ASYMPTOTICS OF CREEPING WAVES IN A DEGENERATED CASE OF MATRIX IMPEDANCE

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Abstract—Creeping waves propagate in the shadow along the surface of a convex body. In the case of a perfectly conducting body coated with high index anisotropic dielectric, this surface can be described by anisotropic impedance boundary condition. In a previous paper the general case of anisotropic impedance was studied. In this paper we discuss a special case characterized by a degenerated impedance matrix. The ansatz for ordinary creeping waves does not allow the asymptotics to be constructed and a new ansatz is suggested. In contrast to the usual one, this ansatz contains an additional quick factor proportional to $k^{1/6}$ (where k is the wavenumber). As a result, the field is described by an asymptotic sequence in inverse powers of $k^{1/6}$. We derive the principal order term of the asymptotics and discuss specific properties of creeping waves on a surface with degenerated impedance.

1. INTRODUCTION

In this paper we study creeping waves on a surface described by an anisotropic impedance boundary condition. That condition is assumed in the form

$$\mathbf{E} - (\mathbf{E}, \mathbf{e}_n)\mathbf{e}_n = \mathbf{Z}\,\mathbf{e}_n \times \mathbf{H}. \tag{1}$$

Here $(\cdot, \cdot)$ denotes scalar product in $\mathbb{R}^3$, cross $\times$ denotes vector product and $\mathbf{e}_n$ stands for the unit vector of the normal to the surface. Creeping waves propagate in the shadow along the surface of a convex body.

They follow the geodesics of the surface. The creeping wave field varies rapidly inside a boundary layer in the vicinity of the surface of the body. The classical method of analysis of creeping waves is the canonical problem method [1]. Among recent contributions to the subject relying on this method are the problem of creeping waves on a circular cylinder with isotropic impedance studied in [2], the problem of a source on a circular cylinder with anisotropic impedance studied in [3] and creeping waves on a coated dielectric sphere treated in [4]. However an efficient way of deriving the creeping wave field on a general convex body is to use the boundary layer method. The main steps of the method are

- choosing an ansatz
- expressing the Maxwell equations and boundary conditions in a body-fitted coordinate system
- substituting the ansatz in above equations and boundary conditions
- sorting the equations by decreasing powers of large parameter
- solving the equations at each order

The method has been successfully applied to both cases of isotropic and anisotropic impedance boundary condition (see e.g., [5]). The body-fitted coordinates are denoted (s, a, n), where s and a are surface coordinates associated with the geodesics followed by creeping waves: s is the arc-length and a is transverse coordinate, as illustrated by Fig. 1, and n is the distance from the surface. When the boundary condition (1) is written in these coordinates, one gets [6]

$$\begin{pmatrix} E_s \\ hE_a \end{pmatrix} = \mathbf{Z} \begin{pmatrix} -hH_a \\ H_s \end{pmatrix}, \qquad (2)$$
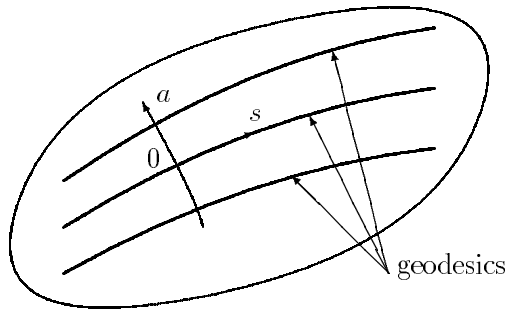

Figure 1. Coordinate system (s, a).

where h measures the divergence of the geodesics pencil. The matrix

$$\mathbf{Z} = \begin{pmatrix} Z_{ss} & Z_{sa} \\ Z_{as} & Z_{aa} \end{pmatrix}$$

is supposed to be symmetric ($Z_{sa} = Z_{as}$) to describe a reciprocal surface This condition is representative of an anisotropic high index layer with symmetric permittivity and permeability tensors covering a perfectly conducting surface A thorough analysis of reciprocity issue for general bianisotropic media is given in [7]. In order to get better approximation capabilities, as explained in [5] one can allow the elements to be of different appropriate orders in the large parameter k (wavenumber of electromagnetic waves).

In our computations as explained below another matrix

$$\mathbf{M} = \begin{pmatrix} Z_{aa}Z_{ss} - Z_{as}^2 & -Z_{as} \\ Z_{as} & 1 \end{pmatrix} \tag{3}$$

appears, which eigenvalues define the attenuation parameters of creeping waves. The general case, where the eigenvalues λ_1, λ_2 of the matrix $\mathbf{M}$ are different ($\lambda_1 \neq \lambda_2$) was examined in [6]. The asymptotics of creeping waves derived in this paper contains terms proportional to $(\lambda_1 - \lambda_2)^{-1}$ which diverge when $\lambda_1 \to \lambda_2$. As a result, when the eigenvalues of the matrix $\mathbf{M}$ coincide, the asymptotics of [6] becomes not valid and therefore specific analysis is required.

Linear algebra [8] says that two subcases are possible. Either the matrix $\mathbf{M}$ is diagonal, or it can not be diagonalized. In the first case, by introducing quantities $J = H + iE$ and $K = H - iE$, one can split the problem into two independent subproblems. The results are similar to the case of isotropic impedance equal to one (see [5]) and one finds the factors

$$\exp\left(\pm i \int \tau \, ds\right)$$

in the amplitudes of two types of creeping waves. The integration in the above formula is carried along the geodesics followed by a creeping wave and τ is the torsion of that geodesics.

The other case, namely when the matrix $\mathbf{M}$ cannot be diagonalized, presents a more difficult problem, analysis of which is the subject of this paper. The ansatz used in [6] is not general enough to handle this specific case. We derive in the section below a new ansatz for the creeping wave field, and show in Sections 3, 4 and 5 how to derive a solution of the problem by using this ansatz. We show that the recurrent procedure can be used to derive the asymptotics up to any desired order, but present formulae at the principal order only.

2. DERIVATION OF THE APPROPRIATE ANSATZ

As explained in the introduction,we use the geodesic coordinate system (s, a, n). The metric matrix of this coordinate system can be found for example in [9]

$$
\begin{pmatrix}
\left(1 + \dfrac{n}{\rho}\right)^2 + \tau^2 n^2 & -h\tau n\left(2 + \dfrac{n}{\rho} + \dfrac{n}{\rho_t}\right) & 0 \\
-h\tau n\left(2 + \dfrac{n}{\rho} + \dfrac{n}{\rho_t}\right) & h^2\left(\left(1 + \dfrac{n}{\rho_t}\right)^2 + \tau^2 n^2\right) & 0 \\
0 & 0 & 1
\end{pmatrix}.
$$

Here ρ is the radius of curvature of the geodesics followed by creeping waves, τ is its torsion, h is the divergence of geodesics pencil and ρ_t is transverse curvature.

One can check (see also section 5) that the usual boundary-layer procedure [10] does not allow the asymptotic series for the field of creeping waves on the surface with degenerated nondiagonalizable impedance to be constructed. Though the principal order ($O(k^{1/3})$) equations can be solved, at the next order $O(k^0)$ one gets a degenerated algebraic system for the amplitudes with incompatible right-hand side. Therefore the analytic form of the asymptotic decomposition (ansatz) for creeping wave on such a surface should be modified. In order to understand why the usual ansatz fails to provide a solution to the problem and to discover how to modify it, we consider briefly the field in a small vicinity of the light-shadow boundary on the surface, i.e. in the Fock domain. It is well known [10] that the field in this domain is solution of parabolic equations and can be written in the form of Fourier integral by spectral parameter ξ. In the principal order one has

$$
\begin{pmatrix} E^a \\ H^a \end{pmatrix} = e^{iks} \int e^{i\sigma\xi} \begin{pmatrix} A_0(\xi) \\ B_0(\xi) \end{pmatrix} w_1(\xi - \nu)\, d\xi,
$$

where E^a and H^a are transverse components of electric and magnetic fields in coordinates (s, a, n), $\sigma = (k\rho/2)^{1/3}\, s/\rho$ is the stretched s coordinate, $\nu_0 = 2\,(k\rho/2)^{2/3}\, n/\rho$ is the stretched normal coordinate and $w_1()$ is the Airy function. The amplitudes $A_0(\xi)$ and $B_0(\xi)$ of the electric and magnetic waves satisfy to the linear algebraic system originated from the boundary condition. This system has the matrix (here and below $\dot{w}_1(\xi) = dw_1(\xi)/d\xi$)

$$
\mathcal{M} = \dot{w}_1(\xi) - \frac{i}{Z_{aa}}\left(\frac{k\rho}{2}\right)^{1/3} w_1(\xi)\mathbf{M}
$$

and its right-hand side contains the incident field. In the case of coincident eigenvalues $\lambda_1 = \lambda_2$ of matrix $\mathbf{M}$ the determinant of $\mathcal{M}$ has zeros of multiplicity two. Therefore the amplitudes A_0, B_0 of the leading term has poles of multiplicity two. These poles are located at the points ξ_ℓ which are solutions of the dispersion equation

$$\mathcal{L}(\xi) \equiv \dot{w}_1(\xi) - \frac{i}{Z_{aa}}\left(\frac{k\rho}{2}\right)^{1/3}\lambda_{1,2}w_1(\xi) = 0.$$

In the next order, that is at $k^{-1/3}$, the amplitudes A_1, B_1 are solutions of the algebraic system with the same matrix $\mathcal{M}$ and its right-hand side contains the incident field and the principal order amplitudes A_0, B_0. Therefore A_1, B_1 have poles of multiplicity four. Further, the multiplicity of poles increases by two each time we proceed to the next order. As a result, the asymptotic series for the field in the Fock domain has the following structure

$$e^{iks}\int\sum_{j=0}^{\infty}k^{-j/3}e^{i\sigma\xi}\frac{\Omega_j(\xi,\sigma,\nu_0,a)}{\mathcal{L}(\xi)^{2j}}d\xi.$$

The functions Ω_j are smooth functions of their arguments and do not contain the large parameter k explicitly. We do not provide particular expressions for Ω_j which are too lengthy and are not needed for the analysis of the structure of the field. If σ is sufficiently large the field in Fock domain should match to creeping waves field. Computing the above integral by the residue theorem, one gets

$$e^{iks+i\sigma\xi_0}\sum_{j=0}^{\infty}k^{-j/3}\sigma^{2j-1}f_j,$$

where ξ_0 is solution of the dispersion equation with minimal imaginary part and f_j are complicated expressions having the order $O(1)$. One can see that the above written series looses its asymptotic character when $\sigma = O(k^{-1/6})$. That means that the field contains dependence not only on ks and $k^{1/3}s$, but also on $k^{1/6}s$.

Therefore one introduces the following ansatz including such a dependence, namely

$$\begin{pmatrix} H \\ E \end{pmatrix} = \exp\left\{iks + i\left(\frac{k}{2}\right)^{1/3}\int_{s_0(a)}^{s}\frac{\xi(s',a)\,ds'}{\rho^{2/3}(s',a)} + ik^{1/6}\Psi(s,a)\right\}$$

$$\times\sum_{j=0}^{\infty}k^{-j/6}\begin{pmatrix} H_j(s,a,N) \\ E_j(s,a,N) \end{pmatrix}, \qquad N = k^{2/3}n. \tag{4}$$

The ansatz (4) is different from the usual ansatz for creeping waves in two respects. First, it contains an additional exponential factor

$$\exp\Big(ik^{1/6}\Psi(s,a)\Big),$$

Second, the decomposition is carried out by inverse powers of $k^{1/6}$ (instead of $k^{1/3}$).

In the next section, we substitute the above ansatz in Maxwell equations and in Section 4 into the boundary condition.

3. MAXWELL EQUATIONS

One rewrites Maxwell equations in the coordinate system (s,a,n) and substitutes in these equations the ansatz (4). Collecting terms of the same order in $k^{-1/6}$, one gets a list of equations for contravariant components of electric (E^s, E^a, E^n) and magnetic (H^s, H^a, H^n) vectors. To compute the principal order terms of the asymptotics, one needs equations at orders $O(k)$, $O(k^{5/6})$, $O(k^{2/3})$, $O(k^{1/2})$, $O(k^{1/3})$, $O(k^{1/6})$, $O(1)$ and $O(k^{-1/6})$. By letting $\mathbf{E}_j = \mathbf{H}_j = 0$ for negative j, one can write these equations in the following way

$$H_j^s = ih\frac{\partial H_{j-2}^a}{\partial N} + PE_{j-4}^n + \frac{1}{h}\frac{\partial\Psi}{\partial a}E_{j-5}^n - \frac{i}{h}\frac{\partial E_{j-6}^n}{\partial a} + 2i\frac{h}{\rho_t}E_{j-6}^a$$

$$+ihN\left(\frac{1}{\rho_t}-\frac{1}{\rho}\right)\frac{\partial E_{j-6}^a}{\partial N}, \tag{5}$$

$$E_j^s = -ih\frac{\partial H_{j-2}^a}{\partial N} + PE_{j-4}^n + \frac{1}{h}\frac{\partial\Psi}{\partial a}E_{j-5}^n - \frac{i}{h}\frac{\partial E_{j-6}^n}{\partial a} + 2i\frac{h}{\rho_t}E_{j-6}^a$$

$$+ihN\left(\frac{1}{\rho_t}-\frac{1}{\rho}\right)\frac{\partial E_{j-6}^a}{\partial N},$$

$$H_j^a + \frac{1}{h}E_j^n = -\frac{i}{h}\frac{\partial E_{j-2}^s}{\partial N} - \frac{1}{h}\frac{\xi}{2^{1/3}\rho^{2/3}}E_{j-4}^n + \frac{1}{h}\left(\frac{N}{\rho}+\frac{N}{\rho_t}\right)E_{j-4}^n$$

$$-\frac{1}{h}\frac{\partial\Psi}{\partial s}E_{j-5}^n + \frac{i}{h}\frac{\partial E_{j-6}^n}{\partial s} + 2i\tau N\frac{\partial E_{j-6}^a}{\partial N} + 2i\tau E_{j-6}^a,$$

$$H_j^n - hE_j^a = -h\left(\frac{N}{\rho}-\frac{N}{\rho_t}\right)E_{j-4}^a - 2\tau NE_{j-4}^s - PE_{j-4}^s + h\frac{\xi}{2^{1/3}\rho^{2/3}}E_{j-4}^a$$

$$+h\frac{\partial\Psi}{\partial s}E_{j-5}^a - \frac{1}{h}\frac{\partial\Psi}{\partial a}E_{j-5}^s - ih\frac{\partial E_{j-6}^a}{\partial s} - 2i\frac{\partial h}{\partial s}E_{j-6}^a,$$

$$E_j^a - \frac{1}{h}H_j^n = \frac{i}{h}\frac{\partial H_{j-2}^s}{\partial N} + \frac{1}{h}\frac{\xi}{2^{1/3}\rho^{2/3}}H_{j-4}^n - \frac{1}{h}\left(\frac{N}{\rho}+\frac{N}{\rho_t}\right)H_{j-4}^n$$

$$+\frac{1}{h}\frac{\partial\Psi}{\partial s}H^n_{j-5}-\frac{i}{h}\frac{\partial H^n_{j-6}}{\partial s}-2i\tau N\frac{\partial H^a_{j-6}}{\partial N}-2i\tau H^a_{j-6},$$

$$E^n_j+hH^a_j=h\left(\frac{N}{\rho}-\frac{N}{\rho_t}\right)H^a_{j-4}+2\tau NH^s_{j-4}+PH^s_{j-4}-h\frac{\xi}{2^{1/3}\rho^{2/3}}H^a_{j-4}$$

$$-h\frac{\partial\Psi}{\partial s}H^a_{j-5}+\frac{1}{h}\frac{\partial\Psi}{\partial a}H^s_{j-5}+ih\frac{\partial H^a_{j-6}}{\partial s}+2i\frac{\partial h}{\partial s}H^a_{j-6}. \quad (6)$$

In the above equations, j takes values $0,1,2,3,4,5,6$ and 7, and P is given by the formula

$$P=\frac{1}{2^{1/3}h}\frac{\partial}{\partial a}\int\limits_{s_0(a)}^{s}\frac{\xi(s',a)\,ds'}{\rho^{2/3}(s',a)}.$$

The first two equations at each order express the longitudinal components of electric and magnetic vectors. That components appear only starting from the order $k^{-1/3}$ (i.e. $E^s_0=H^s_0=E^s_1=H^s_1=0$).

With the help of the other equations one can express $\mathbf{e}_n$ components of the field via transverse components. Starting from the order $O(k^{1/3})$ there are pairs of different expressions for the quantities $H^n_j-hE^a_j$ and $E^n_j+hH^a_j$. The compatibility of these expressions yields

$$L_0\left(\begin{array}{c}H^a_j\\E^a_j\end{array}\right)=\eta\left(\begin{array}{c}H^a_{j-1}\\E^a_{j-1}\end{array}\right)+\left(\frac{\rho}{2}\right)^{2/3}\{iL_1-2\tau\mathbf{S}_2\}\left(\begin{array}{c}H^a_{j-2}\\E^a_{j-2}\end{array}\right), \quad (7)$$

Here we introduced differential operators

$$L_0=\frac{\partial}{\partial\nu^2}+(\nu-\xi), \qquad L_1=\frac{3}{h}\frac{\partial h}{\partial s}+2\frac{\partial}{\partial s},$$

stretched normal ν

$$\nu=2^{1/3}\rho^{-1/3}N=2m^2\frac{n}{\rho}, \qquad m=\left(\frac{k\rho}{2}\right)^{1/3}$$

and function η which is related to Ψ by the formula

$$\Psi=2^{-1/3}\int\limits_{s_0(a)}^{s}\frac{\eta(s',a)\,ds'}{\rho^{2/3}(s',a)}. \quad (8)$$

The matrix $\mathbf{S}_2$ is the Pauli matrix $\mathbf{S}_2=\left(\begin{smallmatrix}0&i\\-i&0\end{smallmatrix}\right)$. The equations (7) should be completed with the boundary conditions presented in the section below and the resulting boundary-value problems are solved in Section 5.

4. DEGENERATED ANISOTROPIC IMPEDANCE BOUNDARY CONDITIONS

Let us now turn to the boundary condition (2). In order that all the elements of the matrix impedance $\mathbf{Z}$ participate in the equation for the attenuation parameter ξ, one chooses the elements of the matrix $\mathbf{Z}$ to be of appropriate orders in k, namely

$$Z_{aa} = O(k^{1/3}), \ Z_{as} = Z_{sa} = O(1), \ Z_{ss} = O(k^{-1/3}).$$

Expressing $\mathbf{e}_s$ and $\mathbf{e}_n$ components of electric and magnetic vectors by means of equations (5)–(6) and taking into account the assumed orders of the impedances yields a set of conditions for E_j^a and H_j^a on the surface. The first component in the boundary condition (2) has the principal order $O(k^{-1/3})$. Conditions at orders $O(k^{-1/2})$, $O(k^{-2/3})$ and $O(k^{-5/6})$ are also needed, these are $(j = 0, 1, 2, 3)$

$$i\frac{\partial H_j^a}{\partial N} + iz_{as}\frac{\partial E_j^a}{\partial N} - z_{aa}H_j^a + PE_{j-2}^a - z_{as}PH_{j-2}^a + \frac{1}{h}\frac{\partial \Psi}{\partial a}E_{j-3}^a$$
$$-z_{as}\frac{1}{h}\frac{\partial \Psi}{\partial a}H_{j-3}^a = 0.$$

Here we introduced $z_{aa} = k^{-1/3}Z_{aa}$, $z_{as} = Z_{as}$ and $z_{ss} = k^{1/3}Z_{ss}$ which all have the order $O(1)$. The second component of (2) is of order $O(1)$. At this and three successive orders, one has $(j = 0, 1, 2, 3)$

$$z_{as}H_j^a + E_j^a - iz_{ss}\frac{\partial E_j^a}{\partial N} + z_{ss}PH_{j-2}^a + z_{ss}\frac{1}{h}\frac{\partial \Psi}{\partial a}H_{j-3}^a = 0.$$

One combines the above equations into the system

$$\frac{\partial}{\partial N}\begin{pmatrix} H_j^a \\ E_j^a \end{pmatrix} + \frac{i}{z_{aa}}\mathbf{M}\begin{pmatrix} H_j^a \\ E_j^a \end{pmatrix} = P\mathbf{S}_2\begin{pmatrix} H_{j-2}^a \\ E_{j-2}^a \end{pmatrix} + \frac{1}{h}\frac{\partial \Psi}{\partial a}\mathbf{S}_2\begin{pmatrix} H_{j-3}^a \\ E_{j-3}^a \end{pmatrix}. \tag{9}$$

Here the matrix $\mathbf{M}$ is defined in (3).

The eigenvalues of the matrix $\mathbf{M}$ as mentioned in the introduction play an important role in the asymptotic procedure. They are defined by the equation

$$(Z - \lambda)(1 - \lambda) + z_{as}^2 = 0.$$

Here $Z = \det(\mathbf{Z}) = Z_{aa}Z_{ss} - Z_{as}^2$. Assuming $\lambda_1 = \lambda_2$ yields

$$\lambda_1 = \lambda_2 = \frac{Z+1}{2}, \qquad Z - 1 = \pm 2Z_{sa}. \tag{10}$$

It is possible to have plus or minus in the last formula of (10). So there are two cases of degenerated matrices. These two cases are treated together by introducing

$$\alpha = \frac{Z-1}{2Z_{sa}} = \pm 1.$$

One introduces new unknowns in order to simplify the boundary conditions. Let K and J be defined by the formulae

$$H_j^a = J_j + K_j, \qquad E_j^a = \alpha\left(J_j - K_j\right). \qquad (11)$$

The formulae (11) can be inverted

$$K_j = \frac{H_j^a - \alpha E_j^a}{2}, \qquad J_j = \frac{H_j^a + \alpha E_j^a}{2}.$$

The boundary conditions (9) yield for new unknowns $(j = 0, 1, 2, 3)$

$$\frac{\partial}{\partial\nu}\left(\begin{array}{c} K_j \\ J_j \end{array}\right) + \frac{im}{2Z_{aa}}\left(\begin{array}{cc} Z+1 & 0 \\ 2Z-2 & Z+1 \end{array}\right)\left(\begin{array}{c} K_j \\ J_j \end{array}\right)$$

$$= \left(\frac{\rho}{2}\right)^{2/3} P\mathbf{S}_2\left(\begin{array}{c} K_{j-2} \\ J_{j-2} \end{array}\right) + \left(\frac{\rho}{2}\right)^{2/3}\frac{1}{h}\frac{\partial\Psi}{\partial a}\mathbf{S}_2\left(\begin{array}{c} K_{j-3} \\ J_{j-3} \end{array}\right). \qquad (12)$$

One rewrites also the equations (7) for new unknowns

$$L_0\left(\begin{array}{c} K_j \\ J_j \end{array}\right) = \eta\left(\begin{array}{c} K_{j-1} \\ J_{j-1} \end{array}\right) + \left(\frac{\rho}{2}\right)^{2/3}\left\{iL_1 - 2\alpha\tau\mathbf{S}_2\right\}\left(\begin{array}{c} K_{j-2} \\ J_{j-2} \end{array}\right). \quad (13)$$

5. SOLVING RECURRENT EQUATIONS

The problem of constructing asymptotics of creeping waves on a surface with degenerated impedance boundary condition is reduced to a set of boundary-value problems for the unknowns K_j and J_j. At each order j there are two equations and two boundary conditions, which together with the radiation condition allow the solution to be determined.

One starts with the principal order equations of (13) which for K_0 and J_0 become

$$L_0\left(\begin{array}{c} K_0 \\ J_0 \end{array}\right) = 0. \qquad (14)$$

From these equations one finds

$$\left(\begin{array}{c} K_0 \\ J_0 \end{array}\right) = \left(\begin{array}{c} A_0 \\ B_0 \end{array}\right) w_1(\xi - \nu).$$

The Airy function $w_1()$ is chosen to satisfy the radiation condition [10]. The amplitudes A_0 and B_0 depend on s and a coordinates and at this step remain arbitrary. Substituting these expressions into the boundary conditions (12) with $j = 0$ allows the attenuation parameter ξ to be determined. Nontrivial solution is possible only if ξ satisfies the dispersion equation ($\dot{w}_1$ denotes derivative of the Airy function w_1)

$$\dot{w}_1(\xi) = i\mu w_1(\xi), \tag{15}$$

where

$$\mu = \frac{m}{Z_{aa}} \frac{Z+1}{2}.$$

That nontrivial solution is

$$K_0 \equiv 0, \qquad J_0 = B_0(s, a) w_1(\xi - \nu).$$

One should note that in the usual case two different types of solutions are possible. In the case of degenerated matrix impedance, solutions with $K_0 \neq 0$ do not exist. When applying the standard scheme of asymptotics construction, this fact prevents solving next order equations and makes the usual ansatz inapplicable.

Now one proceeds to equations in the next order. Solution of the equation (13) which for K_1 and J_1 becomes

$$L_0 \begin{pmatrix} K_1 \\ J_1 \end{pmatrix} = \eta \begin{pmatrix} K_0 \\ J_0 \end{pmatrix}$$

can be written in explicit form

$$K_1 = A_1 w_1(\xi - \nu), \qquad J_1 = B_1 w_1(\xi - \nu) + \eta B_0 \dot{w}_1(\xi - \nu). \tag{16}$$

Here A_1 and B_1 are arbitrary functions of coordinates s and a and the quantity $\eta(s, a)$ is introduced by the formula (8). One substitutes functions (16) into the boundary condition (12) with $j = 1$ and, taking into account the dispersion equation (15), finds

$$\begin{pmatrix} 0 & 0 \\ -\dfrac{im}{Z_{aa}}(Z-1) & 0 \end{pmatrix} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = -\eta \left(\xi + \frac{m^2}{Z_{aa}^2} \frac{(Z+1)^2}{4} \right) \begin{pmatrix} 0 \\ B_0 \end{pmatrix}.$$

Fortunately, the right-hand side of this degenerated system is compatible, and the solution exists, it is

$$A_1 = -i\eta \frac{Z_{aa}}{m} \frac{\xi + \mu^2}{Z - 1} B_0, \tag{17}$$

and B_1 remains arbitrary.

One proceeds to the boundary-value problem for K_2 and J_2. The equations (7) yield

$$L_0 \begin{pmatrix} K_2 \\ J_2 \end{pmatrix} = \eta \begin{pmatrix} K_1 \\ J_1 \end{pmatrix} + \left(\frac{\rho}{2} \right)^{2/3} \{ iL_1 - 2\alpha\tau \mathbf{S}_2 \} \begin{pmatrix} K_0 \\ J_0 \end{pmatrix}. \quad (18)$$

These equations can be solved explicitly. The right-hand sides of the equations (18) are (here and below prime denotes derivative by s)

$$(\text{R.H.S.})_1 = \left(\eta A_1 - i 2^{1/3} \rho^{2/3} \alpha\tau B_0 \right) w_1(\xi - \nu),$$

$$(\text{R.H.S.})_2 = \left(\eta B_1 + i \left(\frac{\rho}{2} \right)^{2/3} \left(3\frac{h'}{h} B_0 + 2\frac{\partial B_0}{\partial s} \right) \right) w_1(\xi - \nu)$$

$$+ \left(\eta^2 + 2i \left(\frac{\rho}{2} \right)^{2/3} \xi' \right) B_0 \dot{w}_1(\xi - \nu)$$

$$+ \frac{2i}{3} \left(\frac{\rho}{2} \right)^{2/3} \frac{\rho'}{\rho} B_0 \nu \dot{w}_1(\xi - \nu).$$

By using the properties of Airy functions, one can write the solution of (18) which contains arbitrary functions $A_2(s, a)$ and $B_2(s, a)$, it is

$$K_2 = A_2 w_1(\xi - \nu) + \left(\eta A_1 - i 2^{1/3} \rho^{2/3} \alpha\tau B_0 \right) \dot{w}_1(\xi - \nu),$$

$$J_2 = B_2 w_1(\xi - \nu) + \eta B_1 \dot{w}_1(\xi - \nu)$$

$$+ i \left(\frac{\rho}{2} \right)^{2/3} \left(3\frac{h'}{h} B_0 + 2\frac{\partial B_0}{\partial s} + \frac{1}{3}\frac{\rho'}{\rho} \right) \dot{w}_1(\xi - \nu)$$

$$- \left(\frac{\eta^2}{2} + i \left(\frac{\rho}{2} \right)^{2/3} \xi' \right) B_0 \nu w_1(\xi - \nu)$$

$$- \frac{i}{6} \left(\frac{\rho}{2} \right)^{2/3} \frac{\rho'}{\rho} B_0 \nu^2 w_1(\xi - \nu).$$

One substitutes these formulae to the boundary conditions (12) with $j = 2$. The first condition can be satisfied only if

$$\left(\frac{\rho}{2} \right)^{2/3} \alpha P - \left(\eta^2 \frac{Z_{aa}}{m} \frac{\xi + \mu^2}{Z - 1} + 2^{1/3} \rho^{2/3} \alpha\tau \right) \left(\xi + \mu^2 \right) = 0. \quad (19)$$

Here the value of A_1, given by the formula (17), is taken into account. The equation (19) allows the function η to be determined, namely

$$\eta = \pm \left(\frac{\rho}{2} \right)^{1/3} \sqrt{\frac{m}{Z_{aa}} \frac{Z - 1}{\xi + \mu^2}} \sqrt{\frac{\alpha P}{\xi + \mu^2} - 2\alpha\tau}. \quad (20)$$

The possibility to choose plus or minus in the expression (20) means that there exist creeping waves of two types. These waves have coincident polarizations in the principal order ($J_0 \neq 0$, $K_0 = 0$), but as can be seen from (17) at the next order $O(k^{-1/6})$ K component appears ($K_1 \neq 0$). In the creeping waves of the two types, this mixed polarization differs in sign.

From the second condition in (12) one finds the amplitude A_2. Using the expression (20) for η, it can be simplified to

$$A_2 = -i\eta f B_1 + 2 \left(\frac{\rho}{2}\right)^{2/3} f \frac{\partial B_0}{\partial s}$$

$$+ \left(\frac{\rho}{2}\right)^{2/3} \left(3\frac{h'}{h}f + \frac{1}{3}\frac{\rho'}{\rho}f + \frac{\xi'}{\xi + \mu^2}f - \frac{i}{2}\frac{\alpha P}{(\xi+\mu^2)^2} + i\frac{\alpha\tau}{\xi+\mu^2} - \frac{\alpha P\mu}{\xi+\mu^2}\right) B_0.$$

Here

$$f = \frac{Z_{aa}}{m}\frac{\xi + \mu^2}{Z - 1}.$$

The dependence of B_0 on s remains undetermined and one considers the boundary-value problem for K_3 and J_3. In order to find how B_0 depends on s it is sufficient to determine K_3 from (7) and then substitute it to the first boundary condition from (12) with $j = 3$.

Consider equations (7). These equations yield the following equation for K_3

$$L_0 K_3 = \eta K_2 + i \left(\frac{\rho}{2}\right)^{2/3} L_1 K_1 - i2^{1/3}\rho^{2/3}\alpha\tau J_1. \tag{21}$$

Substituting here expressions for K_2, K_1 and J_1 one finds the right-hand side of (21)

$$\text{R.H.S.} = \left(\eta A_2 + i \left(\frac{\rho}{2}\right)^{2/3} \left(3\frac{h'}{h}A_1 + 2\frac{\partial A_1}{\partial s} - 2\alpha\tau B_1\right)\right) w_1(\xi - \nu)$$

$$+ \left(\eta^2 A_1 + i \left(\frac{\rho}{2}\right)^{2/3} (2\xi' A_1 - 4\eta\alpha\tau B_0)\right) \dot{w}_1(\xi - \nu)$$

$$+ \frac{2i}{3} \left(\frac{\rho}{2}\right)^{2/3} \frac{\rho'}{\rho} A_1 \nu \dot{w}_1(\xi - \nu),$$

Using properties of Airy functions, the solution of (21) can be found

$$K_3 = A_3 w_1(\xi - \nu)$$

$$+ \left(\eta A_2 + i \left(\frac{\rho}{2}\right)^{2/3} \left(3\frac{h'}{h}A_1 + 2\frac{\partial A_1}{\partial s} + \frac{1}{3}\frac{\rho'}{\rho}A_1 - 2\alpha\tau B_1\right)\right) \dot{w}_1(\xi - \nu)$$

$$- \left(\frac{\eta^2}{2} A_1 + i \left(\frac{\rho}{2} \right)^{2/3} (\xi' A_1 - 2\eta\alpha\tau B_0) \right) \nu \, w_1(\xi - \nu)$$

$$- \frac{i}{6} \left(\frac{\rho}{2} \right)^{2/3} \frac{\rho'}{\rho} A_1 \nu^2 w_1(\xi - \nu).$$

At this step A_3 is as usually an arbitrary function. One substitutes this solution to the first boundary condition from (12). Expressing $\dot{w}_1(\xi)$ in that formula with the help of dispersion equation (15) and taking into account the introduced function μ, one gets

$$i\eta \left(\frac{2}{\rho} \right)^{2/3} A_2 - 3\frac{h'}{h} A_1 - 2\frac{\partial A_1}{\partial s} - \frac{1}{3}\frac{\rho'}{\rho} A_1 + i\frac{\eta^2}{2} \left(\frac{2}{\rho} \right)^{2/3} \frac{A_1}{\xi + \mu^2} - \frac{\xi'}{\xi + \mu^2} A_1$$

$$+ 2\alpha\tau B_1 - \frac{\alpha P}{\xi + \mu^2} B_1 + \left(2\alpha\tau\eta - i\alpha P\eta\mu - \frac{\alpha}{h}\frac{\partial \Psi}{\partial a} \right) \frac{B_0}{\xi + \mu^2} = 0. \quad (22)$$

Now one substitutes expressions for the functions A_1 and A_2. It is not difficult to see that due to the choice of function η, terms proportional to B_1 disappear from the equation (22). Dividing the above equation by $4i\eta f B_0$ allows it to be rewritten in the form convenient for integration, namely

$$\frac{B_0'}{B_0} + \frac{3}{2}\frac{h'}{h} + \frac{1}{6}\frac{\rho'}{\rho} + \frac{1}{2}\frac{(\eta f)'}{\eta f} + \frac{\xi'}{\xi + \mu^2} - \frac{i}{4}\frac{\alpha P}{(\xi + \mu^2)^2} - \frac{1}{2}\frac{\alpha P\mu}{(\xi + \mu^2)f}$$

$$+ \frac{i}{4}\frac{\alpha}{h}\frac{\partial \Psi}{\partial a}\frac{1}{(\xi + \mu^2)f} = 0.$$

One notes that

$$\frac{\xi'}{\xi + \mu^2} = \frac{d'}{d}, \qquad d = \dot{w}_1^2(\xi) - \xi \, w_1^2(\xi) = -(\xi + \mu^2)w_1^2(\xi).$$

This allows the solution to be written in the form

$$B_0(s, a) = \frac{B_0(a)}{h^{3/2}\rho^{1/6}\sqrt{\eta f d}} \exp \left(\alpha \int_{s_0(a)}^{s} F(s', a)ds' \right), \quad (23)$$

where

$$F(s, a) = \frac{i}{4}\frac{P}{(\xi + \mu^2)^2} + \frac{1}{2}\frac{P\mu}{(\xi + \mu^2)f} - \frac{i}{4}\frac{1}{h}\frac{\partial \Psi}{\partial a}\frac{1}{(\xi + \mu^2)f}.$$

Now one can write the principal order term of the creeping waves asymptotics. Introducing the above determined functions in the ansatz (4), one finally gets

$$
\begin{pmatrix} H \\ E \end{pmatrix} =
$$

$$
\exp \left\{ iks + i \left(\frac{k}{2} \right)^{1/3} \int\limits_{s_0(a)}^{s} \frac{\xi(s',a)\, ds'}{\rho^{2/3}(s',a)} + i \frac{k^{1/6}}{2^{1/3}} \int\limits_{s_0(a)}^{s} \frac{\eta\, ds}{\rho^{2/3}} \right\} B_0 w_1(\xi - \nu),
$$

(24)

where ξ is a solution of (15), η is given by (20), and B_0 is presented in (23).

We stop our derivations at this step. However it is easy to check that the asymptotic procedure allows solutions K_j and J_j to be found at any order. Each time one starts solving the corresponding boundary-value problem, there are undetermined amplitudes A_{j-1}, B_{j-3}, B_{j-2} and B_{j-1}. One finds general solutions K_j and J_j of the inhomogeneous Airy equations. These solutions contain arbitrary at this step amplitudes A_j and B_j. Then one considers the boundary conditions. The first boundary condition (for K) yields differential equation for the amplitude B_{j-3}. The second gives expression for the amplitude A_j via known and yet undetermined amplitudes. Thus at each step number j one introduces new unknown amplitude B_j and determines s dependence in the amplitude B_{j-3}.

6. CONCLUSION

The principal order approximation for the electromagnetic creeping waves on an anisotropic impedance surface with nondiagonalizable degenerated impedance matrix is given by (24). One can see that such parameters as h, ρ, $\mathbf{Z}$, τ and P are present in the principal order of this asymptotics. Thus in order to define the amplitude dependence one needs to perform some preliminary matching procedure, and to fix the lower bound of integration given by function $s_0(a)$ in (4).

One can note also that there appears another specific case, namely if

$$
P = 2\tau(\xi + \mu^2)
$$

the function η becomes equal to zero. In this case the even terms in the asymptotic series (4) disappear. Moreover the equation (22) is automatically satisfied and the dependence of B_0 on s remains at this

step undetermined. However it can be determined from the further order equations.

Finally let us compare the derived asymptotics of creeping waves on a surface with anisotropic impedance boundary condition described by degenerated matrix that has coincident eigenvalues and can not be diagonalized with the usual asymptotics of electromagnetic creeping waves. The difference is in the following aspects:

(i) Additional exponential factor

$$\exp\left(ik^{1/6}\Psi(s,a)\right)$$

appears. Function $\Psi(s,a)$ depends on torsion τ and torsion-like quantity P (P is equal to torsion for the circular cylinder case).

(ii) Asymptotic series is carried out by $k^{-j/6}$ rather than $k^{-j/3}$ as in the usual case.

(iii) Creeping waves of two types do not differ in attenuation parameters ξ (similar effect was observed for isotropic case if the impedance Z is equal to 1).

(iv) Creeping waves of two types do not differ in polarization, such difference appears only at the next order, i.e. at $O(k^{-1/6})$.

(v) Torsion τ and torsion-like quantity P are present in quick factor (function Ψ), i.e. these quantities play more important role than in the general case when only τ appears and only in the amplitude factor.

(vi) Some additional factors appear also in the amplitude B_0.

Let us also note that using the ansatz (4) in the nondegenerated case results in the asymptotic decomposition which represents the sum of two usual asymptotics for electromagnetic creeping waves, one of which is multiplied by $k^{-1/6}$. In this case one has $\eta = 0$, $\Psi = 0$, and terms of even and odd orders do not interact with each other. One therefore checks that our procedure provides in this case the same results as previous analysis.

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