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ELECTROMAGNETIC FIELDS IN SELF-DUAL MEDIA IN DIFFERENTIAL-FORM REPRESENTATION

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Abstract—Four-dimensional differential-form formalism is applied to define the duality transformation between electromagnetic fields and sources. The class of linear media invariant in any non-trivial duality transformation is labeled as that of self-dual media. It is shown that the medium dyadic of a self-dual medium, which represents a mapping between the two electromagnetic field two-forms, satisfies a quadratic algebraic equation. Further, it is shown that fields and sources in a self-dual medium can be decomposed in two uncoupled sets each self-dual with respect to a duality transformation. Also, for each of the decomposed fields the original medium can be replaced by a simpler effective medium. Splitting the electromagnetic problem in two self-dual parts can be used to simplify the solution process because differential equations for fields are reduced to those with second-order scalar operators. This is applied to find plane-wave solutions for the general self-dual medium.

1. INTRODUCTION

Differential forms is a mathematical formalism [1–3] which can be used to replace classical Gibbsian vector and dyadic analysis [4, 5]. While the latter is associated with the three-dimensional space, the former has no such a limitation and can be applied to a space of any dimension. In practice, though, physical interest lies in the Minkowskian four-space. In particular, the Maxwell equations can be given a compact form in terms of the four-dimensional differential form representation [6–10]. Recently, it has been demonstrated that, in addition to the basic electromagnetic laws, classes of linear electromagnetic media whose analytical treatment with Gibbsian formalism leads to quite complicated expressions can be defined and analyzed with less effort in terms of the four-dimensional multivector and dyadic algebra associated with the differential-form formalism [11, 12]. Also, new concepts like the perfect electromagnetic conductor (PEMC) have emerged through the four-dimensional representation [13].

The notation applied here follows that of [10]. A short summary has been given in the appendices of [11] and [12]. The basic Maxwell equations are given in the simple form

$$\mathbf{d}\wedge\mathbf{\Phi}=\boldsymbol{\gamma}_m,\qquad \mathbf{d}\wedge\mathbf{\Psi}=\boldsymbol{\gamma}_e, \tag{1}$$

where the three-dimensional expansions of the electromagnetic two-forms $\mathbf{\Phi}$, $\mathbf{\Psi}$, members of the space $\mathbb{F}_2$, are

$$\mathbf{\Phi}=\mathbf{B}+\mathbf{E}\wedge\boldsymbol{\varepsilon}_4,\qquad \mathbf{\Psi}=\mathbf{D}-\mathbf{H}\wedge\boldsymbol{\varepsilon}_4. \tag{2}$$

In terms of a dual-vector basis $\{\boldsymbol{\varepsilon}_i\}\in\mathbb{F}_1$ the differential operator is represented as

$$\mathbf{d}=\sum_{i=1}^{4}\boldsymbol{\varepsilon}_i\partial_{x_i}, \tag{3}$$

where x_1, x_2, x_3 are the three spatial coordinates and $x_4=\tau=ct$ is the (normalized) temporal coordinate. The reciprocal vector basis is denoted by $\{\mathbf{e}_i\}\in\mathbb{E}_1$ and it satisfies

$$\mathbf{e}_i|\boldsymbol{\varepsilon}_j=\boldsymbol{\varepsilon}_j|\mathbf{e}_i=\delta_{ij}. \tag{4}$$

The electric and magnetic source three-forms $\boldsymbol{\gamma}_e$, $\boldsymbol{\gamma}_m\in\mathbb{F}_3$ are

$$\boldsymbol{\gamma}_e=\varrho_e-\mathbf{J}_e\wedge\boldsymbol{\varepsilon}_4,\qquad \boldsymbol{\gamma}_m=\varrho_m-\mathbf{J}_m\wedge\boldsymbol{\varepsilon}_4. \tag{5}$$

They satisfy the charge conservation equations

$$\mathbf{d}\wedge\boldsymbol{\gamma}_e=0,\qquad \mathbf{d}\wedge\boldsymbol{\gamma}_m=0. \tag{6}$$

Magnetic sources are here assumed to be equivalent sources without any assumptions on the existence or non-existence of the magnetic monopole.

Any linear electromagnetic medium is defined through a medium dyadic $\overline{\overline{\mathsf{M}}}\in\mathbb{F}_2\mathbb{E}_2$ which maps two-forms to two-forms and, thus, has 36 scalar parameters in general. The relation is written as

$$\mathbf{\Psi}=\overline{\overline{\mathsf{M}}}|\mathbf{\Phi}. \tag{7}$$

A number of classes of media, defined by certain restrictions on the medium dyadic, have been recently under study. The present effort defines another class of media through the duality transformation.

2. DUALITY

Historically, duality in electromagnetics was introduced by Oliver Heaviside [7, 10] in 1886 to reflect the formal symmetry of the Maxwell equations in electric and magnetic quantities. In its general form, it can be defined through a linear transformation $\mathbf{\Psi} \to \mathbf{\Psi}_d$, $\mathbf{\Phi} \to \mathbf{\Phi}_d$ with four parameters as [10]

$$\begin{pmatrix} \mathbf{\Psi}_d \\ \mathbf{\Phi}_d \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathbf{\Psi} \\ \mathbf{\Phi} \end{pmatrix}. \tag{8}$$

It is more convenient to define the duality transformation in terms of another set of four parameters λ, Z_d, Z and θ, in the form

$$\begin{pmatrix} Z_d\mathbf{\Psi}_d \\ \mathbf{\Phi}_d \end{pmatrix} = \lambda\mathcal{R}(\theta) \begin{pmatrix} Z\mathbf{\Psi} \\ \mathbf{\Phi} \end{pmatrix}, \tag{9}$$

where

$$\mathcal{R}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{10}$$

is a rotation matrix. When the electric and magnetic source three-forms are transformed as

$$\begin{pmatrix} Z_d\gamma_{ed} \\ \gamma_{md} \end{pmatrix} = \lambda\mathcal{R}(\theta) \begin{pmatrix} Z\gamma_e \\ \gamma_m \end{pmatrix}, \tag{11}$$

the Maxwell equations are invariant in form:

$$\begin{pmatrix} Z_d\mathbf{d}\wedge\mathbf{\Psi}_d \\ \mathbf{d}\wedge\mathbf{\Phi}_d \end{pmatrix} = \lambda\mathcal{R}(\theta) \begin{pmatrix} Z\mathbf{d}\wedge\mathbf{\Psi} \\ \mathbf{d}\wedge\mathbf{\Phi} \end{pmatrix}$$

$$= \lambda\mathcal{R}(\theta) \begin{pmatrix} Z\gamma_e \\ \gamma_m \end{pmatrix} = \begin{pmatrix} Z_d\gamma_{ed} \\ \gamma_{md} \end{pmatrix}, \tag{12}$$

or

$$\mathbf{d}\wedge\mathbf{\Phi}_d = \gamma_{md}, \qquad \mathbf{d}\wedge\mathbf{\Psi}_d = \gamma_{ed}. \tag{13}$$

Assuming that the transformed fields satisfy medium equations of the form (7)

$$\mathbf{\Psi}_d = \overline{\overline{\mathsf{M}}}_d|\mathbf{\Phi}_d, \tag{14}$$

and writing

$$Z_d\mathbf{\Psi}_d = \lambda(\sin\theta\bar{\bar{\mathsf{I}}}^{(2)T} + \cos\theta Z\overline{\overline{\mathsf{M}}})|\mathbf{\Phi}, \tag{15}$$

$$\mathbf{\Phi}_d = \lambda(\cos\theta\bar{\bar{\mathsf{I}}}^{(2)T} - \sin\theta Z\overline{\overline{\mathsf{M}}})|\mathbf{\Phi}, \tag{16}$$

after eliminating $\mathbf{\Phi}$ we obtain the following transform rule for the medium dyadic [10]

$$Z_d\overline{\overline{\mathsf{M}}}_d = (\sin\theta\overline{\overline{\mathsf{I}}}^{(2)T} + \cos\theta Z\overline{\overline{\mathsf{M}}})|(\cos\theta\overline{\overline{\mathsf{I}}}^{(2)T} - \sin\theta Z\overline{\overline{\mathsf{M}}})^{-1}. \qquad (17)$$

The unit dyadic $\overline{\overline{\mathsf{I}}} \in \mathbb{E}_1\mathbb{F}_1$ maps vectors to themselves and $\overline{\overline{\mathsf{I}}}^{(2)T} \in \mathbb{F}_2\mathbb{E}_2$ does the same for two-forms. Because the medium dyadic describes ratios of fields, its transformation rule (17) does not involve the magnitude parameter λ of the duality transformation.

3. SELF-DUAL MEDIUM

The electromagnetic medium is self-dual if it does not change in some duality transformation, so that $\overline{\overline{\mathsf{M}}}_d = \overline{\overline{\mathsf{M}}}$ is satisfied. From (17) we then obtain a restricting equation for the medium dyadic $\overline{\overline{\mathsf{M}}}$:

$$\sin\theta Z Z_d\overline{\overline{\mathsf{M}}}^2 + (Z - Z_d)\cos\theta\overline{\overline{\mathsf{M}}} + \sin\theta\overline{\overline{\mathsf{I}}}^{(2)T} = 0. \qquad (18)$$

However, here we must exclude the set of trivial transformations with $Z = Z_d$ and $\sin\theta = 0$ for which (18) is satisfied by any dyadic $\overline{\overline{\mathsf{M}}}$. In such a trivial duality transformation all fields and sources are just multiplied by the same scalar.

Because (18) is a quadratic equation in the dyadic $\overline{\overline{\mathsf{M}}}$, for any self-dual medium there exist two scalars M_+, M_- such that $\overline{\overline{\mathsf{M}}}$ satisfies

$$(\overline{\overline{\mathsf{M}}} - M_+\overline{\overline{\mathsf{I}}}^{(2)T})|(\overline{\overline{\mathsf{M}}} - M_-\overline{\overline{\mathsf{I}}}^{(2)T}) = (\overline{\overline{\mathsf{M}}} - M_-\overline{\overline{\mathsf{I}}}^{(2)T})|(\overline{\overline{\mathsf{M}}} - M_+\overline{\overline{\mathsf{I}}}^{(2)T}) = 0. \quad (19)$$

Conversely, one can show that, for a medium satisfying (19), one can define a duality transformation in terms of which the medium is self-dual. Such a transformation is not unique. For example, we could choose any value for λ, assume $\theta = \pi/4$ and solve Z, Z_d from

$$M_+ + M_- = \frac{Z_d - Z}{Z_d Z}, \qquad M_+M_- = 1/Z_d Z \qquad (20)$$

in the form

$$Z = -\frac{1}{2M_+M_-}(M_+ + M_- + \sqrt{(M_+ + M_-)^2 + 4M_+M_-}\ ), \quad (21)$$

$$Z_d = \frac{1}{2M_+M_-}(M_+ + M_- - \sqrt{(M_+ + M_-)^2 + 4M_+M_-}\ ). \quad (22)$$

For a self-dual medium, the inverse of the medium dyadic can be expressed from (19) as a linear combination of $\overline{\overline{\mathsf{M}}}$ and $\overline{\overline{\mathsf{I}}}^{(2)T}$,

$$\overline{\overline{\mathsf{M}}}^{-1} = -\frac{1}{M_+M_-}\overline{\overline{\mathsf{M}}} + \frac{M_+ + M_-}{M_+M_-}\overline{\overline{\mathsf{I}}}^{(2)T}, \tag{23}$$

For the modified (metric) medium dyadic defined as $\overline{\overline{\mathsf{M}}}_g = \mathbf{e}_N\lfloor\overline{\overline{\mathsf{M}}} \in \mathbb{E}_2\mathbb{E}_2$ [10], the condition (23) can be shown to take the form

$$\overline{\overline{\mathsf{M}}}_g + M_+M_-\overline{\overline{\mathsf{M}}}_g^{-1}\rfloor\rfloor\mathbf{e}_N\mathbf{e}_N = (M_+ + M_-)\mathbf{e}_N\lfloor\overline{\overline{\mathsf{I}}}^{(2)T}. \tag{24}$$

4. THREE-DIMENSIONAL REPRESENTATION

The modified medium dyadic $\overline{\overline{\mathsf{M}}}_g$ corresponding to the general bi-anisotropic medium can be expressed in terms of Gibbsian three-dimensional medium dyadics $\overline{\overline{\epsilon}}_g, \overline{\overline{\xi}}_g, \overline{\overline{\zeta}}_g, \overline{\overline{\mu}}_g \in \mathbb{E}_1\mathbb{E}_1$ as [10]

$$\begin{aligned}\overline{\overline{\mathsf{M}}}_g \;=\; & -\mathbf{e}_{123}\mathbf{e}_{123}\lfloor\lfloor\overline{\overline{\mu}}_g^{-1} - \mathbf{e}_4\wedge\overline{\overline{\xi}}_g|\overline{\overline{\mu}}_g^{-1}\rfloor\mathbf{e}_{123} \\ & + \mathbf{e}_{123}\lfloor\overline{\overline{\mu}}_g^{-1}|\overline{\overline{\zeta}}_g\wedge\mathbf{e}_4 + (\overline{\overline{\epsilon}}_g - \overline{\overline{\xi}}_g|\overline{\overline{\mu}}_g^{-1}|\overline{\overline{\zeta}}_g)^{\wedge}_{\wedge}\mathbf{e}_4\mathbf{e}_4,\end{aligned} \tag{25}$$

and, the inverse satisfies

$$\begin{aligned}\mathbf{e}_N\mathbf{e}_N\lfloor\lfloor\overline{\overline{\mathsf{M}}}_g^{-1} \;=\; & \mathbf{e}_{123}\mathbf{e}_{123}\lfloor\lfloor\overline{\overline{\epsilon}}_g^{-1} - \mathbf{e}_4\wedge\overline{\overline{\zeta}}_g|\overline{\overline{\epsilon}}_g^{-1}\rfloor\mathbf{e}_{123} \\ & + \mathbf{e}_{123}\lfloor\overline{\overline{\epsilon}}_g^{-1}|\overline{\overline{\xi}}_g\wedge\mathbf{e}_4 - (\overline{\overline{\mu}}_g - \overline{\overline{\zeta}}_g|\overline{\overline{\epsilon}}_g^{-1}|\overline{\overline{\xi}}_g)^{\wedge}_{\wedge}\mathbf{e}_4\mathbf{e}_4.\end{aligned} \tag{26}$$

Writing (24) in the form

$$\overline{\overline{\mathsf{M}}}_g + M_+M_-\overline{\overline{\mathsf{M}}}_g^{-1}\rfloor\rfloor\mathbf{e}_N\mathbf{e}_N = (M_+ + M_-)(\mathbf{e}_{123}\lfloor\overline{\overline{\mathsf{I}}}_s^T\wedge\mathbf{e}_4 - \mathbf{e}_4\wedge(\mathbf{e}_{123}\lfloor\overline{\overline{\mathsf{I}}}_s^{(2)T})), \tag{27}$$

it can be used to derive the corresponding self-dual conditions for the four Gibbsian medium dyadics. Here

$$\overline{\overline{\mathsf{I}}}_s = \sum_{i=1}^{3}\mathbf{e}_i\boldsymbol{\varepsilon}_i = \overline{\overline{\mathsf{I}}} - \mathbf{e}_4\boldsymbol{\varepsilon}_4 \tag{28}$$

denotes the three-dimensional (spatial) unit dyadic. Equating spatial-spatial, spatial-temporal, temporal-spatial and temporal-temporal parts separately, leads after lengthy but straightforward manipulations to the following three conditions between the Gibbsian medium dyadics:

$$\overline{\overline{\epsilon}}_g = M_+ M_- \overline{\overline{\mu}}_g, \tag{29}$$

$$\overline{\overline{\xi}}_g + \overline{\overline{\zeta}}_g = (M_+ + M_-)\overline{\overline{\mu}}_g, \tag{30}$$

$$\overline{\overline{\xi}}_g|\overline{\overline{\mu}}_g^{-1}|\overline{\overline{\zeta}}_g = M_+ M_- \overline{\overline{\zeta}}_g|\overline{\overline{\epsilon}}_g^{-1}|\overline{\overline{\xi}}_g. \tag{31}$$

It is quite easy to show that the third condition is valid whenever the first and the second ones are valid. The conclusion is that, for a self-dual medium, the three Gibbsian dyadics $\overline{\overline{\epsilon}}_g, \overline{\overline{\mu}}_g$ and $\overline{\overline{\xi}}_g + \overline{\overline{\zeta}}_g$ must be multiples of the same dyadic, while $\overline{\overline{\xi}}_g - \overline{\overline{\zeta}}_g$ may be a multiple of any other dyadic. Thus, we can write

$$\overline{\overline{\mu}}_g = \overline{\overline{\mathsf{Q}}}, \quad \overline{\overline{\epsilon}}_g = M_+ M_- \overline{\overline{\mathsf{Q}}}, \tag{32}$$

$$\overline{\overline{\xi}}_g = \frac{M_+ + M_-}{2}(\overline{\overline{\mathsf{Q}}} - \overline{\overline{\mathsf{T}}}), \qquad \overline{\overline{\zeta}}_g = \frac{M_+ + M_-}{2}(\overline{\overline{\mathsf{Q}}} + \overline{\overline{\mathsf{T}}}), \tag{33}$$

for some dyadics $\overline{\overline{\mathsf{Q}}}, \overline{\overline{\mathsf{T}}} \in \mathbb{E}_1\mathbb{E}_1$. This result corresponds to that obtained through Gibbsian dyadic analysis in [14].

5. AB-MEDIA

As a special case of a self-dual medium let us consider the class of media previously defined in [15] and labeled as that of AB-media (for affine bi-anisotropic). Such media have the property that their 3D medium dyadics are form-invariant in any spatial affine transformations. The medium dyadic of an AB-medium was expressed the form

$$\overline{\overline{\mathsf{M}}} = \alpha \overline{\overline{\mathsf{I}}}_s^{(2)T} + \epsilon' \overline{\overline{\mathsf{B}}} \wedge \mathbf{e}_4 + \frac{1}{\mu}\boldsymbol{\varepsilon}_4 \wedge \overline{\overline{\mathsf{B}}}^{-1} + \beta \boldsymbol{\varepsilon}_4 \wedge \overline{\overline{\mathsf{I}}}_s^T \wedge \mathbf{e}_4, \tag{34}$$

where $\alpha, \epsilon', \mu, \beta$ are scalars. $\overline{\overline{\mathsf{B}}} \in \mathbb{F}_2\mathbb{E}_1$ is a three-dimensional dyadic mapping vectors to two-forms and $\overline{\overline{\mathsf{B}}}^{-1} \in \mathbb{F}_1\mathbb{E}_2$ its three-dimensional inverse satisfying

$$\overline{\overline{\mathsf{B}}}|\overline{\overline{\mathsf{B}}}^{-1} = \overline{\overline{\mathsf{I}}}_s^{(2)T}, \qquad \overline{\overline{\mathsf{B}}}^{-1}|\overline{\overline{\mathsf{B}}} = \overline{\overline{\mathsf{I}}}_s^T. \tag{35}$$

To prove that AB-media are actually self-dual, it is sufficient to show that the medium dyadic (34) satisfies a quadratic equation. To see this, the square of (34) consisting of eight non-null terms can be compressed as

$$\begin{aligned}\overline{\overline{\mathsf{M}}}^2 &= (\alpha^2 - \frac{\epsilon'}{\mu})\bar{\bar{\mathsf{I}}}_s^{(2)T} + (\alpha\epsilon' - \epsilon'\beta)\overline{\overline{\mathsf{B}}} \wedge \mathbf{e}_4 \\ &\quad + (\frac{\alpha}{\mu} - \frac{\beta}{\mu})\boldsymbol{\varepsilon}_4 \wedge \overline{\overline{\mathsf{B}}}^{-1} + (\frac{\epsilon'}{\mu} - \beta^2)\boldsymbol{\varepsilon}_4 \wedge \bar{\bar{\mathsf{I}}}_s^T \wedge \mathbf{e}_4 \\ &= (\alpha - \beta)\overline{\overline{\mathsf{M}}} + (\alpha\beta - \frac{\epsilon'}{\mu})\bar{\bar{\mathsf{I}}}^{(2)T}, \end{aligned} \tag{36}$$

which has the required form (18). Actually, a self-dual medium falls to the class of AB-media if the dyadic $\overline{\overline{\mathsf{T}}}$ in (33) happens to be a multiple of the dyadic $\overline{\overline{\mathsf{Q}}}$. A medium whose four Gibbsian medium dyadics are multiples of the same dyadic was previously studied in [16].

6. EIGENSOLUTIONS

It was shown in [15] that electromagnetic problems associated with an AB-medium could be decomposed in terms of eigenexpansions into two simpler problems involving simpler media. It is now shown that the decomposition method can be generalized to self-dual media. The dual eigenbivectors $\boldsymbol{\Phi}_i$ of the medium satisfy

$$\overline{\overline{\mathsf{M}}}|\boldsymbol{\Phi}_i = M_i\boldsymbol{\Phi}_i, \tag{37}$$

with corresponding eigenvalues M_i. Applying the left-hand side of (19) on $\boldsymbol{\Phi}_i$ leaves us with the scalar equation

$$(M_i - M_+)(M_i - M_-) = 0, \tag{38}$$

from which we see that, for a self-dual medium, there are only two eigenvalues M_+ and M_-. Let us denote the corresponding dual eigenbivectors by $\boldsymbol{\Phi}_+, \boldsymbol{\Phi}_-$.

Assuming now $M_+ \neq M_-$, let us define two dyadics $\overline{\overline{\mathsf{P}}}_\pm$ as

$$\overline{\overline{\mathsf{P}}}_+ = \frac{\overline{\overline{\mathsf{M}}} - M_+\bar{\bar{\mathsf{I}}}^{(2)T}}{M_- - M_+}, \qquad \overline{\overline{\mathsf{P}}}_- = \frac{\overline{\overline{\mathsf{M}}} - M_-\bar{\bar{\mathsf{I}}}^{(2)T}}{M_+ - M_-}, \tag{39}$$

satisfying

$$\bar{\bar{\mathsf{I}}}^{(2)T} = \overline{\overline{\mathsf{P}}}_+ + \overline{\overline{\mathsf{P}}}_-, \qquad \overline{\overline{\mathsf{M}}} = M_-\overline{\overline{\mathsf{P}}}_+ + M_+\overline{\overline{\mathsf{P}}}_-. \tag{40}$$

From (19) we can easily check the properties

$$\overline{\overline{\mathsf{P}}}_+^2 = \overline{\overline{\mathsf{P}}}_+, \qquad \overline{\overline{\mathsf{P}}}_-^2 = \overline{\overline{\mathsf{P}}}_-, \qquad \overline{\overline{\mathsf{P}}}_+|\overline{\overline{\mathsf{P}}}_- = \overline{\overline{\mathsf{P}}}_-|\overline{\overline{\mathsf{P}}}_+ = 0. \tag{41}$$

whence $\overline{\overline{\mathsf{P}}}_+$ and $\overline{\overline{\mathsf{P}}}_-$ are projection dyadics. Because they satisfy

$$\overline{\overline{\mathsf{P}}}_+|\mathbf{\Phi}_+ = 0, \quad \overline{\overline{\mathsf{P}}}_+|\mathbf{\Phi}_- = \mathbf{\Phi}_-, \tag{42}$$

$$\overline{\overline{\mathsf{P}}}_-|\mathbf{\Phi}_- = 0, \quad \overline{\overline{\mathsf{P}}}_-|\mathbf{\Phi}_+ = \mathbf{\Phi}_+, \tag{43}$$

we can decompose any given dual bivector $\mathbf{\Phi}$ as

$$\mathbf{\Phi} = \mathbf{\Phi}_+ + \mathbf{\Phi}_-, \tag{44}$$

defining

$$\mathbf{\Phi}_+ = \overline{\overline{\mathsf{P}}}_-|\mathbf{\Phi}, \qquad \mathbf{\Phi}_- = \overline{\overline{\mathsf{P}}}_+|\mathbf{\Phi}. \tag{45}$$

In fact, the component fields satisfy the eigenequations

$$\overline{\overline{\mathsf{P}}}_\pm|\mathbf{\Phi}_\pm = \overline{\overline{\mathsf{P}}}_\pm|\overline{\overline{\mathsf{P}}}_\mp|\mathbf{\Phi} = 0 \quad \Rightarrow \quad (\overline{\overline{\mathsf{M}}} - M_\pm \overline{\overline{\mathsf{I}}}^{(2)T})|\mathbf{\Phi}_\pm = 0. \tag{46}$$

One can further show that the eigen-two-forms $\mathbf{\Phi}_\pm$ and $\mathbf{\Psi}_\pm$ are actually self-dual fields, i.e., each of them is invariant in a duality transformation. To see this, we first apply (18) to $\mathbf{\Phi}_\pm$ whence we obtain the scalar equation

$$\sin\theta Z\ Z_d M_\pm^2 + (Z - Z_d)\cos\theta M_\pm + \sin\theta = 0 \tag{47}$$

or

$$\cos\theta - ZM_\pm \sin\theta = \frac{Z}{Z_d}\cos\theta + \frac{1}{Z_d M_\pm}\sin\theta. \tag{48}$$

From (15), (16), the duality transformation of eigenfields can be written as

$$(\mathbf{\Psi}_\pm)_d = \frac{\lambda}{Z_d}(\sin\theta\overline{\overline{\mathsf{I}}}^{(2)T} + \cos\theta Z\overline{\overline{\mathsf{M}}})|\mathbf{\Phi}_\pm = \lambda(\frac{1}{Z_d M_\pm}\sin\theta + \frac{Z}{Z_d}\cos\theta)\mathbf{\Psi}_\pm, \tag{49}$$

$$(\mathbf{\Phi}_\pm)_d = \lambda(\cos\theta\overline{\overline{\mathsf{I}}}^{(2)T} - \sin\theta Z\overline{\overline{\mathsf{M}}})|\mathbf{\Phi}_\pm = \lambda(\cos\theta - \sin\theta Z M_\pm)\mathbf{\Phi}_\pm, \tag{50}$$

which are not yet of self-dual form. However, the factor λ in (9) can be freely chosen because its value does not affect the self-dual property of the medium. Actually, we can choose $\lambda = \lambda_\pm$ separately for each of the eigen-two-forms $\mathbf{\Phi}_\pm$. Taking (48) into account, we can set

$$\frac{1}{\lambda_\pm} = \cos\theta - ZM_\pm\sin\theta = \frac{Z}{Z_d}\cos\theta + \frac{1}{Z_d M_\pm}\sin\theta. \tag{51}$$

Denoting now the two duality transformations thus defined by the subscripts $d_\pm$, we have

$$(\mathbf{\Psi}_\pm)_{d_\pm} = \mathbf{\Psi}_\pm, \qquad (\mathbf{\Phi}_\pm)_{d_\pm} = \mathbf{\Phi}_\pm, \tag{52}$$

and, indeed, the eigenfields are self-dual with respect to the corresponding duality transformation. With respect to the other transformation the rules are

$$(\mathbf{\Psi}_\pm)_{d_\mp} = \frac{\lambda_\mp}{\lambda_\pm}\mathbf{\Psi}_\pm, \qquad (\mathbf{\Phi}_\pm)_{d_\mp} = \frac{\lambda_\mp}{\lambda_\pm}\mathbf{\Phi}_\pm. \tag{53}$$

7. EFFECTIVE MEDIA

The condition for the eigenfields

$$\mathbf{\Psi}_\pm = \overline{\overline{\mathsf{M}}}|\mathbf{\Phi}_\pm = M_\pm \mathbf{\Phi}_\pm \tag{54}$$

can be decomposed in its three-dimensional components as

$$\mathbf{D}_\pm = M_\pm \mathbf{B}_\pm, \qquad \mathbf{H}_\pm = -M_\pm \mathbf{E}_\pm. \tag{55}$$

From the medium relation satisfied by the eigenfields,

$$\mathbf{e}_{123}\lfloor \mathbf{D}_\pm = \overline{\overline{\epsilon}}_g|\mathbf{E}_\pm + \overline{\overline{\xi}}_g|\mathbf{H}_\pm, \tag{56}$$

we obtain a relation between the eigenfields $\mathbf{B}_\pm$ and $\mathbf{E}_\pm$,

$$\mathbf{e}_{123}\lfloor \mathbf{B}_\pm = \frac{1}{M_\pm}(\overline{\overline{\epsilon}}_g - M_\pm \overline{\overline{\xi}}_g)|\mathbf{E}_\pm = \overline{\overline{\mathsf{X}}}_\pm|\mathbf{E}_\pm. \tag{57}$$

The dyadic $\overline{\overline{\mathsf{X}}}_\pm \in \mathbb{E}_1\mathbb{E}_1$ can be expressed in terms of (32) and (33) as

$$\overline{\overline{\mathsf{X}}}_\pm = \frac{\overline{\overline{\epsilon}}_g}{M_\pm} - \overline{\overline{\xi}}_g = \pm\frac{M_- - M_+}{2}\overline{\overline{\mathsf{Q}}} + \frac{M_+ + M_-}{2}\overline{\overline{\mathsf{T}}}. \tag{58}$$

(57) corresponds to the conditions (46). It shows how the eigenfields are related in the self-dual medium and will be called the polarization condition.

Because the eigenfields are not general fields but restricted by the polarization condition, for each eigenfield one can replace the original medium by an effective medium which is simpler than the original medium. The effective Gibbsian permittivity and permeability dyadics denoted by $\overline{\overline{\epsilon}}_{g\pm}, \overline{\overline{\mu}}_{g\pm}$ can be introduced by applying (55) in the Gibbsian medium equations as

$$\mathbf{e}_{123}\lfloor \mathbf{D}_\pm = \overline{\overline{\epsilon}}_g|\mathbf{E}_\pm + \overline{\overline{\xi}}_g|\mathbf{H}_\pm = (\overline{\overline{\epsilon}}_g - M_\pm\overline{\overline{\xi}}_g)|\mathbf{E}_\pm = \overline{\overline{\epsilon}}_{g\pm}|\mathbf{E}_\pm, \tag{59}$$

$$\mathbf{e}_{123}\lfloor\mathbf{B}_\pm = \overline{\overline{\zeta}}_g|\mathbf{E}_\pm + \overline{\overline{\mu}}_g|\mathbf{H}_\pm = (\overline{\overline{\mu}}_g - M_\pm^{-1}\overline{\overline{\zeta}}_g)|\mathbf{H}_\pm = \overline{\overline{\mu}}_{g\pm}|\mathbf{H}_\pm. \quad (60)$$

Thus, the four Gibbsian medium dyadics can be replaced by the four effective medium dyadics

$$\begin{pmatrix} \overline{\overline{\epsilon}}_{g\pm} & \overline{\overline{\xi}}_{g\pm} \\ \overline{\overline{\zeta}}_{g\pm} & \overline{\overline{\mu}}_{g\pm} \end{pmatrix} = \begin{pmatrix} \overline{\overline{\epsilon}}_g - M_\pm\overline{\overline{\xi}}_g & 0 \\ 0 & \overline{\overline{\mu}}_g - M_\pm^{-1}\overline{\overline{\zeta}}_g \end{pmatrix}$$
$$= \begin{pmatrix} M_\pm & 0 \\ 0 & -M_\pm^{-1} \end{pmatrix} \overline{\overline{\mathsf{X}}}_\pm. \quad (61)$$

The effective Gibbsian medium dyadics satisfy the relation

$$\overline{\overline{\epsilon}}_{g\pm}/M_\pm = -M_\pm\overline{\overline{\mu}}_{g\pm} = \overline{\overline{\mathsf{X}}}_\pm. \quad (62)$$

Obviously, the effective self-dual media are simplified special cases of the AB-medium. Green dyadics in 3D representation for such media have been previously studied in [14].

In four-dimensional representation, the effective medium dyadics take the form

$$\overline{\overline{\mathsf{M}}}_\pm = \overline{\overline{\epsilon}}_{g\pm}{}^\wedge_\wedge\mathbf{e}_4\mathbf{e}_4 - \mathbf{e}_{123}\mathbf{e}_{123}\lfloor\lfloor\overline{\overline{\mu}}_{g\pm}^{-1} = \overline{\overline{\epsilon}}_{g\pm}{}^\wedge_\wedge\mathbf{e}_4\mathbf{e}_4 - \frac{\overline{\overline{\mu}}_{g\pm}^{(2)T}}{\mathbf{e}_{123}\mathbf{e}_{123}||\overline{\overline{\mu}}_{g\pm}^{(3)}}$$
$$= M_\pm\left(\overline{\overline{\mathsf{X}}}_\pm{}^\wedge_\wedge\mathbf{e}_4\mathbf{e}_4 + \frac{\overline{\overline{\mathsf{X}}}_\pm^{(2)T}}{\mathbf{e}_{123}\mathbf{e}_{123}||\overline{\overline{\mathsf{X}}}_\pm^{(3)}}\right). \quad (63)$$

8. DECOMPOSED FIELDS

For a self-dual medium, the two Maxwell equations

$$\mathbf{d}\wedge\mathbf{\Phi} = \mathbf{d}\wedge\mathbf{\Phi}_+ + \mathbf{d}\wedge\mathbf{\Phi}_- = \boldsymbol{\gamma}_m, \quad (64)$$

$$\mathbf{d}\wedge\mathbf{\Psi} = \mathbf{d}\wedge\overline{\overline{\mathsf{M}}}|\mathbf{\Phi} = M_+\mathbf{d}\wedge\mathbf{\Phi}_+ + M_-\mathbf{d}\wedge\mathbf{\Phi}_- = \boldsymbol{\gamma}_e, \quad (65)$$

can be decomposed to two uncoupled equation sets as

$$\mathbf{d}\wedge\mathbf{\Phi}_\pm = \boldsymbol{\gamma}_{m\pm}, \quad (66)$$

$$\mathbf{d}\wedge\mathbf{\Psi}_\pm = \boldsymbol{\gamma}_{e\pm}, \quad (67)$$

with

$$\boldsymbol{\gamma}_{m\pm} = \pm\frac{1}{M_+ - M_-}(\boldsymbol{\gamma}_e - M_\mp\boldsymbol{\gamma}_m), \qquad \boldsymbol{\gamma}_{m+} + \boldsymbol{\gamma}_{m-} = \boldsymbol{\gamma}_m, \quad (68)$$

$$\boldsymbol{\gamma}_{e\pm} = \pm\frac{M_\pm}{M_+ - M_-}(\boldsymbol{\gamma}_e - M_\mp\boldsymbol{\gamma}_m), \qquad \boldsymbol{\gamma}_{e+} + \boldsymbol{\gamma}_{e-} = \boldsymbol{\gamma}_e. \tag{69}$$

The sources $\boldsymbol{\gamma}_{e\pm}, \boldsymbol{\gamma}_{m\pm}$ can be easily shown to be self-dual with respect to the same duality transformations defined by (51) as the fields $\boldsymbol{\Phi}_\pm, \boldsymbol{\Psi}_\pm$. The decomposed sources are related by

$$\boldsymbol{\gamma}_{e\pm} = M_\pm\boldsymbol{\gamma}_{m\pm}, \tag{70}$$

whence the equations (67) are actually the same as (66). Thus, it is sufficient to take only one of the equations, say (66), together with the polarization condition (46) or (57) into account.

To be able to solve the fields due to given sources in a self-dual medium, we expand (66) in three-dimensions as

$$\mathbf{d}\wedge\boldsymbol{\Phi}_\pm = (\mathbf{d}_s + \partial_\tau\boldsymbol{\varepsilon}_4)\wedge(\mathbf{B}_\pm + \mathbf{E}_\pm\wedge\boldsymbol{\varepsilon}_4) = \boldsymbol{\varrho}_{m\pm} - \mathbf{J}_{m\pm}\wedge\boldsymbol{\varepsilon}_4. \tag{71}$$

Substituting the polarization condition

$$\mathbf{B}_\pm = \boldsymbol{\varepsilon}_{123}\lfloor\overline{\overline{\mathsf{X}}}_\pm|\mathbf{E}_\pm \tag{72}$$

and eliminating $\mathbf{B}_\pm$, the spatial two-form equation for $\mathbf{E}_\pm$ reads

$$\mathbf{d}_s\wedge\mathbf{E}_\pm + \partial_\tau\boldsymbol{\varepsilon}_{123}\lfloor\overline{\overline{\mathsf{X}}}_\pm|\mathbf{E}_\pm = -\mathbf{J}_{m\pm}. \tag{73}$$

Contraction by $\mathbf{e}_{123}\lfloor$ gives a vector-valued equation of the form

$$\overline{\overline{\mathsf{H}}}_\pm(\mathbf{d}_s,\partial_t)|\mathbf{E}_\pm = -\mathbf{e}_{123}\lfloor\mathbf{J}_{m\pm}, \tag{74}$$

where the two operator dyadics $\overline{\overline{\mathsf{H}}}_\pm(\mathbf{d}_s,\partial_\tau)\in\mathbb{E}_1\mathbb{E}_1$ are defined by

$$\overline{\overline{\mathsf{H}}}_\pm(\mathbf{d}_s,\partial_\tau) = \mathbf{e}_{123}\lfloor(\mathbf{d}_s\wedge\overline{\overline{\mathsf{I}}}^T + \boldsymbol{\varepsilon}_{123}\lfloor\overline{\overline{\mathsf{X}}}_\pm\partial_\tau) = (\mathbf{e}_{123}\lfloor\mathbf{d}_s)\lfloor\overline{\overline{\mathsf{I}}}^T + \overline{\overline{\mathsf{X}}}_\pm\partial_\tau. \tag{75}$$

It is seen that the operators depend on the medium only through the corresponding dyadics $\overline{\overline{\mathsf{X}}}_\pm$. If these are split in their respective symmetric and antisymmetric parts as

$$\overline{\overline{\mathsf{X}}}_\pm = \overline{\overline{\mathsf{S}}}_\pm + (\mathbf{e}_{123}\lfloor\boldsymbol{\alpha}_\pm)\lfloor\overline{\overline{\mathsf{I}}}^T, \tag{76}$$

where the $\boldsymbol{\alpha}_\pm$ are two dual vectors, we can write

$$\overline{\overline{\mathsf{H}}}_\pm(\mathbf{d}_s,\partial_\tau) = (\mathbf{e}_{123}\lfloor\mathbf{d}_\pm)\lfloor\overline{\overline{\mathsf{I}}}^T + \overline{\overline{\mathsf{S}}}_\pm\partial_\tau, \qquad \mathbf{d}_\pm = \mathbf{d}_s + \boldsymbol{\alpha}_\pm\partial_\tau. \tag{77}$$

Note that the differential operators $\mathbf{d}_\pm$ involve both spatial and temporal differentiation. The inverse operator dyadic can be expanded as [10]

$$\overline{\overline{\mathsf{H}}}_\pm^{-1} = \frac{1}{H_\pm}\boldsymbol{\varepsilon}_{123}\boldsymbol{\varepsilon}_{123}\lfloor\lfloor\overline{\overline{\mathsf{H}}}_\pm^{(2)T}, \qquad H_\pm = \boldsymbol{\varepsilon}_{123}\boldsymbol{\varepsilon}_{123}||\overline{\overline{\mathsf{H}}}_\pm^{(3)}, \tag{78}$$

whence the decomposed fields satisfying (74) are solutions to respective equations with a scalar operator,

$$H_\pm(\mathbf{d}_s,\partial_\tau)\mathbf{E}_\pm = -\boldsymbol{\varepsilon}_{123}\lfloor\overline{\overline{\mathsf{H}}}{}_\pm^{(2)T}(\mathbf{d}_s,\partial_\tau)|\mathbf{J}_{m\pm}. \tag{79}$$

The two scalar operators can be expanded as

$$\begin{aligned} H_\pm(\mathbf{d}_s,\partial_\tau) &= \boldsymbol{\varepsilon}_{123}\boldsymbol{\varepsilon}_{123}||((\mathbf{e}_{123}\lfloor\mathbf{d}_\pm)\lfloor\overline{\overline{\mathsf{I}}}{}^T + \overline{\overline{\mathsf{S}}}_\pm\partial_\tau)^{(3)} \\ &= \boldsymbol{\varepsilon}_{123}\boldsymbol{\varepsilon}_{123}||\Big(((\mathbf{e}_{123}\lfloor\mathbf{d}_\pm)\lfloor\overline{\overline{\mathsf{I}}}{}^T)^{(3)} + ((\mathbf{e}_{123}\lfloor\mathbf{d}_\pm)\lfloor\overline{\overline{\mathsf{I}}}{}^T)^{(2)}{}^\wedge_\wedge\overline{\overline{\mathsf{S}}}_\pm\partial_\tau \\ &\quad + ((\mathbf{e}_{123}\lfloor\mathbf{d}_\pm)\lfloor\overline{\overline{\mathsf{I}}}{}^T)^\wedge_\wedge\overline{\overline{\mathsf{S}}}{}_\pm^{(2)}\partial_\tau^2 + \overline{\overline{\mathsf{S}}}{}_\pm^{(3)}\partial_\tau^3\Big). \end{aligned} \tag{80}$$

Because of interplay of symmetric and antisymmetric dyadics, two of the terms vanish. In fact, inserting

$$((\mathbf{e}_{123}\lfloor\mathbf{d}_\pm)\lfloor\overline{\overline{\mathsf{I}}}{}^T)^{(2)} = \mathbf{e}_{123}\mathbf{e}_{123}\lfloor\lfloor\mathbf{d}_\pm\mathbf{d}_\pm, \tag{81}$$

$$((\mathbf{e}_{123}\lfloor\mathbf{d}_\pm)\lfloor\overline{\overline{\mathsf{I}}}{}^T)^{(3)} = 0, \tag{82}$$

$$\boldsymbol{\varepsilon}_{123}\boldsymbol{\varepsilon}_{123}||(((\mathbf{e}_{123}\lfloor\mathbf{d}_\pm)\lfloor\overline{\overline{\mathsf{I}}}{}^T)^\wedge_\wedge\overline{\overline{\mathsf{S}}}{}_\pm^{(2)}) = 0, \tag{83}$$

the double operator takes the form

$$\begin{aligned} H_\pm(\mathbf{d}_s,\partial_\tau) &= (\overline{\overline{\mathsf{S}}}_\pm||\mathbf{d}_\pm\mathbf{d}_\pm + \boldsymbol{\varepsilon}_{123}\boldsymbol{\varepsilon}_{123}||\overline{\overline{\mathsf{S}}}{}_\pm^{(3)}\partial_\tau^2)\partial_\tau \\ &= (\overline{\overline{\mathsf{S}}}_\pm||(\mathbf{d}_s+\boldsymbol{\alpha}_\pm\partial_\tau)(\mathbf{d}_s+\boldsymbol{\alpha}_\pm\partial_\tau) + \boldsymbol{\varepsilon}_{123}\boldsymbol{\varepsilon}_{123}||\overline{\overline{\mathsf{S}}}{}_\pm^{(3)}\partial_\tau^2)\partial_\tau, \end{aligned} \tag{84}$$

which is a second-order operator multiplied by ∂_τ. Obviously, scalar Green functions corresponding to these operators can be formed and the solutions for (79) corresponding to the decomposed sources $\mathbf{J}_{m\pm}$ can be expressed in integral form. Instead of studying the general problem in more detail, let us consider plane waves in a self-dual medium.

9. PLANE WAVES

Plane-wave fields are solutions for source-free Maxwell equations of the form

$$\boldsymbol{\Phi}(\mathbf{x}) = \boldsymbol{\Phi}_o\exp(\boldsymbol{\nu}|\mathbf{x}), \tag{85}$$

where $\nu \in \mathbb{F}_1$ is the dual wave vector. In a self-dual medium, the plane waves are decomposed in two self-dual sets corresponding to the subscripts $\pm$. Let us use three-dimensional expansion

$$\boldsymbol{\nu} = \boldsymbol{\beta} + k\boldsymbol{\varepsilon}_4, \tag{86}$$

where $\boldsymbol{\beta}$ is the spatial part of $\boldsymbol{\nu}$ and k is the wavenumber of the wave. In this case, the equation for the decomposed fields requires

$$H_\pm(\boldsymbol{\beta}, k) = 0, \tag{87}$$

which from (84) has the form

$$\overline{\overline{\mathsf{S}}}_\pm||(\boldsymbol{\beta} + k\boldsymbol{\alpha}_\pm)(\boldsymbol{\beta} + k\boldsymbol{\alpha}_\pm) + k^2\boldsymbol{\varepsilon}_{123}\boldsymbol{\varepsilon}_{123}||\overline{\overline{\mathsf{S}}}_\pm^{(3)} = 0. \tag{88}$$

This is known as the dispersion equation which gives the relation between the plane-wave quantities $\boldsymbol{\beta}$ and k. For a given $\boldsymbol{\beta}$, the wavenumbers $k = k_\pm(\boldsymbol{\beta})$ corresponding to the two eigenwaves can be readily solved from the quadratic equation

$$k_\pm^2 + 2k_\pm \frac{\boldsymbol{\beta}|\overline{\overline{\mathsf{S}}}_\pm|\boldsymbol{\alpha}_\pm}{\boldsymbol{\varepsilon}_{123}\boldsymbol{\varepsilon}_{123}||\overline{\overline{\mathsf{S}}}_\pm^{(3)}} + \frac{\overline{\overline{\mathsf{S}}}_\pm||\boldsymbol{\beta}\boldsymbol{\beta}}{\boldsymbol{\varepsilon}_{123}\boldsymbol{\varepsilon}_{123}||\overline{\overline{\mathsf{S}}}_\pm^{(3)}} = 0. \tag{89}$$

After solving the wavenumbers $k_\pm(\boldsymbol{\beta})$, the eigenfields $\mathbf{E}_\pm$ satisfying the equations

$$\overline{\overline{\mathsf{H}}}_\pm(\boldsymbol{\beta}, k_\pm)|\mathbf{E}_\pm = 0, \tag{90}$$

corresponding to a given $\boldsymbol{\beta}$, can be found. In fact, applying the identity [10]

$$\overline{\overline{\mathsf{H}}}_\pm^{(2)}\lfloor\mathbf{E}_\pm = (\overline{\overline{\mathsf{H}}}_\pm|\mathbf{E}_\pm) \wedge \overline{\overline{\mathsf{H}}}_\pm, \tag{91}$$

valid for any dyadics $\overline{\overline{\mathsf{H}}}_\pm \in \mathbb{E}_1\mathbb{E}_1$, we obtain

$$\overline{\overline{\mathsf{H}}}_\pm^{(2)}(\boldsymbol{\beta}, k_\pm)\lfloor\mathbf{E}_\pm = 0. \tag{92}$$

At this point we can apply another identity [10]

$$\mathbf{C}_\pm\rfloor(\mathbf{E}_\pm \wedge \mathbf{\Pi}_\pm) = \mathbf{E}_\pm(\mathbf{C}_\pm|\mathbf{\Pi}_\pm) + \mathbf{\Pi}_\pm\lfloor(\mathbf{C}_\pm\lfloor\mathbf{E}_\pm), \tag{93}$$

valid for any bivector $\mathbf{C}_\pm$, dual vector $\mathbf{E}_\pm$ and dual bivector $\mathbf{\Pi}_\pm$. Choosing $\mathbf{C}_\pm = \mathbf{\Gamma}_\pm|\overline{\overline{\mathsf{H}}}_\pm^{(2)}$ for some dual bivector $\mathbf{\Gamma}_\pm \in \mathbb{F}_2$ and noting that the spatial trivector $\mathbf{E}_\pm \wedge \mathbf{\Pi}_\pm$ is a scalar multiple of $\boldsymbol{\varepsilon}_{123}$, from (93) and (92) we can solve $\mathbf{E}_\pm$ in the simple form

$$\mathbf{E}_\pm = \mathbf{\Gamma}_\pm|\overline{\overline{\mathsf{H}}}_\pm^{(2)}\lfloor\boldsymbol{\varepsilon}_{123}. \tag{94}$$

The magnitudes of the two eigenfields $\mathbf{E}_\pm$ depend on the two dual bivectors $\mathbf{\Gamma}_\pm$ which must be chosen so that the right-hand side of (94) does not vanish. This is always possible if $\overline{\overline{\mathsf{H}}}_\pm^{(2)} \neq 0$. The converse (degenerate) case is omitted here.

The expression (94), whenever nonzero, gives the 3D eigenfields $\mathbf{E}_\pm$ of the plane wave. The corresponding 4D eigen-two-forms $\mathbf{\Phi}_\pm$ can be found from

$$\begin{aligned}\mathbf{\Phi}_\pm &= \mathbf{B}_\pm + \mathbf{E}_\pm \wedge \varepsilon_4 = (\varepsilon_{123}\lfloor\overline{\overline{\mathsf{X}}}_\pm - \varepsilon_4 \wedge \bar{\mathsf{I}}^T)|\mathbf{E}_\pm \\ &= (\varepsilon_{123}\lfloor\overline{\overline{\mathsf{X}}}_\pm - \varepsilon_4 \wedge \bar{\mathsf{I}}^T)|(\mathbf{\Gamma}_\pm|\overline{\overline{\mathsf{H}}}_\pm^{(2)}(\boldsymbol{\beta}, k_\pm)\rfloor\varepsilon_{123}). \\ &= (\varepsilon_{123}\varepsilon_{123}\lfloor\lfloor\overline{\overline{\mathsf{X}}}_\pm - \varepsilon_4 \wedge \bar{\mathsf{I}}^T\rfloor\varepsilon_{123})|(\mathbf{\Gamma}_\pm|\overline{\overline{\mathsf{H}}}_\pm^{(2)}(\boldsymbol{\beta}, k_\pm)) \end{aligned} \qquad (95)$$

and, finally, the other two-forms as $\mathbf{\Psi}_\pm = M_\pm \mathbf{\Phi}_\pm$.

10. CONCLUSION

Using differential-form representation, the class of self-dual media was defined in four-dimensional formalism as consisting of media which are invariant in a duality transformation other than the trivial one. It was shown that a medium is self-dual exactly when its 4D medium dyadic $\overline{\overline{\mathsf{M}}} \in \mathbb{E}_2\mathbb{F}_2$ satisfies an algebraic equation of the second degree. Corresponding conditions for the four 3D (Gibbsian) medium dyadics were shown to coincide with those obtained earlier through Gibbsian vector analysis. It was further shown that, in a self-dual medium, any electromagnetic field can be decomposed in two noncoupled self-dual parts corresponding to decomposed self-dual sources. It is possible to define two effective media which can replace the original self-dual medium for the respective decomposed fields. Because the effective media are simpler than the original medium, problem solution can be simplified through the self-dual decomposition of fields and sources. As an example, plane-wave solutions were considered for the general self-dual medium.

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