}{\omega\mu_2 h} Ms_m^4(q_2,u) \left\{ \begin{array}{c} I_{m,n}^{2,e,\dot{o}} \\ I_{m,n}^{2,o,\dot{o}} \end{array} \right\} \tag{A45}$$

$$\left\{ \begin{array}{c} me_{4,1}^{m,n} \\ mo_{4,1}^{m,n} \end{array} \right\} = \left\{ \begin{array}{c} me_{4,2}^{m,n} \\ mo_{4,2}^{m,n} \end{array} \right\} = 0 \tag{A46}$$

$$\left\{ \begin{array}{c} me_{4,3}^{m,n} \\ mo_{4,3}^{m,n} \end{array} \right\} = \frac{k_2^2 - k_z^2}{j\omega\mu_2} Mc_m^4(q_2,u) \left\{ \begin{array}{c} I_{m,n}^{2,e,e} \\ 0 \end{array} \right\} \tag{A47}$$

$$\left\{ \begin{array}{c} me_{4,4}^{m,n} \\ mo_{4,4}^{m,n} \end{array} \right\} = \frac{k_2^2 - k_z^2}{j\omega\mu_2} Ms_m^4(q_2,u) \left\{ \begin{array}{c} 0 \\ I_{m,n}^{2,e,o} \end{array} \right\} \tag{A48}$$

The vectors in expressions (45) are organized as follows $(l = 1 \dots N)$

$$C_0^1 = \begin{bmatrix} e_{0,3}^{M,1} \\ e_{0,3}^{E,1} \end{bmatrix} \quad C_0^2 = \begin{bmatrix} e_{0,4}^{M,2} \\ e_{0,4}^{E,2} \end{bmatrix} \quad C_l^1 = \begin{bmatrix} e_{l,3}^{M,1} \\ o_{l,3}^{M,1} \\ e_{l,3}^{E,1} \\ o_{l,3}^{E,1} \end{bmatrix} \quad C_l^2 = \begin{bmatrix} e_{l,4}^{M,2} \\ o_{l,4}^{M,2} \\ e_{l,4}^{E,2} \\ o_{l,4}^{E,2} \end{bmatrix} \tag{A49}$$

$$JE_0 = \begin{bmatrix} \int_0^{2\pi} ce_0(q_2,v)\tilde{J}_z dv \\ -\int_0^{2\pi} ce_0(q_2,v)\tilde{J}_v dv \end{bmatrix} \tag{A50}$$

$$\left\{ \begin{array}{c} JE_l \\ JO_l \end{array} \right\} = \begin{bmatrix} 0 \\ 0 \\ \int_0^{2\pi} \left\{ \begin{array}{c} ce_l(q_2,v) \\ se_l(q_2,v) \end{array} \right\} \tilde{J}_z dv \\ -\int_0^{2\pi} \left\{ \begin{array}{c} ce_l(q_2,v) \\ se_l(q_2,v) \end{array} \right\} \tilde{J}_v dv \end{bmatrix} \tag{A51}$$

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