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ANALYSIS OF LOSSY INHOMOGENEOUS PLANAR LAYERS USING FINITE DIFFERENCE METHOD

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Abstract—A general method is introduced to frequency domain analysis of lossy Inhomogeneous Planar Layers (IPLs). In this method, the IPLs are subdivided to several thin homogeneous layers, at first. Then the electric and magnetic fields are obtained using second order finite difference method. The accuracy of the method is studied using analysis of some special types of IPLs.

1. INTRODUCTION

Inhomogeneous Planar Layers (IPL) are widely used in electromagnetics as optimum shields and filters and etc. Also, the IPLs potentially provide less scattering, less stress, larger bandwidth and better coupling effects than homogeneous planar layers [1–8]. The differential equations describing IPLs have non-constant coefficients and so except for a few special cases no analytical solution exists for them. The IPLs with variations such as inverse of distance $(1/z)$, inverse of distance with power two $(1/z^2)$ and exponential of distance (e^z) are some of these special cases [8]. Of course, the most straightforward method to analyze IPLs is subdividing them into many thin homogeneous layers then using the concept of analysis of multilayer structures [9].

The subject of this paper is using second order finite difference method to analyze lossy and dispersive IPLs. In this method, the IPLs are subdivided to several thin homogeneous layers, at first. Then the electric and magnetic fields are obtained using second order finite difference method. Some closed relations, for which second derivative of the fields has been considered, are obtained for this purpose. This method is applicable to all arbitrary lossy and dispersive IPLs. The accuracy of the method is studied using analysis of some special kinds of IPLs.

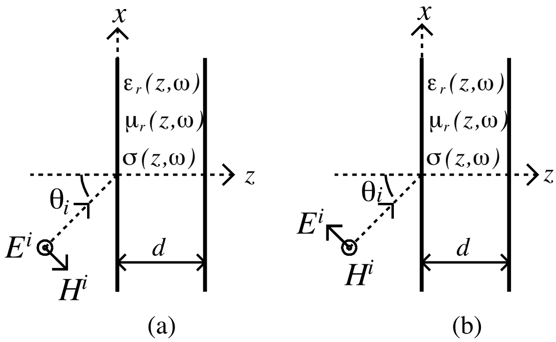

Figure 1. Incident plane wave to IPL structures a) TE^z polarization mode b) TM^z polarization mode.

2. THE EQUATIONS OF IPLS

In this section, the frequency domain equations of the IPLs are reviewed. Figure 1 shows a typical IPL with the thickness of d. Two different polarizations are possible, one the TM^z and other the TE^z. It is assumed that the incident plane wave propagates obliquely towards positive x and z direction with an angle of incidence θ_i , electric filed strength of E^i and the velocity of c (the velocity of the light in the free space).

It is evident that all components of the electric and magnetic fields can be expressed as follows

$$f(x, y, z, \omega) = g(z, \omega) \exp(-jk_x x) \tag{1}$$

in which

$$k_x = \frac{\omega}{c} \sin(\theta_i) \tag{2}$$

First, two following parameters are defined versus the angular frequency, ω, and the distance from the first surface of the layer, z.

$$\hat{Z}(z, \omega) = j\omega\mu_0\mu_r(z, \omega) \tag{3}$$

$$\hat{Y}(z, \omega) = \sigma(z, \omega) + j\omega\varepsilon_0\varepsilon_r(z, \omega) \tag{4}$$

The differential equations describing lossy and dispersive IPLs are given by

$$\frac{dE_y(z, \omega)}{dz} = \hat{Z}(z, \omega)H_x(z, \omega) \tag{5}$$

$$\frac{dH_x(z,\omega)}{dz} = \left(\hat{Y}(z,\omega) + k_x^2 \hat{Z}^{-1}(z,\omega)\right) E_y(z,\omega) \tag{6}$$

$$H_z = jk_x \hat{Z}^{-1}(z,\omega) E_y \tag{7}$$

for TE^z polarization and

$$\frac{dE_x(z,\omega)}{dz} = -\left(\hat{Z}(z,\omega) + k_x^2 \hat{Y}^{-1}(z,\omega)\right) H_y(z,\omega) \tag{8}$$

$$\frac{dH_y(z,\omega)}{dz} = -\hat{Y}(z,\omega) E_x(z,\omega) \tag{9}$$

$$E_z = -jk_x \hat{Y}^{-1}(z,\omega) H_y \tag{10}$$

for TM^z polarization.

Furthermore, there are two boundary conditions as follows

$$E_y(0,\omega) - \frac{\eta_0}{\cos(\theta_i)} H_x(0,\omega) = 2E^i(\omega) \tag{11}$$

$$E_y(d,\omega) + \frac{\eta_0}{\cos(\theta_i)} H_x(d,\omega) = 0 \tag{12}$$

for TE^z polarization and

$$E_x(0,\omega) + \eta_0 \cos(\theta_i) H_y(0,\omega) = 2E^i(\omega)\cos(\theta_i) \tag{13}$$

$$E_x(d,\omega) - \eta_0 \cos(\theta_i) H_y(d,\omega) = 0 \tag{14}$$

for TM^z polarization. In (11)–(14), $\eta_0 = \sqrt{\mu_0/\varepsilon_0}$ is the wave impedance in the free space. After determining the electric and magnetic fields along the IPLs, other electromagnetic functions of the structure will be obtainable using their defined relations. For example the reflection and the transmission coefficients will be determined as follows

$$\Gamma_{in}(\omega) = \begin{cases} \dfrac{1}{E^i} E_y(0,\omega) - 1, & \mathrm{TE} \\ \dfrac{1}{E^i \cos(\theta_i)} E_x(0,\omega) - 1, & \mathrm{TM} \end{cases} \tag{15}$$

$$T(\omega) = \begin{cases} \dfrac{1}{E^i} E_y(d,\omega), & \mathrm{TE} \\ \dfrac{1}{E^i \cos(\theta_i)} E_x(d,\omega), & \mathrm{TM} \end{cases} \tag{16}$$

Combining (5) with (6) and (8) with (9), gives the following general differential equations for IPLs.

$$\frac{d^2 F(z,\omega)}{dz^2} - f(z,\omega) \frac{dF(z,\omega)}{dz} - g(z,\omega) F(z,\omega) = 0 \tag{17}$$

$$G(z,\omega) = -h^{-1}(z,\omega)\frac{dF(z,\omega)}{dz} \tag{18}$$

Where

$$F(z,\omega) \triangleq \begin{cases} E_y(z,\omega); & \text{TE} \\ H_y(z,\omega); & \text{TM} \end{cases} \tag{19}$$

$$G(z,\omega) \triangleq \begin{cases} H_x(z,\omega); & \text{TE} \\ E_x(z,\omega); & \text{TM} \end{cases} \tag{20}$$

$$g(z,\omega) = \hat{Z}(z,\omega)\hat{Y}(z,\omega) + k_x^2 \tag{21}$$

$$h(z,\omega) = \begin{cases} -\hat{Z}(z,\omega); & \text{TE} \\ \hat{Y}(z,\omega); & \text{TM} \end{cases} \tag{22}$$

$$f(z,\omega) = \frac{dh(z,\omega)}{dz}h^{-1}(z,\omega) \tag{23}$$

Furthermore, the boundary conditions in (11)–(14) can be written as follows

$$F(0,\omega) + PG(0,\omega) = Q(\omega) \tag{24}$$

$$F(d,\omega) - PG(d,\omega) = 0 \tag{25}$$

where

$$P = \begin{cases} -\dfrac{\eta_0}{\cos(\theta_i)}; & \text{TE} \\ \dfrac{1}{\eta_0\cos(\theta_i)}; & \text{TM} \end{cases} \tag{26}$$

$$Q(\omega) = \begin{cases} 2E^i(\omega); & \text{TE} \\ \dfrac{2E^i(\omega)}{\eta_0}; & \text{TM} \end{cases} \tag{27}$$

One sees from (17)–(25) that, analytically solving the equations of general type IPLs is a very hard problem.

3. ANALYSIS OF IPLS USING FINITE DIFFERENCE METHOD

In this section, the analysis of arbitrary IPLs using finite difference method is proposed. First, the IPLs are subdivided to N thin homogeneous layers with thickness of $\Delta z = d/N$. Then, two differential Equations (17) and (18) are discretized to obtain the following difference equations, respectively

$$\begin{aligned} & F(d-(n+1)\Delta z,\omega) \\ \cong\ & 2F(d-n\Delta z,\omega) - F(d-(n-1)\Delta z,\omega) + \frac{d^2F(z,\omega)}{dz^2}\Big|_{z=d-n\Delta z}\Delta z^2 \\ =\ & \Big(2+g(d-n\Delta z,\omega)\Delta z^2 - f(d-n\Delta z,\omega)\Delta z\Big)F(d-n\Delta z,\omega) \\ & +\left(f(d-n\Delta z,\omega)\Delta z-1\right)F(d-(n-1)\Delta z,\omega); \\ & n=1,2,\cdots,N-1 \end{aligned} \tag{28}$$

$$\begin{aligned} G(d-n\Delta z,\omega) &= -h^{-1}(d-n\Delta z,\omega)\frac{dF(z,\omega)}{dz}\Big|_{z=d-n\Delta z} \\ &\cong -h^{-1}(d-n\Delta z,\omega)\left(F(d-(n-1)\Delta z,\omega)\right. \\ &\quad \left.-F(d-n\Delta z,\omega)\right)/\Delta z;\quad n=1,2,\cdots,N \end{aligned} \tag{29}$$

To obtain (28)–(29), the forward difference and three points approximations has been used for the first and second derivatives of the function $F(z)$, respectively. To use (28), the function $F(z)$ at $z = d - \Delta z$ is required. It can be found using (17)–(18) and the boundary condition (25) as follows

$$\begin{aligned} & F(d-\Delta z,\omega) \\ \cong\ & F(d,\omega) - \frac{dF(z,\omega)}{dz}\Big|_{z=d}\Delta z + \frac{d^2F(z,\omega)}{dz^2}\Big|_{z=d}\Delta z^2/2 \\ =\ & F(d,\omega) + h(d,\omega)G(d,\omega)\Delta z \\ & +\left(-f(d,\omega)h(d,\omega)G(d,\omega)+g(d,\omega)F(d,\omega)\right)\Delta z^2/2 \\ =\ & \Big(1+h(d,\omega)P^{-1}\Delta z+0.5g(d,\omega)\Delta z^2 \\ & -0.5f(d,\omega)h(d,\omega)P^{-1}\Delta z^2\Big)F(d,\omega) \end{aligned} \tag{30}$$

Using (28)–(30), the electric and magnetic fields of all thin layers are obtained step-by-step from $z = d$ to $z = 0$. However, $F(d,\omega)$, which behaves like a scale factor, is required to be known in this process. This unknown parameter can be assumed unit, at first. Then its correct value is obtained so that the boundary condition (24) is satisfied. In this way, we will have

$$F(d,\omega) = \frac{Q(\omega)}{F(0,\omega)+PG(0,\omega)} \tag{31}$$

4. EXAMPLES AND RESULTS

In this section, two special types of IPLs are considered to analyze using the presented method. The time consumed for the examples was less than 1.0 sec. using a Pentium-4 PC and MATLAB program.

Type 1: (*Lossy and Homogeneous Planar Layer*)

Consider a lossy and homogeneous planar layer with the following parameters

$$\mu_r(z) = \mu_{r0} \tag{32}$$
$$\varepsilon_r(z) = \varepsilon_{r0} \tag{33}$$
$$\sigma(z) = \sigma_0 \tag{34}$$

It is simple to show that the exact electric field of this type of planar layers is as follows

$$E_y(z,\omega) = 2E^i \frac{\eta\cos(\theta_i)}{\eta\cos(\theta_i)+\eta_0} \frac{1}{1-\Gamma^2\exp(-2\gamma_z d)} (\exp(-\gamma_z z)+\Gamma\exp(\gamma_z(z-2d))) \tag{35}$$

for TE^z polarization and

$$E_x(z,\omega) = 2E^i \frac{\eta\cos(\theta_i)}{\eta+\eta_0\cos(\theta_i)} \frac{1}{1-\Gamma^2\exp(-2\gamma_z d)} (\exp(-\gamma_z z)+\Gamma\exp(\gamma_z(z-2d))) \tag{36}$$

for TM^z polarization, where

$$\gamma_z = \sqrt{j\omega\mu_0\mu_{r0}(\sigma_0+j\omega\varepsilon_0\varepsilon_{r0})+k_x^2} \tag{37}$$

$$\eta = \begin{cases} \dfrac{\hat{Z}}{\gamma_z}; & \text{TE}^z \\ \dfrac{\hat{Z}+k_x^2\hat{Y}^{-1}}{\gamma_z}; & \text{TM}^z \end{cases} \tag{38}$$

$$\Gamma = \begin{cases} \dfrac{\eta_0-\eta\cos(\theta_i)}{\eta_0+\eta\cos(\theta_i)}; & \text{TE}^z \\ \dfrac{\eta_0\cos(\theta_i)-\eta}{\eta_0\cos(\theta_i)+\eta}; & \text{TM}^z \end{cases} \tag{39}$$

Now, consider a lossy homogeneous planar layer with parameters of $\varepsilon_{r0} = 4$, $\mu_{r0} = 1.5$, $\sigma_0 = 0.02$ and $d = 40$ cm. It is exposed to a plane wave with the angle of incidence of $\theta_i = 60°$, the excitation frequency of $f = 1.0$ GHz and the electric field strength of $E^i = 1.0$ V/m.

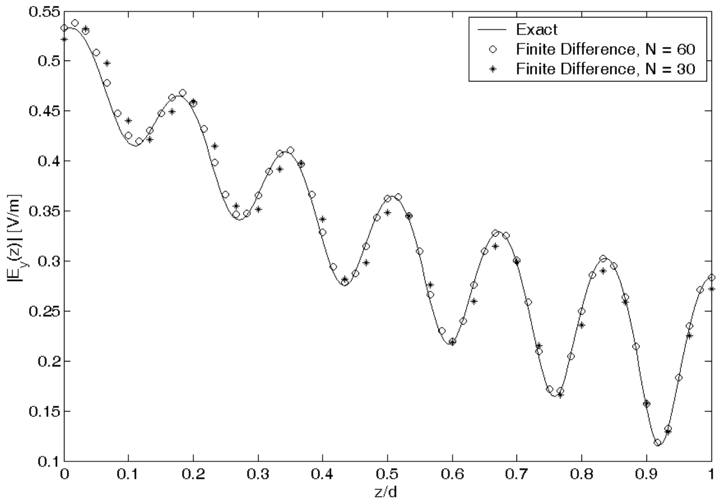

Figure 2. The amplitude of the transverse component of the electric field for TE^z polarization, obtained from exact formulas and from the presented method with $N = 30$ and $N = 60$ layers (for lossy and homogeneous planar layer).

Figures 2, 3, compare the amplitude of the transverse component of the electric field, obtained from (35)–(36) and from the presented method considering $N = 30$ and $N = 60$ layers, for TE^z and TM^z polarizations, respectively. One sees a good agreement between the exact solutions and the solutions obtained from the proposed method. It is seen and also evident that, as the number of layers, N, increases the accuracy of the obtained solutions increases. Also, the error has been spread along the whole thickness of the layer.

Type 2: (*Lossless and Exponential Inhomogeneous Layer*)

Consider a lossless and exponential IPL with the following parameters

$$\mu_r(z) = \mu_{r0} \tag{40}$$

$$\varepsilon_r(z) = \varepsilon_{r0} \exp(Kz) \tag{41}$$

$$\sigma(z) = 0 \tag{42}$$

Now, assume that $\varepsilon_{r0} = 4$, $\mu_{r0} = 1.0$, $d = 20\,\mathrm{cm}$ and $K = 1$. A plane wave with TE^z polarization, the angle of incidence of $\theta_i = 60°$ and the electric field strength of $E^i = 1.0\,\mathrm{V/m}$ illuminates the assumed structure. Figures 4, 5 compare the amplitude of the transverse component of the electric field, obtained from the exact solution

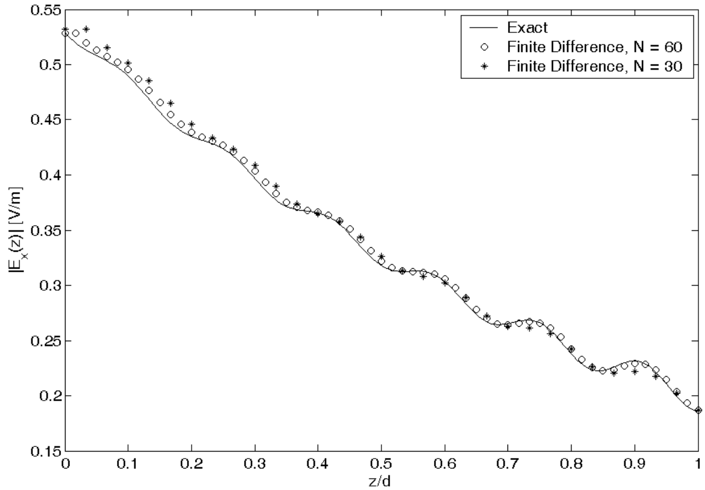

Figure 3. The amplitude of the transverse component of the electric field for TM^z polarization, obtained from exact formulas and from the presented method with $N = 30$ and $N = 60$ layers (for lossy and homogeneous planar layer).

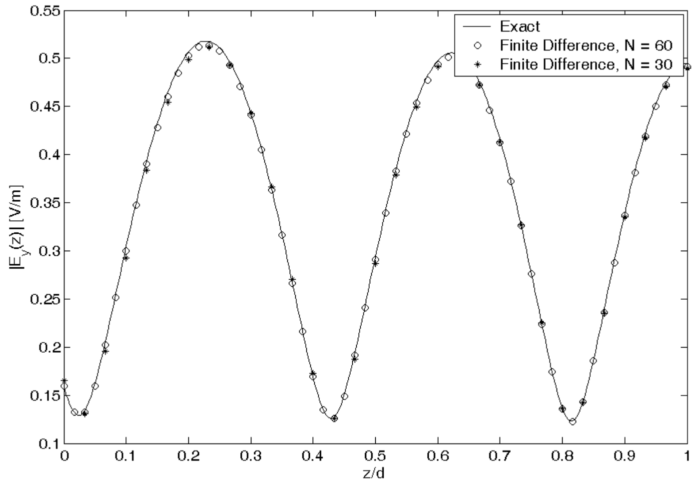

Figure 4. The amplitude of the transverse component of the electric field for TE^z polarization at frequency of $f = 1.0\,\mathrm{GHz}$, obtained from exact formulas and from the presented method with $N = 30$ and $N = 60$ layers (for lossless and exponential inhomogeneous planar layer).

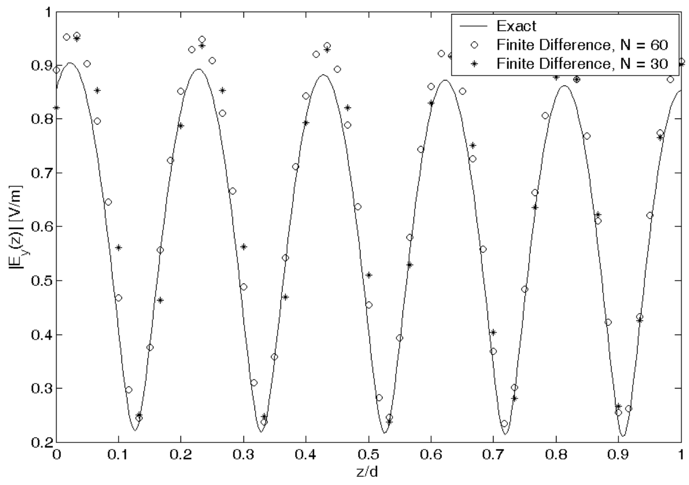

Figure 5. The amplitude of the transverse component of the electric field for TE^z polarization at frequency of $f = 2.0\,\text{GHz}$, obtained from exact formulas and from the presented method with $N = 30$ and $N = 60$ layers (for lossless and exponential inhomogeneous planar layer).

(in the Appendix) and from the presented method with $N = 30$ and $N = 60$ layers for the excitation frequency of 1.0 and 2.0 GHz, respectively. Also, Fig. 6 compares the amplitude of the reflection and transmission coefficients written in (15)–(16), versus the angle of incidence, obtained from the exact solution and from the presented method for the excitation frequency of $f = 1.0\,\text{GHz}$. Again, one sees a good agreement between the results from exact solution and the results from the presented method. Furthermore, as the source frequency increases, the accuracy of the method decreases. The better accuracy for larger angles of incidence, may be due to larger wavelength along the thickness of IPL for these angles ($\lambda_z = \lambda / \cos\theta_i$, in which λ is the wavelength in IPL).

According the above examples, one may conclude that the proposed method is applicable to all arbitrary IPLs. Also, it is concluded that as the excitation frequency, the length of the line (with respect to the wavelength) and the variations of the primary parameters increase, the necessary number of layers increases. To obtain a crude relation for the amount of error, consider a lossless and homogeneous planar layer. The relative error in (28) will be as

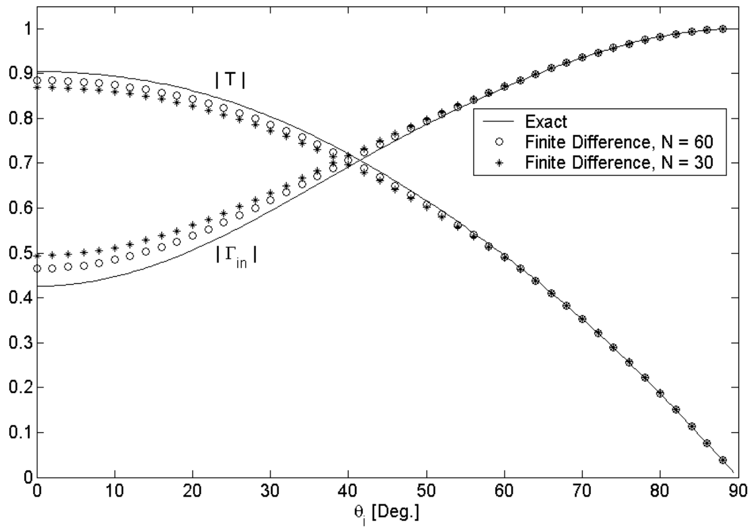

Figure 6. The amplitude of the reflection and transmission coefficients for TE^z polarization, obtained from exact formulas and from the presented method with $N = 30$ and $N = 60$ layers (for lossless and exponential inhomogeneous planar layer).

follows

$$\begin{aligned} Error &\cong \frac{1}{12F}\left|\frac{d^4F}{dz^4}\right|\Delta z^4 = \frac{1}{12}\left(\hat{Z}\hat{Y}+k_x^2\right)^2\Delta z^4 = \frac{1}{12}k_z^4\Delta z^4 \\ &= \frac{1}{12}\left(2\pi\frac{\Delta z}{\lambda}\cos\theta_i\right)^4 = \frac{130}{N^4}\left(\frac{d}{\lambda}\cos\theta_i\right)^4 \end{aligned} \tag{43}$$

For example, to have the relative error less than 10^{-3} for $d/\lambda = 2.2$ and $\theta_i = 60^\circ$ (as in Example 1), N must be greater than 37.

5. CONCLUSION

The second order finite difference method was used to analyze Inhomogeneous Planar Layers (IPLs). In this proposed method, the IPLs are subdivided to many homogeneous thin layers, at first. Then the electric and magnetic fields are obtained using step-by-step numerical integration. Some closed relations, for which second derivative of the voltage has been considered, are obtained for this purpose. It was seen that, as the variations of the IPL parameters, the excitation frequency and the thickness of the IPL (with respect to the wavelength) increases and the angle of incidence decreases, the necessary number of layers increases. The method is evaluated using analysis of some special kinds of IPLs. The method is very simple and fast and can be used for all arbitrary lossy and dispersive IPLs.

APPENDIX A.

The exact electric and magnetic fields of lossless exponential IPLs for TE^z polarization are determined. Using (40)–(42) in (5)–(6), the following second order differential equation is obtained.

$$\frac{d^2E_y(z,\omega)}{dz^2} - \left(k_x^2 - k_0^2\varepsilon_{r0}\exp(Kz)\right)E_y(z,\omega) = 0 \tag{A1}$$

Using the solution of (A1) in [8] and also (5), the following electric and magnetic fields are obtained.

$$\begin{aligned} E_y(z,\omega) &= A_1 J\left[\frac{2k_x}{K}, \frac{2k_0\sqrt{\varepsilon_{r0}}}{K}\exp\left(\frac{K}{2}z\right)\right] \\ &\quad + A_2 J\left[-\frac{2k_x}{K}, \frac{2k_0\sqrt{\varepsilon_{r0}}}{K}\exp\left(\frac{K}{2}z\right)\right] \end{aligned} \tag{A2}$$

$$\begin{aligned} H_x(z,\omega) &= \frac{-j\sqrt{\varepsilon_{r0}}}{\eta_0}\exp\left(\frac{K}{2}z\right)\left\{A_1 J'\left[\frac{2k_x}{K}, \frac{2k_0\sqrt{\varepsilon_{r0}}}{K}\exp\left(\frac{K}{2}z\right)\right]\right. \\ &\quad \left. + A_2 J'\left[-\frac{2k_x}{K}, \frac{2k_0\sqrt{\varepsilon_{r0}}}{K}\exp\left(\frac{K}{2}z\right)\right]\right\} \end{aligned} \tag{A3}$$

where the function $J[\alpha, \beta]$ is the Bessel function J of order α and argument β and the primes indicate the first derivative of the function with respect to its argument. From (A2)–(A3) and the boundary conditions (11)–(12), we have

$$a_1A_1 + a_2A_2 = 2E^i \tag{A4}$$

$$a_3A_1 + a_4A_2 = 0 \tag{A5}$$

in which

$$a_1 = J\left[\frac{2k_x}{K}, \frac{2k_0\sqrt{\varepsilon_{r0}}}{K}\right] + \frac{j\sqrt{\varepsilon_{r0}}}{\cos(\theta_i)}J'\left[\frac{2k_x}{K}, \frac{2k_0\sqrt{\varepsilon_{r0}}}{K}\right] \tag{A6}$$

$$a_2 = J\left[\frac{-2k_x}{K}, \frac{2k_0\sqrt{\varepsilon_{r0}}}{K}\right] + \frac{j\sqrt{\varepsilon_{r0}}}{\cos(\theta_i)}J'\left[\frac{-2k_x}{K}, \frac{2k_0\sqrt{\varepsilon_{r0}}}{K}\right] \tag{A7}$$

$$a_3 = J\left[\frac{2k_x}{K}, \frac{2k_0\sqrt{\varepsilon_{r0}}}{K}\exp\left(\frac{K}{2}d\right)\right]$$

$$-\frac{j\sqrt{\varepsilon_{r0}}}{\cos(\theta_i)}J'\left[\frac{2k_x}{K},\frac{2k_0\sqrt{\varepsilon_{r0}}}{K}\exp\left(\frac{K}{2}d\right)\right]\exp\left(\frac{K}{2}d\right) \quad \text{(A8)}$$

$$a_4 = J\left[\frac{-2k_x}{K},\frac{2k_0\sqrt{\varepsilon_{r0}}}{K}\exp\left(\frac{K}{2}d\right)\right] -\frac{j\sqrt{\varepsilon_{r0}}}{\cos(\theta_i)}J'\left[\frac{-2k_x}{K},\frac{2k_0\sqrt{\varepsilon_{r0}}}{K}\exp\left(\frac{K}{2}d\right)\right]\exp\left(\frac{K}{2}d\right) \quad \text{(A9)}$$

Finally, the unknown coefficients A_1 and A_2 are determined using (A4)–(A5) as follows

$$A_1 = \frac{2a_4}{a_1a_4 - a_2a_3}E^i \quad \text{(A10)}$$

$$A_2 = \frac{-2a_3}{a_1a_4 - a_2a_3}E^i \quad \text{(A11)}$$

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