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FRACTIONAL CURL OPERATOR IN CHIRAL MEDIUM AND FRACTIONAL NON-SYMMETRIC TRANSMISSION LINE

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Abstract—Fractional curl operator has been utilized to wave propagation in lossless, isotropic, homogeneous and reciprocal chiral medium when it contains interfaces. The fractional solutions for the corresponding standing wave solution and transverse impedance are determined. Equivalent fractional non-symmetric transmission line has also been analyzed.

1. INTRODUCTION

Fractional calculus is a branch of mathematics that deals with operators having non-integer and/or complex order, e.g., fractional derivative and fractional integral [1]. Tools of fractional calculus have various applications in different disciplines of science and engineering, e.g., Optics, Control and Mechanics etc. Mathematical recipe to fractionalize a linear operator is available in [2, 3]. Recently, while exploring the roles and applications of fractional calculus in electromagnetics a new fractional operator has been introduced [2]. The new fractional operator is termed as fractional curl operator.

Fractional curl operator has been utilized to find the new set of solutions to Maxwell's equations by fractionalizing the principle of duality [2]. New set of solutions is named as fractional dual solutions to the Maxwell equations. In electromagnetics, principle of duality states that if $(\underline{E}, \eta\underline{H})$ is one set of solutions (original solutions) to Maxwell equations, then other set of solutions (dual to the original solutions) is $(\eta\underline{H}, -\underline{E})$, where η is the impedance of the medium. The solutions which may be regarded as intermediate step between the original and dual to the original solutions may be obtained using the following relations [2]

$$\underline{E}_{fd} = \frac{1}{(ik)^\alpha}(\nabla\times)^\alpha \underline{E}$$
$$\eta \underline{H}_{fd} = \frac{1}{(ik)^\alpha}(\nabla\times)^\alpha \eta \underline{H}$$

where $(\nabla\times)^\alpha$ means fractional curl operator and $k = \omega\sqrt{\mu\epsilon}$ is the wavenumber of the medium. It may be noted that fd means fractional dual solutions. Naqvi et al. [4] afterward extended the work [2] and discussed the behavior of fractional dual solutions in an unbounded chiral medium. Lakhtakia [5] derived theorem which shows that a dyadic operator which commutes with curl operator can be used to find new solutions of the Faraday and Ampere-Maxwell equations. Veliev and Engheta [6] utilized the fractional curl operator to a fixed solution and obtained the fractional fields that represent the solution of reflection problem from an anisotropic surface. Naqvi and Rizvi [7] determined the sources corresponding to the fractional dual solutions. Naqvi and Abbas [8, 9] extended the work for metamaterial and for complex and higher order fractional curl operator respectively. In present work, we have determined the fractional solutions when chiral medium contains interfaces. Equivalent fractional non-symmetric transmission line has also been studied.

2. UNBOUNDED CHIRAL MEDIUM

Consider a uniform plane wave propagating along z-axis in an unbounded, lossless, isotropic and reciprocal chiral medium. According to field decomposition approach [10], field quantities, $\underline{E}$ and $\underline{H}$ may be pictured as consisting of two parts, i.e., $(\underline{E}_+, \underline{H}_+)$ and $(\underline{E}_-, \underline{H}_-)$. Two parts are termed as wavefields. The electric fields corresponding to two wavefields are

$$\underline{E}_\pm(z) = \underline{E}_\pm(0)\exp(ik_\pm z) \tag{1}$$

where $k_\pm = k(1 \pm \kappa_r)$ are wavenumbers of the two wavefields. $k = \omega\sqrt{\mu\epsilon}$ and $\kappa_r = \kappa\sqrt{\mu_0\epsilon_0/\mu\epsilon}$. κ is the chirality parameter. Using the following relation

$$\eta_\pm \underline{H}_\pm(z) = \pm i \underline{E}_\pm(z) \tag{2}$$

corresponding magnetic field may be obtained. In above expression

$$\eta_\pm = \sqrt{\frac{\mu_\pm}{\epsilon_\pm}} = \eta$$

This means that each wavefield sees chiral medium as achiral medium with equivalent constitutive parameters (ϵ_+, μ_+) and (ϵ_-, μ_-). Medium parameters of the equivalent isotropic media are related to the parameters of chiral medium by the following relations

$$\epsilon_\pm = \epsilon(1 \pm \kappa_r), \qquad \mu_\pm = \mu(1 \pm \kappa_r)$$

Simple expressions for the wavefields can be written as

$$\begin{aligned} \underline{E}_+ &= \frac{1}{2}\left(\underline{E} - j\eta\underline{H}\right) \\ \underline{E}_- &= \frac{1}{2}\left(\underline{E} + j\eta\underline{H}\right) \\ \underline{H}_+ &= \frac{1}{2}\left(\underline{H} + \frac{j}{\eta}\underline{E}\right) \\ \underline{H}_- &= \frac{1}{2}\left(\underline{H} - \frac{j}{\eta}\underline{E}\right) \end{aligned}$$

The total fields in chiral medium are

$$\begin{aligned} \underline{E}(z) &= \underline{E}^i_+(0)\exp(ik_+z) + \underline{E}^i_-(0)\exp(ik_-z) \\ \eta\underline{H}(z) &= i\left[\underline{E}^i_+(0)\exp(ik_+z) - \underline{E}^i_-(0)\exp(ik_-z)\right] \end{aligned} \tag{3}$$

Fractionalizing the electric fields $\underline{E}_+(z)$ and $\underline{E}_-(z)$

$$\begin{aligned} \underline{E}_{fd+}(z) &= \frac{1}{(ik_+)^\alpha}(\nabla\times)^\alpha\underline{E}_+(z) \\ &= \frac{1}{(ik_+)^\alpha}\left\{(\hat{z}\times)^\alpha\underline{E}^i_+(0)\right\}\left\{\frac{d^\alpha}{dz^\alpha}\exp(ik_+z)\right\} \\ &= \underline{E}^i_+(0)\exp\left\{i\left(k_+z + \frac{\alpha\pi}{2}\right)\right\} \end{aligned}$$

Similarly

$$\underline{E}_{fd-}(z) = \underline{E}^i_-(0)\exp\left\{i\left(k_-z - \frac{\alpha\pi}{2}\right)\right\}$$

Fractional dual fields corresponding to the original fields given in equation (3) may be written as

$$\begin{aligned}\underline{E}_{fd}(z) &= \underline{E}^i_+(0)\exp\left\{i\left(k_+z+\frac{\alpha\pi}{2}\right)\right\}\\ &\quad+\underline{E}^i_-(0)\exp\left\{i\left(k_-z-\frac{\alpha\pi}{2}\right)\right\}\\ \eta\underline{H}_{fd}(z) &= i\left[\underline{E}^i_+(0)\exp\left\{i\left(k_+z+\frac{\alpha\pi}{2}\right)\right\}\right.\\ &\quad\left.-\underline{E}^i_-(0)\exp\left\{i\left(k_-z-\frac{\alpha\pi}{2}\right)\right\}\right]\end{aligned} \tag{4}$$

It is obvious from equation (4) that for $\alpha = 0$

$$\underline{E}_{fd}(z) = \underline{E}(z), \qquad \eta\underline{H}_{fd}(z) = \eta\underline{H}(z)$$

and for $\alpha = 1$

$$\underline{E}_{fd}(z) = \eta\underline{H}(z), \qquad \eta\underline{H}_{fd}(z) = -\underline{E}(z)$$

which is consistent with electromagnetics principle of duality. For $0 < \alpha < 1$, solutions may be regarded as intermediate between the original and dual to the original solutions.

3. STANDING WAVES IN CHIRAL MEDIUM

Assume a plane wave hits normally a chiral-chiral interface located at $z = 0$. The intrinsic impedance before the interface is η while after the interface is η_1. Total electric field of wavefields before the interface may be written as

$$\underline{E}_\pm(z) = \underline{E}^i_\pm(0)\exp(ik_\pm z) + R_{\mp\pm}\underline{E}^i_\pm(0)\exp(-ik_\mp z)$$

where R_+- is the reflection co-efficient for negative incident wavefield and positive reflected wavefield while R_-+ is the reflection coefficient for positive incident wavefield and negative reflected wave field. For reciprocal chiral media, the reflection coefficient becomes $R_{\mp\pm} = R = \frac{\eta_1-\eta}{\eta_1+\eta}$. So, we can write

$$\begin{aligned}\underline{E}_\pm(z) &= \underline{E}^i_\pm(0)\exp(ik_\pm z) - R\underline{E}^i_\pm(0)\exp(-ik_\mp z)\\ &= \underline{E}^i_\pm(0)\{\exp(ik_\pm z) + R\exp(-ik_\mp z)\}\end{aligned}$$

The corresponding magnetic field is

$$\eta\underline{H}_\pm(z) = \pm i\underline{E}^i_\pm(0)\{\exp(ik_\pm z) - R\exp(-ik_\mp z)\}$$

Using above expressions, total fields may be written as

$$\begin{aligned}\underline{E}(z) &= \underline{E}_{+}^{i}(0)\{\exp(ik_{+}z) + R\exp(-ik_{-}z)\} \\ &\quad +\underline{E}_{-}^{i}(0)\{\exp(ik_{-}z) + R\exp(-ik_{+}z)\} \\ &= E_1 + E_2 + E_3 + E_4 \end{aligned} \tag{5}$$

$$\begin{aligned}\eta\underline{H}(z) &= i\underline{E}_{+}^{i}(0)\{\exp(ik_{+}z) - R\exp(-ik_{-}z)\} \\ &\quad -i\underline{E}_{-}^{i}(0)\{\exp(ik_{-}z) - R\exp(-ik_{+}z)\} \\ &= \eta H_1 + \eta H_2 + \eta H_3 + \eta H_4 \end{aligned} \tag{6}$$

Let us write fractional dual solution to each of the field components as

$$\begin{aligned}\underline{E}_{1fd} &= \frac{1}{(ik_{+})^{\alpha}}(\bigtriangledown\times)^{\alpha}\{\underline{E}_{+}^{i}(0)\exp(ik_{+}z)\} \\ &= (i)^{\alpha}\underline{E}_{+}^{i}(0)\exp(ik_{+}z)\end{aligned}$$

Similarly other components can be written as

$$\begin{aligned}\underline{E}_{2fd} &= -(i)^{\alpha}(-1)^{\alpha}\underline{E}_{+}^{i}(0)\exp(-ik_{-}z) \\ \underline{E}_{3fd} &= (-i)^{\alpha}\underline{E}_{-}^{i}(0)\exp(ik_{-}z) \\ \underline{E}_{4fd} &= -(-i)^{\alpha}(-1)^{\alpha}\underline{E}_{-}^{i}(0)\exp(-ik_{+}z) \\ \eta\underline{H}_{1fd} &= i(i)^{\alpha}\underline{E}_{+}^{i}(0)\exp(ik_{+}z) \\ \eta\underline{H}_{2fd} &= i(i)^{\alpha}(-1)^{\alpha}\underline{E}_{+}^{i}(0)\exp(-ik_{-}z) \\ \eta\underline{H}_{3fd} &= -i(-i)^{\alpha}\underline{E}_{-}^{i}(0)\exp(ik_{-}z) \\ \eta\underline{H}_{4fd} &= -i(-i)^{\alpha}(-1)^{\alpha}\underline{E}_{-}^{i}(0)\exp(-ik_{+}z)\end{aligned}$$

Using these values in equation (5) and equation (6), we can write fractional dual solutions as

$$\begin{aligned}\underline{E}_{fd}(z) &= (i)^{\alpha}\underline{E}_{+}^{i}(0)\left[\exp\{i(k_{+}z)\} + R(-1)^{\alpha}\exp\{-i(k_{-}z)\}\right] \\ &\quad +(-i)^{\alpha}\underline{E}_{-}^{i}(0)\left[\exp\{i(k_{-}z)\} + R(-1)^{\alpha}\exp\{-i(k_{+}z)\}\right] \\ \eta\underline{H}_{fd}(z) &= i(i)^{\alpha}\underline{E}_{+}^{i}(0)\left[\exp\{i(k_{+}z)\} - R(-1)^{\alpha}\exp\{-i(k_{-}z)\}\right] \\ &\quad -i(-i)^{\alpha}\underline{E}_{-}^{i}(0)\left[\exp\{i(k_{-}z)\} - R(-1)^{\alpha}\exp\{-i(k_{+}z)\}\right]\end{aligned}$$

on simplifying, we can write

$$\underline{E}_{fd}(z) = \exp\left(\frac{i\alpha\pi}{2}\right)$$

$$\left\{(i)^\alpha \underline{E}^i_+(0)\left[\exp\left\{i\left(k_+z-\frac{\alpha\pi}{2}\right)\right\}+R\exp\left\{-i\left(k_-z-\frac{\alpha\pi}{2}\right)\right\}\right]\right.$$
$$\left.+(-i)^\alpha \underline{E}^i_-(0)\left[\exp\left\{i\left(k_-z-\frac{\alpha\pi}{2}\right)\right\}+R\exp\left\{-i\left(k_+z-\frac{\alpha\pi}{2}\right)\right\}\right]\right\} \tag{7}$$

$$\eta\underline{H}_{fd}(z) = i\exp\left(\frac{i\alpha\pi}{2}\right)$$
$$\left\{(i)^\alpha \underline{E}^i_+(0)\left[\exp\left\{i\left(k_+z-\frac{\alpha\pi}{2}\right)\right\}-R\exp\left\{-i\left(k_-z-\frac{\alpha\pi}{2}\right)\right\}\right]\right.$$
$$\left.-(-i)^\alpha \underline{E}^i_-(0)\left[\exp\left\{i\left(k_-z-\frac{\alpha\pi}{2}\right)\right\}-R\exp\left\{-i\left(k_+z-\frac{\alpha\pi}{2}\right)\right\}\right]\right\} \tag{8}$$

It is obvious that for $\alpha = 0$

$$\underline{E}_{fd}(z) = \underline{E}(z) \qquad \eta\underline{H}_{fd}(z) = \eta\underline{H}(z)$$

which is original set of solutions given by equations (5) and (6). For $\alpha = 1$

$$\underline{E}_{fd}(z) = \eta\underline{H}(z) \qquad \eta\underline{H}_{fd}(z) = -\underline{E}(z)$$

which is dual to the original set of solutions. For $0 < \alpha < 1$, solutions set given by (7) and (8) may be regarded as fractional dual solutions corresponding to the original solutions set given by (5) and (6).

4. TRANSVERSE WAVE IMPEDANCE

Transverse impedance of fractional dual fields is defined as

$$Z_{fdxy} = \frac{E_{fdx}}{H_{fdy}} = -\frac{E_{fdy}}{H_{fdx}} \tag{9}$$

where

$$E_{fdx}(z) = \exp(\frac{i\alpha\pi}{2})$$
$$\left\{(i)^\alpha E^i_+(0)\left[\exp\left\{i\left(k_+z-\frac{\alpha\pi}{2}\right)\right\}+R\exp\left\{-i\left(k_-z-\frac{\alpha\pi}{2}\right)\right\}\right]\right.$$
$$\left.+(-i)^\alpha E^i_-(0)\left[\exp\left\{i\left(k_-z-\frac{\alpha\pi}{2}\right)\right\}+R\exp\left\{-i\left(k_+z-\frac{\alpha\pi}{2}\right)\right\}\right]\right\}$$

and

$$\eta H_{fdy}(z) = \exp\left(\frac{i\alpha\pi}{2}\right)$$

$$\left\{(i)^\alpha E_+^i(0)\left[\exp\left\{i\left(k_+z-\frac{\alpha\pi}{2}\right)\right\}-R\exp\left\{-i\left(k_-z-\frac{\alpha\pi}{2}\right)\right\}\right]\right.$$
$$\left.+(-i)^\alpha E_-^i(0)\left[\exp\left\{i\left(k_-z-\frac{\alpha\pi}{2}\right)\right\}-R\exp\left\{-i\left(k_+z-\frac{\alpha\pi}{2}\right)\right\}\right]\right\}$$

Using relations $k_+ = k + k\kappa_r$ and $k_- = k - k\kappa_r$, above equations take the following form

$$\begin{aligned} E_{fdx}(z) &= \exp\left(\frac{i\alpha\pi}{2}\right)\left[\exp\left\{i\left(kz-\frac{\alpha\pi}{2}\right)\right\}+R\exp\left\{-i\left(kz-\frac{\alpha\pi}{2}\right)\right\}\right] \\ &\quad \left\{(i)^\alpha E_+^i(0)\exp(+ik\kappa_r z)+(-i)^\alpha E_-^i(0)\exp(-ik\kappa_r z)\right\} \\ H_{fdy}(z) &= \frac{1}{\eta}\exp\left(\frac{i\alpha\pi}{2}\right)\left[\exp\left\{i\left(kz-\frac{\alpha\pi}{2}\right)\right\}-R\exp\left\{-i\left(kz-\frac{\alpha\pi}{2}\right)\right\}\right] \\ &\quad \left\{(i)^\alpha E_+^i(0)\exp(+ik\kappa_r z)+(-i)^\alpha E_-^i(0)\exp(-ik\kappa_r z)\right\} \end{aligned}$$

Substituting the E_{fdx} and H_{fdy} in equation (9), following is obtained

$$Z_{fdxy} = \eta\frac{\left[\exp\left\{i\left(kz-\frac{\alpha\pi}{2}\right)\right\}+R\exp\left\{-i\left(kz-\frac{\alpha\pi}{2}\right)\right\}\right]}{\left[\exp\left\{i\left(kz-\frac{\alpha\pi}{2}\right)\right\}-R\exp\left\{-i\left(kz-\frac{\alpha\pi}{2}\right)\right\}\right]}$$

For PEC interface, above yields the following

$$Z_{fdxy} = i\eta\tan\left(kz-\frac{\alpha\pi}{2}\right) \tag{10a}$$

and at $z = 0$

$$Z_{fdxy} = -i\eta\tan\left(\frac{\alpha\pi}{2}\right) \tag{10b}$$

Similarly

$$\begin{aligned} Z_{fdyx} &= -\frac{E_{fdy}}{H_{fdx}} \\ &= i\eta\tan\left(kz-\frac{\alpha\pi}{2}\right) \end{aligned} \tag{10c}$$

and at $z = 0$

$$Z_{fdyx} = -i\eta\tan\left(\frac{\alpha\pi}{2}\right) \tag{10d}$$

From equation (10b) and (10d), it is obvious that for $\alpha = 0$, value of transverse wave impedance at $z = 0$ is $Z_{fd} = 0$, which describes the situation as if a PEC interface is placed at $z = 0$. For $\alpha = 1$, the transverse wave impedance is $Z_{fd} = \infty$, which describes a situation as if a PMC interface is placed at $z = 0$. Hence for $0 < \alpha < 1$, equation (10b) and (10d) describe a surface located at $z = 0$ whose impedance may be regarded intermediate step of the impedance of a PEC interface and PMC interface.

It is of interest to extend the above results for the case of multiple boundary mediums. For this purpose, geometry with two interfaces are considered. Similarly geometries having more than two interfaces may be treated. Consider a chiral slab which is sandwiched between two different chiral media. The width and intrinsic impedance of the chiral slab are L and η_3 respectively. Intrinsic impedance of the medium before the slab is η_1 while after the slab is η_2. Front end of the slab is located at $z = 0$ while back end is located at $z = -L$. Transverse impedance, for the fractional dual solution corresponding to present case, before the slab may be written as

$$Z_{1fdxy} = \eta_1 \frac{\left[\exp\left\{i\left(k_1 z - \frac{\alpha\pi}{2}\right)\right\} + R_1 \exp\left\{-i\left(k_1 z - \frac{\alpha\pi}{2}\right)\right\}\right]}{\left[\exp\left\{i\left(k_1 z - \frac{\alpha\pi}{2}\right)\right\} - R_1 \exp\left\{-i\left(k_1 z - \frac{\alpha\pi}{2}\right)\right\}\right]} \tag{11}$$

where

$$\begin{aligned} R_1 &= \frac{Z_{2fdxy} - \eta_1}{Z_{2fdxy} + \eta_1} \\ Z_{2fdxy} &= \eta_1 \frac{\left[\exp\left\{i\left(k_2 z - \frac{\alpha\pi}{2}\right)\right\} + R_2 \exp\left\{-i\left(k_2 z - \frac{\alpha\pi}{2}\right)\right\}\right]}{\left[\exp\left\{i\left(k_2 z - \frac{\alpha\pi}{2}\right)\right\} - R_2 \exp\left\{-i\left(k_2 z - \frac{\alpha\pi}{2}\right)\right\}\right]} \\ R_2 &= \frac{\eta_3 - \eta_2}{\eta_3 + \eta_2} \end{aligned}$$

It is obvious from above expression that for value of fractional parameter $\alpha = \frac{2k_2L}{\pi}$, the situation behaves as if there is no chiral slab.

Above result may be easily extended for the case of chiral slab backed by PEC interface and the result is given by

$$Z_{1fdxy} = \eta_1 \frac{\left[\exp\left\{i\left(k_1 z - \frac{\alpha\pi}{2}\right)\right\} + R_1 \exp\left\{-i\left(k_1 z - \frac{\alpha\pi}{2}\right)\right\}\right]}{\left[\exp\left\{i\left(k_1 z - \frac{\alpha\pi}{2}\right)\right\} - R_1 \exp\left\{-i\left(k_1 z - \frac{\alpha\pi}{2}\right)\right\}\right]} \tag{11a}$$

where

$$R_1 = \frac{Z_{2fdxy} - \eta_1}{Z_{2fdxy} + \eta_1}$$
$$Z_{2fdxy} = i\eta_2 \tan\left(k_2 L - \frac{\alpha\pi}{2}\right)$$

It may be noted that for $\alpha = 0$, equation (11a) deals with situation of a slab backed by PEC interface while for $\alpha = 1$, equation (11a) deals with situation of a slab backed by PMC interface. For $0 < \alpha < 1$, equation (11a), deals with fractional dual situation.

5. NON-SYMMETRIC TRANSMISSION LINE

When an electromagnetic wave of any polarization propagates through a bi-isotropic medium, solution of Maxwell equations in bi-isotropic medium gives rise to two circularly polarized waves. One of the waves is right circularly polarized (RCP) while the other one is left circularly polarized (LCP). These RCP and LCP components move with different phase velocities and may be represented in terms of wave numbers k_+ and k_- respectively. A plane wave propagating through a plane-parallel structure of bi-isotropic medium can be analyzed in terms of two non-interacting scalar transmission lines with two eigen waves much in the same way as simple isotropic medium. The main difference is that in case of bi-isotropic medium re ected wave has wave number different than the incident wave. This means that a left circularly polarized wave will become right circularly polarized upon re ection from the interface and vice versa. Thus incident and re ected components of the wave will see different effective media and hence the corresponding transmission line becomes non symmetric with different parameters for the waves propagating in the opposite directions. For the two circularly polarized TEM eigen waves depending only upon z-coordinate, the source free Maxwell equations can be written as

$$u_z \times \underline{E'}_{\pm}(z) = -i\omega\mu_{\pm}\underline{H}_{\pm}(z), \qquad u_z \times \underline{H'}_{\pm}(z) = i\omega\epsilon_{\pm}\underline{E}_{\pm}(z) \quad (12)$$

where the primes denote differentiation with respect to z. In terms of circular polarization (CP) unit vectors $u_{\pm}$ satisfying $u_z \times u_{\pm} = \pm i u_{\pm}$ we can write

$$\underline{E\pm} = u_{\pm}E_{\pm} \qquad \text{and} \qquad \underline{H\pm} = u_{\pm}H_{\pm}.$$

Using these values, we may write the scalar form of equation (12) as

$$E'_{\pm}(z) = -i\omega\mu_{\pm}\{\mp iH_{\pm}(z)\}, \qquad \mp iH'_{\pm}(z) = -i\omega\epsilon_{\pm}E_{\pm}(z)$$

They resemble the transmission line equations

$$V'(z) = -i\omega LI(z), \qquad I'(z) = -i\omega CV(z) \tag{13}$$

If we identify electric field with voltage $V_{\pm} = E_{\pm}$, the current must be recognized as $I_{\pm} = \mp iH_{\pm}$.

6. FRACTIONAL NON-SYMMETRIC TRANSMISSION LINE

The positive and negative wave-fields are represented by their respective voltage and current components. These components are added to get total voltage and current. For a boundary with reflection coefficient $\Gamma_{\mp\pm}$, we can write wave fields as

$$V_{\pm}(z) = V_{\pm}(0)\exp(ik_{\pm}z) + \Gamma_{\mp\pm}V_{\pm}(0)\exp(-ik_{\mp}z)$$

$$ZI_{\pm}(z) = V_{\pm}(0)\exp(ik_{\pm}z) - \Gamma_{\mp\pm}V_{\pm}(0)\exp(-ik_{\mp}z)$$

For a reciprocal chiral medium $\Gamma_{\mp\pm} = \Gamma = \frac{Z_L - Z}{Z_L + Z}$. So, we can write

$$\begin{aligned} V_{\pm}(z) &= V_{\pm}(0)\{\exp(ik_{\pm}z) + \Gamma\exp(-ik_{\mp}z)\} \\ ZI_{\pm}(z) &= V_{\pm}(0)\{\exp(ik_{\pm}z) - \Gamma\exp(-ik_{\mp}z)\} \end{aligned}$$

Now total voltage and current equations can be written as

$$\begin{aligned} V(z) &= V_{+}(0)\{\exp(ik_{+}z) + \Gamma\exp(-ik_{-}z)\} \\ &\quad +V_{-}(0)\{\exp(ik_{-}z) + \Gamma\exp(-ik_{+}z)\} \\ &= V_1 + V_2 + V_3 + V_4 \qquad (14a) \\ ZI(z) &= V_{+}(0)\{\exp(ik_{+}z) - \Gamma\exp(-ik_{-}z)\} \\ &\quad +V_{-}(0)\{\exp(ik_{-}z) - \Gamma\exp(-ik_{+}z)\} \\ &= ZI_1 + ZI_2 + ZI_3 + ZI_4 \qquad (14b) \end{aligned}$$

Let us write fractional dual solution to each component of equation(14a) and equation (14b) as

$$\begin{aligned} V_{1fd} &= \frac{1}{(ik_{+})^{2\alpha}}V_{+}(0)\frac{d^{2\alpha}}{dz^{2\alpha}}\exp(ik_{+}z) \\ &= \exp(i\alpha\pi)V_{+}(0)\exp\{i(k_{+}z - \alpha\pi)\} \end{aligned}$$

Similarly other components can be written as

$$V_{2fd} = \exp(i\alpha\pi)V_{+}(0)\Gamma\exp\{-i(k_{-}z - \alpha\pi)\}$$

$$\begin{aligned}
V_{3fd} &= \exp(i\alpha\pi)V_-(0)\exp\{i(k_-z-\alpha\pi)\}\\
V_{4fd} &= \exp(i\alpha\pi)V_-(0)\Gamma\exp\{-i(k_+z-\alpha\pi)\}\\
ZI_{1fd} &= \exp(i\alpha\pi)V_+(0)\exp\{i(k_+z-\alpha\pi)\}\\
ZI_{2fd} &= \exp(i\alpha\pi)V_+(0)\Gamma\exp\{-i(k_-z-\alpha\pi)\}\\
ZI_{3fd} &= \exp(i\alpha\pi)V_-(0)\exp\{i(k_-z-\alpha\pi)\}\\
ZI_{4fd} &= \exp(i\alpha\pi)V_-(0)\Gamma\exp\{-i(k_+z-\alpha\pi)\}
\end{aligned}$$

Using these values in equation (14a) and equation (14b) we have

$$\begin{aligned}
V_{fd}(z) = \exp(i\alpha\pi)V_+(0)[\exp\{i(k_+z-\alpha\pi)\}+\Gamma\exp\{-i(k_-z-\alpha\pi)\}]\\
+\exp(i\alpha\pi)V_-(0)[\exp\{i(k_+z-\alpha\pi)\}+\Gamma\exp\{-i(k_+z-\alpha\pi)\}]
\end{aligned} \tag{15a}$$

$$\begin{aligned}
ZI_{fd}(z) = \exp(i\alpha\pi)V_+(0)[\exp\{i(k_+z-\alpha\pi)\}-\Gamma\exp\{-i(k_-z-\alpha\pi)\}]\\
+\exp(i\alpha\pi)V_-(0)[\exp\{i(k_-z-\alpha\pi)\}-\Gamma\exp\{-i(k_+z-\alpha\pi)\}]
\end{aligned} \tag{15b}$$

For $\alpha = 0$

$$V_{fd}(z) = V(z) \qquad ZI_{fd}(z) = ZI(z)$$

and $\alpha = \frac{1}{2}$

$$V_{fd}(z) = ZI(z) \qquad ZI_{fd}(z) = V(z)$$

which is original and dual to the original set of solutions for a transmission line. For $0 < \alpha < 1/2$, solutions set $(V_{\mathrm{fd}}, ZI_{\mathrm{fd}})$ may be regarded as intermediate step between the original and dual to the original solutions of a transmission line. Transmission line corresponding to solutions set $(V_{\mathrm{fd}}, ZI_{\mathrm{fd}})$ may be termed as fractional dual transmission line.

7. INPUT IMPEDANCE OF TERMINATED FRACTIONAL NON-SYMMETRIC LINE

Input impedance of the fractional dual transmission line is defined as

$$Z_{fd} = \frac{V_{fd}(z)}{I_{fd}(z)} \tag{16}$$

Using the following relations

$$k_+ = k + k\kappa_r \qquad k_- = k - k\kappa_r$$

We can write

$$\begin{aligned}\exp(ik_+z) &= exp(ikz) \times \exp(ik\kappa_r)\\ \exp(ik_-z) &= \exp(ikz) \times \exp(-ik\kappa_r)\end{aligned}$$

Using these values in equation (15a) and equation (15b), it is obtained

$$\begin{aligned}V_{fd}(z) &= \frac{1}{(Z_L+Z)}\exp(i\alpha\pi)\{V_+(0)\exp(ik\kappa_r z)+V_-(0)\exp(-ik\kappa_r z)\}\\ &\quad 2\cos(kz-\alpha\pi)\{Z_L+iZ\tan(kz-\alpha\pi)\}\\ ZI_{fd}(z) &= \frac{1}{(Z_L+Z)}\exp(i\alpha\pi)\{V_+(0)\exp(ik\kappa_r z)+V_-(0)\exp(-ik\kappa_r z)\}\\ &\quad 2\cos(kz-\alpha\pi)\{iZ_L\tan(kz-\alpha\pi)+Z\}\end{aligned}$$

Putting these values of $V_{fd}(z)$ and $I_{fd}(z)$ in equation (16), we get

$$Z_{fd} = Z\frac{Z_L + iZ\tan(kz-\alpha\pi)}{Z + iZ_L\tan(kz-\alpha\pi)} \tag{17}$$

For $\alpha = 0$

$$Z_{fd} = Z\frac{Z_L + iZ\tan(kz)}{Z + iZ_L\tan(kz)} \tag{17a}$$

which is the relation for input impedance at any point for a transmission line having characteristic impedance Z. We consider it as original transmission line. Now for $\alpha = \frac{1}{2}$ equation (17) gives

$$Z_{fd} = Z\frac{Z_L + iZ\cot(kz)}{Z + iZ_L\cot(kz)} \tag{17b}$$

which is the relation for input impedance at any point for dual to the original transmission line. It may be noted that at a particular point along the original transmission line input admittance becomes the corresponding input impedance of the dual transmission line. So, for $\alpha = 0$, equation (17) represents input impedance of the original transmission line and $\alpha = \frac{1}{2}$ represents the dual to the original transmission line, while $0 < \alpha < \frac{1}{2}$, equation (17) represents input impedance of the fractional dual transmission line. It may also be noted that variation in the value of fractional parameter α along the fractional transmission line corresponds to variation in the observation point along the original transmission line. So behavior along a transmission line may be studied by changing the fractional parameter along the corresponding fractional transmission line while keeping the observation point constant.

Setting observation point at $z = 0$, that is at the load, equation (17) yields following

$$Z_{fd} = Z\frac{Z_L - iZ\tan(\alpha\pi)}{Z - iZ_L\tan(\alpha\pi)}$$

It is obvious from above equation that

$$Z_{fd}|_{\alpha=0} = Z_L, \qquad Z_{fd}|_{\alpha=\frac{1}{2}} = \frac{Z^2}{Z_L}$$

or

$$\overline{Z}_{fd}|_{\alpha=0} = \overline{Z}_L, \quad \overline{Z}_{fd}|_{\alpha=\frac{1}{2}} = \frac{1}{\overline{Z}_L} = \overline{Y}_L$$

This means if original transmission line, that is $\alpha = 0$, deals with transmission line which is terminated by a load Z_L then $\alpha = 1/2$ deals with transmission line which is terminated by a load having impedance Y_L. For $0 < \alpha < \frac{1}{2}$ represents new line terminated by load, which may be regarded as intermediate step between the loads Z_L and Y_L.

Now consider the following two special cases

$$\text{If} \quad Z_L = 0, \quad \text{then} \quad Z_{fd}|_{\alpha=0} = 0, \quad Z_{fd}|_{\alpha=\frac{1}{2}} = \infty$$

and

$$\text{If} \quad Z_L = \infty, \quad \text{then} \quad Z_{fd}|_{\alpha=0} = \infty, \quad Z_{fd}|_{\alpha=\frac{1}{2}} = 0$$

Which shows that if the original transmission line is a short-circuited line then dual to the original line will be an open-circuited transmission line and vice versa. For $0 < \alpha < \frac{1}{2}$, represents a line terminated by an inductive type load if the original line is a short circuited line. For $0 < \alpha < \frac{1}{2}$, represents a line terminated by capacitive type load if the original line is open circuited line.

8. INPUT IMPEDANCE OF MULTIPLE-SECTIONS FRACTIONAL NON-SYMMETRIC LINE

Consider a transmission line having characteristic impedance Z_1. The transmission line is connected to another transmission line having characteristic impedance Z_2. The length of line having characteristic impedance Z_2 is L and the line is terminated by a load Z_L. It is assumed that load is located at $z = 0$. $Z_{2\text{in}}$ is the input impedance just before the junction of two lines, that is at $z = -L$. We can write input impedance Z_{in} along the transmission line having intrinsic impedance Z_1 as

$$Z_{\text{in}} = Z_1\frac{Z_{2\text{in}} + iZ_1\tan(k_1 z)}{Z_1 + iZ_{2\text{in}}\tan(k_1 z)}, \quad z < -L \tag{18a}$$

where

$$Z_{2\text{in}} = Z_2 \frac{Z_L + iZ_2 \tan(k_2 L)}{Z_2 + iZ_L \tan(k_2 L)} \tag{18b}$$

Using the same treatment as done in previous section we can write the input impedance of two sections fractional dual transmission line as

$$Z_{\text{infd}} = Z_1 \frac{Z_{2\text{infd}} + iZ_1 \tan(k_1 z - \alpha\pi)}{Z_1 + iZ_{2\text{infd}} \tan(k_1 z - \alpha\pi)} \tag{19a}$$

where

$$Z_{2\text{infd}} = Z_2 \frac{Z_L + iZ_2 \tan(k_2 L - \alpha\pi)}{Z_2 + iZ_L \tan(k_2 L - \alpha\pi)} \tag{19b}$$

For $\alpha = 0$, equation (19) gives

$$Z_{\text{infd}}\big|_{\alpha=0} = Z_1 \frac{Z_{2\text{infd}}\big|_{\alpha=0} + iZ_1 \tan(k_1 z)}{Z_1 + iZ_{2\text{infd}}\big|_{\alpha=0} \tan(k_1 z)} \tag{20a}$$

where

$$Z_{2\text{infd}}\big|_{\alpha=0} = Z_2 \frac{Z_L + iZ_2 \tan(k_2 L)}{Z_2 + iZ_L \tan(k_2 L)} \tag{20b}$$

which is given in equation (18). Similarly, for $\alpha = \frac{1}{2}$

$$Z_{\text{infd}}\big|_{\alpha=\frac{1}{2}} = Z_1 \frac{Z_{2\text{infd}}\big|_{\alpha=\frac{1}{2}} + iZ_1 \cot(k_1 z)}{Z_1 + iZ_{2\text{infd}}\big|_{\alpha=\frac{1}{2}} \cot(k_1 z)} \tag{21a}$$

where

$$Z_{2\text{infd}}\big|_{\alpha=\frac{1}{2}} = Z_2 \frac{Z_L + iZ_2 \cot(k_2 L)}{Z_2 + iZ_L \cot(k_2 L)} \tag{21b}$$

which is dual to the input impedance given in (18) or (20). For $0 < \alpha < 1/2$, input impedance Z_{infd} may be regarded as intermediate step of the two input impedances given by (20) and (21).

Condition for impedance matching of transmission line network described by (19) is

$$Z_{2\text{infd}} = Z_1 = Z_2 \frac{Z_L + iZ_2 \tan(k_2 L - \alpha\pi)}{Z_2 + iZ_L \tan(k_2 L - \alpha\pi)} \tag{22}$$

The value of fractional parameter α, in terms of the impedances, required for the impedance matching is given below

$$\alpha = \frac{1}{\pi}\left[k_2 L - \tan^{-1}\left(\frac{Z_2(Z_1 - Z_L)}{i\left(Z_2^2 - Z_1 Z_L\right)}\right)\right] \tag{23}$$

It may be noted that for $\alpha = \frac{k_2 L}{\pi}$, equation (19) yields input impedance as if in the circuit there is no transmission line having characteristic impedance Z_2. From equation (23) if $Z_1 = Z_L$, then

$$\alpha = \frac{k_2 L}{\pi}$$

is the condition for impedance matching.

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