Progress In Electromagnetics Research, PIER 60, 107–117, 2006

ANALYSIS OF COUPLED OR SINGLE NONUNIFORM TRANSMISSION LINES USING TAYLOR'S SERIES EXPANSION

M. Khalaj-Amirhosseini

College of Electrical Engineering
Iran University of Science and Technology
Tehran, Iran

Abstract—A method is proposed for analysis of arbitrarily loaded lossy and dispersive nonuniform single or coupled transmission lines. In this method, all the per-unit-length parameters or matrices of the single or coupled lines and also the voltages and currents along the length of them are expanded in Taylor's series. The solutions of voltages and currents are obtained after finding unknown coefficients of their series. The accuracy of the method is studied using analysis of some special types of single and coupled transmission lines.

1. INTRODUCTION

Single and coupled nonuniform transmission lines (NTLs) are widely used in RF and microwave circuits as resonators, impedance matching [1], size matching [2], delay equalizers [3], filters [4], wave shaping [5], analog signal processing [6], VLSI interconnect [7] and etc. The differential equations describing these structures have non-constant coefficients because their per-unit-length parameters or matrices vary along their length. The most used method is subdividing the nonuniform lines into many short sections [8–13]. In each section a uniform [8–11], linear [12], exponential [13] or other types of lines may be inserted. These type of differential equations have been solved analytically and without approximation only for a few special types of NTLs such as linear [12], exponential [13], power-law [14, 15], binomial [16], exponential power law [17] and hermite [18] types.

The subject of this paper is using Taylor's series expansion to analyze coupled or single NTLs. In this method, all the per-unit-length parameters and also the voltage and current distributions along the structure are expanded in Taylor's series. First, the unknown coefficients of the series are obtained from some recursive relations. Then the voltages and currents of the lines along the structure will be obtained. The solutions obtained from this method are exact but they are expressed by means of infinite linear equations. This method is applicable to many arbitrarily loaded lossy and dispersive coupled and single NTLs. The accuracy of the method is studied using analysis of some special kinds of single and coupled NTLs.

2. THE EQUATIONS OF NTLS

In this section, the frequency domain equations of loaded coupled or single NTLs are reviewed. It is assumed that the principal propagation mode of the lines is TEM or quasi-TEM. This assumption is valid when the widths in the cross section are small enough compared to the wavelength. Figure 1 shows typical coupled and single NTLs consisting of M ($M = 1$ for single NTLs) lines with length of d and with arbitrary terminal loads of $Z_{S,m}(\omega)$ and $Z_{L,m}(\omega)$, in which $m = 1, 2, \ldots, M$.

The partial differential equations describing lossy and dispersive

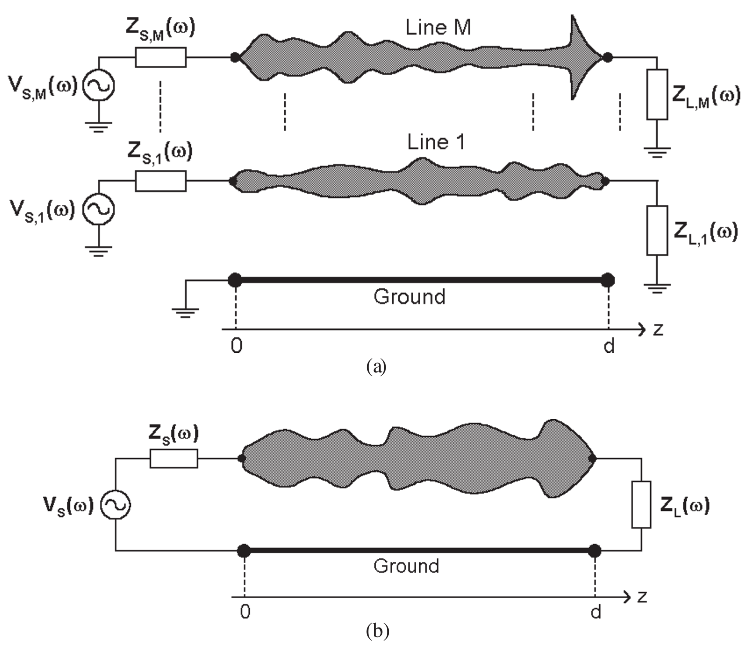

Figure 1. A typical nonuniform transmission line terminated by arbitrary loads a) Coupled NTL with M lines b) Single NTL.

NTLs in the frequency domain are given by

$$\frac{d\boldsymbol{V}(z,\omega)}{dz} = -\boldsymbol{Z}(z,\omega)\boldsymbol{I}(z,\omega) \tag{1}$$

$$\frac{d\boldsymbol{I}(z,\omega)}{dz} = -\boldsymbol{Y}(z,\omega)\boldsymbol{V}(z,\omega) \tag{2}$$

in which $\boldsymbol{V}$ and $\boldsymbol{I}$ are $M \times 1$ voltage and current vectors, respectively. Also we have

$$\boldsymbol{Z}(z,\omega) = \boldsymbol{R}(z,\omega) + j\omega\boldsymbol{L}(z,\omega) \tag{3}$$

$$\boldsymbol{Y}(z,\omega) = \boldsymbol{G}(z,\omega) + j\omega\boldsymbol{C}(z,\omega) \tag{4}$$

In (3)–(4), $\boldsymbol{R}, \boldsymbol{L}, \boldsymbol{G}$ and $\boldsymbol{C}$ are the per-unit-length matrices of the coupled transmission lines, whose dimensions are $M \times M$. These matrices are reduced to the distributed primary parameters R, L, G and C, for the single transmission lines. Also, the characteristic impedance and the propagation coefficient of the single lines will be as follows, respectively

$$Z_C(z,\omega) = \sqrt{\frac{Z(z,\omega)}{Y(z,\omega)}} = \sqrt{\frac{R(z,\omega)+j\omega L(z,\omega)}{G(z,\omega)+j\omega C(z,\omega)}} \tag{5}$$

$$\begin{aligned}\gamma(z,\omega) &= \alpha(z,\omega) + j\beta(z,\omega) = \sqrt{Z(z,\omega)Y(z,\omega)} \\ &= \sqrt{[R(z,\omega)+j\omega L(z,\omega)][G(z,\omega)+j\omega C(z,\omega)]}\end{aligned} \tag{6}$$

Combining (1) and (2) with each other, gives the following differential equations for the voltage and current vectors of NTLs.

$$\frac{d^2\boldsymbol{V}(z,\omega)}{dz^2} - \boldsymbol{f}(z,\omega)\frac{d\boldsymbol{V}(z,\omega)}{dz} - \boldsymbol{g}(z,\omega)\boldsymbol{V}(z,\omega) = \boldsymbol{0} \tag{7}$$

$$\boldsymbol{I}(z,\omega) = -\boldsymbol{Z}^{-1}(z,\omega)\frac{d\boldsymbol{V}(z,\omega)}{dz} \tag{8}$$

Where

$$\boldsymbol{g}(z,\omega) = \boldsymbol{Z}(z,\omega)\boldsymbol{Y}(z,\omega) \tag{9}$$

$$\boldsymbol{f}(z,\omega) = \frac{d\boldsymbol{Z}(z,\omega)}{dz}\boldsymbol{Z}^{-1}(z,\omega) \tag{10}$$

Furthermore, the terminal conditions for loaded NTLs are as follows

$$\boldsymbol{V}(0,\omega) + \boldsymbol{Z}_S(\omega)\boldsymbol{I}(0,\omega) = \boldsymbol{V}_S(\omega) \tag{11}$$

$$\boldsymbol{V}(d,\omega) - \boldsymbol{Z}_L(\omega)\boldsymbol{I}(d,\omega) = \boldsymbol{0} \tag{12}$$

Where $\boldsymbol{Z}_S$ and $\boldsymbol{Z}_L$ are diagonal source and load matrices, respectively. One sees from (7)–(12) that, solving analytically the equations of general type coupled or single NTLs is a very hard problem.

3. ANALYSIS OF NTLS USING TAYLOR'S SERIES

In this section, the analysis of arbitrary coupled or single NTLs using Taylor's series expansion is proposed. It is assumed that each of four per-unit-length matrices existing in (3)–(4), could be expressed by Taylor's series as follows

$$\boldsymbol{L}(z,\omega) = \sum_{n=0}^{\infty} \boldsymbol{L}_n(\omega)(z/d)^n \tag{13}$$

$$\boldsymbol{C}(z,\omega) = \sum_{n=0}^{\infty} \boldsymbol{C}_n(\omega)(z/d)^n \tag{14}$$

$$\boldsymbol{R}(z,\omega) = \sum_{n=0}^{\infty} \boldsymbol{R}_n(\omega)(z/d)^n \tag{15}$$

$$\boldsymbol{G}(z,\omega) = \sum_{n=0}^{\infty} \boldsymbol{G}_n(\omega)(z/d)^n \tag{16}$$

The frequency dependent matrices $\boldsymbol{L}_n$, $\boldsymbol{C}_n$, $\boldsymbol{R}_n$ and $\boldsymbol{G}_n$ are assumed to be known. These matrices are related to the n-th differential of their own matrices with respect to z. We also consider the voltage and current vectors of NTLs in Taylor's series as follows

$$\boldsymbol{V}(z,\omega) = \sum_{n=0}^{\infty} \boldsymbol{V}_n(\omega)(z/d)^n \tag{17}$$

$$\boldsymbol{I}(z,\omega) = \sum_{n=0}^{\infty} \boldsymbol{I}_n(\omega)(z/d)^n \tag{18}$$

Where the frequency dependent coefficients $\boldsymbol{V}_n(\omega)$ and $\boldsymbol{I}_n(\omega)$ are unknown $M \times 1$ vectors, which have to be determined. Using (13)–(18) in (1)–(2), the following relations are obtained.

$$\sum_{n=0}^{\infty}(n+1)\boldsymbol{V}_{n+1}(\omega)(z/d)^n = -d\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\boldsymbol{Z}_m(\omega)\boldsymbol{I}_k(\omega)(z/d)^{k+m} \tag{19}$$

$$\sum_{n=0}^{\infty}(n+1)\boldsymbol{I}_{n+1}(\omega)(z/d)^n = -d\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\boldsymbol{Y}_m(\omega)\boldsymbol{V}_k(\omega)(z/d)^{k+m} \tag{20}$$

In (19)–(20), we have

$$\boldsymbol{Z}_m(\omega) = \boldsymbol{R}_m(\omega) + j\omega\boldsymbol{L}_m(\omega) \tag{21}$$

$$\boldsymbol{Y}_m(\omega) = \boldsymbol{G}_m(\omega) + j\omega\boldsymbol{C}_m(\omega) \tag{22}$$

Also, using (17)–(18) in terminal conditions (11)–(12), the following relations are obtained.

$$\boldsymbol{V}_0(\omega) + \boldsymbol{Z}_S(\omega)\boldsymbol{I}_0(\omega) = \boldsymbol{V}_S(\omega) \tag{23}$$

$$\sum_{n=0}^{\infty}(\boldsymbol{V}_n(\omega) - \boldsymbol{Z}_L(\omega)\boldsymbol{I}_n(\omega)) = \boldsymbol{0} \tag{24}$$

Equating the coefficients of the same power terms in two sides of (19)–(20), gives us the following recursive relations for $n = 0, 1, 2, \ldots$

$$\boldsymbol{V}_{n+1}(\omega) = \frac{-d}{n+1}\sum_{k=0}^{n}\boldsymbol{Z}_{n-k}(\omega)\boldsymbol{I}_k(\omega) \tag{25}$$

$$\boldsymbol{I}_{n+1}(\omega) = \frac{-d}{n+1}\sum_{k=0}^{n}\boldsymbol{Y}_{n-k}(\omega)\boldsymbol{V}_k(\omega) \tag{26}$$

To find the unknown coefficients $\boldsymbol{V}_n$ and $\boldsymbol{I}_n$ in a specified angular frequency ω, we truncate the maximum power in Taylor's series to N, i.e., $n \leq N$ at first. Consequently, there will be $2N + 2$ equations of (23)–(26) to find $2N + 2$ unknown coefficients. From mathematical theorems, one may conclude that to converge the solutions (17)–(18), the Taylor's series of each of the matrices in (13)–(16) have to be converged at all points on the interval $z = [0, d]$.

4. EXAMPLES AND RESULTS

In this section, three special types of single and coupled transmission lines (uniform and linearly nonuniform) are analyzed both using analytical formulas and using the proposed method.

Example 1: (Uniform Single Transmission Line)

Consider a lossless and uniform single transmission line ($M = 1$, $R = G = 0$). Assume $Z_0 = \sqrt{L/C} = 50\,\Omega$, $\gamma = j\beta = j\omega\sqrt{LC} = j\omega/c$ (c is the velocity of the light), $d = 20\,\text{cm}$, $f = 1.0\,\text{GHz}$, $Z_S = 50\,\Omega$, $Z_L = 100\,\Omega$ and $V_S = 1\,\text{V}$. Figure 2, compares the magnitude of voltage of the line, obtained from analytical formulas and from the introduced method with $N = 10$ and $N > 15$ coefficients. One sees an excellent agreement between the results from analytical solution and the results from the introduced method with $N > 15$ coefficients. It is seen and also evident that, as the number of unknown coefficients, N, increases, the accuracy of the obtained solution increases. Furthermore, for insufficient number of unknown coefficients (such as $N = 10$ in this example), the obtained voltages

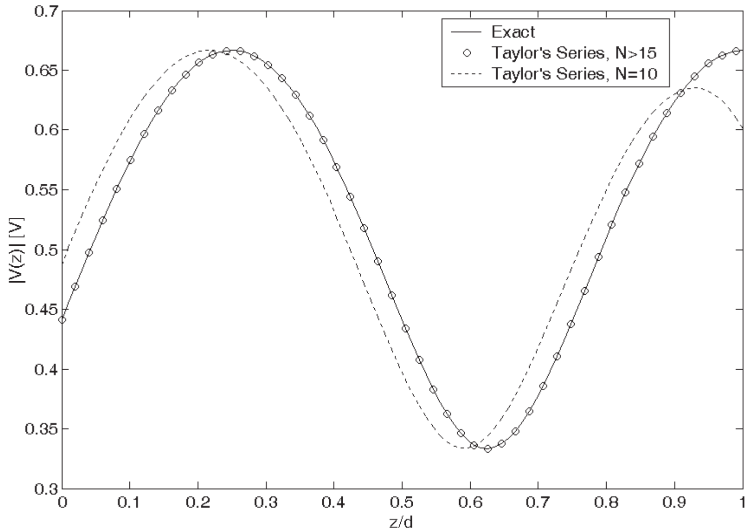

Figure 2. The magnitude of the voltage of uniform single line, obtained from exact formulas and from the introduced method with $N = 10$ and $N > 15$ coefficients.

of all points degrade from the exact ones. Also, Fig. 3 shows the absolute of the unknown coefficients V_n for two cases of ($d = 20\,\text{cm}$, $f = 1.0\,\text{GHz}$) and ($d = 40\,\text{cm}$, $f = 1.0\,\text{GHz}$) or ($d = 20\,\text{cm}$, $f = 2.0\,\text{GHz}$). One sees that, as the length of the line or equivalently the source frequency increases, the necessary number of unknown coefficients, N, increases.

Example 2: (Uniform Coupled Transmission Lines)

Consider a lossless uniform coupled microstrip structure with $M = 2$ strips. The substrate permittivity is $\varepsilon_r = 10$, the width of strips and the gap between them are equal to the thickness of the substrate. This inhomogeneous structure will has the following the per-unit-length matrices.

$$\boldsymbol{L}(z) = \boldsymbol{L}_0 = \begin{bmatrix} 425.6 & 74.83 \\ 74.83 & 425.6 \end{bmatrix} \text{nH/m} \tag{27}$$

$$\boldsymbol{C}(z) = \boldsymbol{C}_0 = \begin{bmatrix} 174.9 & -14.25 \\ -14.25 & 174.9 \end{bmatrix} \text{pF/m} \tag{28}$$

$$\boldsymbol{R}(z) = \boldsymbol{G}(z) = \boldsymbol{0} \tag{29}$$

Assume that $d = 20\,\text{cm}$, $f = 1.0\,\text{GHz}$, $Z_{S,1} = Z_{S,2} = 50\,\Omega$, $Z_{L,1} =$

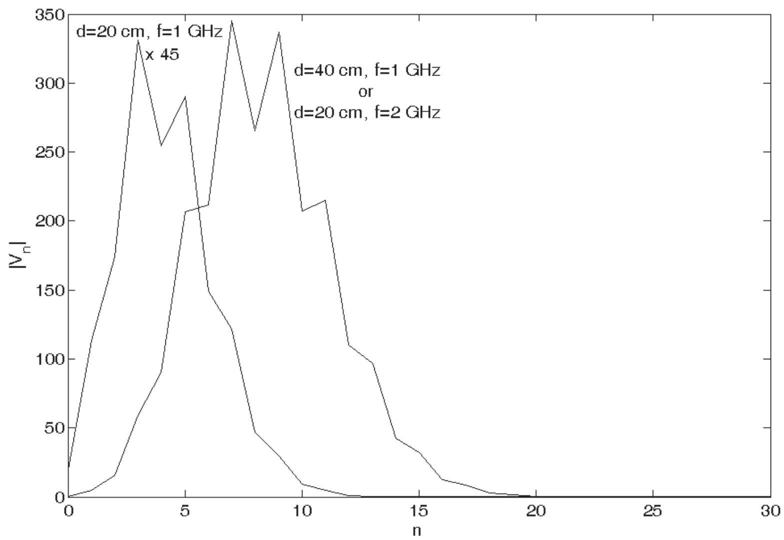

Figure 3. The absolute of the unknown coefficients for two cases ($d = 20\,\text{cm}$, $f = 1.0\,\text{GHz}$) and ($d = 40\,\text{cm}$, $f = 1.0\,\text{GHz}$) or ($d = 20\,\text{cm}$, $f = 2.0\,\text{GHz}$).

$Z_{L,2} = 50\,\Omega$, $V_{S,1} = 1\,\text{V}$ and $V_{S,2} = 0$. The exact voltages of this structure can be determined using the modal decomposing method [10]. Figure 4 compares the magnitude of voltages of two lines, obtained from the modal decomposing method and from the proposed method with $N = 25$ and $N = 50$ coefficients. Again, one sees an excellent agreement between the exact solutions and the solutions obtained from the proposed method with sufficient number of coefficients.

Example 3: (Linearly Nonuniform Single Line)

Consider a lossless and linearly varied single NTL with the following distributed primary parameters

$$L(z) = L_0(1 + k(z/d)) \tag{30}$$

$$C(z) = \frac{C_0}{1 + k(z/d)} = C_0 \sum_{n=0}^{\infty} (-k)^n (z/d)^n \tag{31}$$

$$R(z) = G(z) = 0 \tag{32}$$

This type of transmission line will have the following secondary

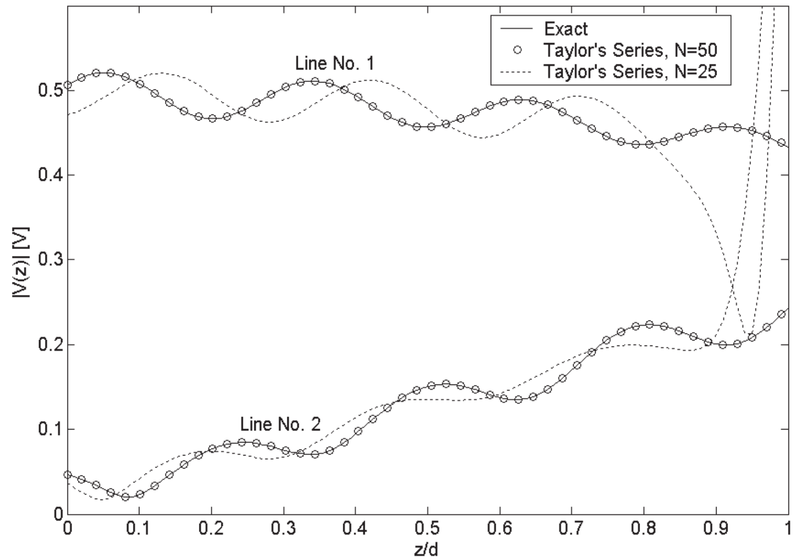

Figure 4. The magnitude of the voltage of uniform coupled lines, obtained from the exact formulas and from the introduced method with $N = 25$ and $N = 50$ coefficients.

parameters defined in (5)–(6)

$$Z_C(z) = \sqrt{L_0/C_0}(1 + k(z/d)) \tag{33}$$

$$\gamma = j\beta = j\omega\sqrt{L_0C_0} \tag{34}$$

Assume that $Z_C(0) = \sqrt{L_0/C_0} = 50\,\Omega$, $\beta = \omega\sqrt{L_0C_0} = \omega/c$, $d = 20\,\text{cm}$, $f = 1.0\,\text{GHz}$, $Z_S = 50\,\Omega$, $Z_L = 100\,\Omega$, $V_S = 1\,\text{V}$ and $k = 0.5$, 1.0 or 1.1. Figure 5 compares the magnitude of voltage of the line obtained in [12] and from the proposed method with $N = 100$ coefficients. One sees the agreement between the results from analytical solution and the results from the introduced method is excellent, good and bad for the cases of $k = 0.5$, $k = 1.0$ and $k = 1.1$, respectively. The degradation in the case of $k = 1.1$, is due to the divergence of the Taylor's series of the primary parameter $C(z)$ in (31) at some points, in which $kz/d > 1$.

From the above examples, one may conclude that the introduced method is applicable for all NTLs, whose parameters can be expressed by a converged Taylor's series. Also, it is concluded that as the excitation frequency, the length of the line (with respect to the wavelength) and the variations of the primary parameters increase, the necessary number of coefficients increases.

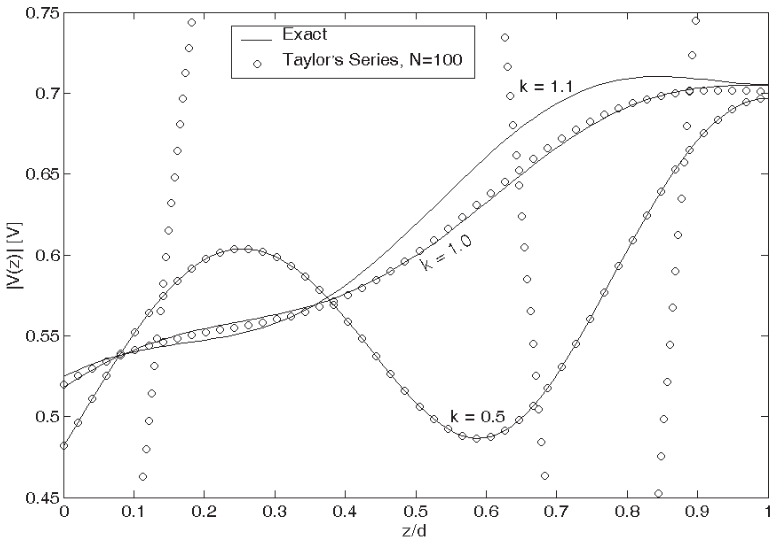

Figure 5. The magnitude of the voltage of linear nonuniform line, obtained from exact formulas and from the introduced method with $N = 100$ coefficients.

5. CONCLUSIONS

A method was introduced to frequency domain analysis of arbitrarily loaded lossy and dispersive nonuniform transmission lines. In this method, all distributed primary parameters of the lines and also the voltage and current distribution along the line are considered as a Taylor's series. It was seen that, as the number of unknown coefficients increases the accuracy of the obtained solution increases. Also, as the length of the lines with respect to the wavelength or the variations of the primary parameters increase, the necessary number of unknown coefficients increases. The validity of the method was verified using analysis of some special types of lines. This method is very simple and fast and can be used for all NTLs, whose primary parameters can be expressed by a converged Taylor's series.

REFERENCES

1. Collin, R. E., *Foundations for Microwave Engineering*, McGraw-Hill, New York, 1996.
2. Gholam-Tamimi, M. and M. Khalaj-Amirhosseini, "Conical

coupled stripline transmission lines," *IEEE Symp. Mic. Ant. Prop. EMC (MAPE 2005)*, 1084–1087, Beijing, China, Aug. 8–12, 2005.

3. Tang, C. C. H., "Delay equalization by tapered cutoff waveguides," *IEEE Trans. Mic. Theory and Tech.*, 608–615, Nov. 1964.
4. Roberts, P. P. and G. E. Town, "Design of microwave filters by inverse scattering," *IEEE Trans. Mic. Theory and Tech.*, 739–743, Apr. 1995.
5. Burkhart, S. C. and R. B. Wilcox, "Arbitrary pulse shape synthesis via nonuniform transmission lines," *IEEE Trans. Mic. Theory and Tech.*, 1514–1518, Oct. 1990.
6. Hayden, L. A. and V. K. Tripathi, "Nonuniform coupled microstrip transversal filters for analog signal processing," *IEEE Trans. Mic. Theory and Tech.*, 47–53, Jan. 1991.
7. Dhaene, T., L. Martens, and D. D. Zutter, "Transient simulation of arbitrary nonuniform interconnection structures characterized by scattering parameters," *IEEE Trans. Circuits and Sys.-I*, 928–937, Nov. 1992.
8. Khalaj Amirhosseini, M., "Analysis of coupled or single nonuniform transmission lines using step-by-step numerical integration," *Progress in Electromagnetics Research*, PIER 58, 187–198, 2006.
9. Cheldavi, A., M. Kamarei, and S. Safavi-Naeini, "First order approximation of the exact solution of arbitrary nonuniform transmission lines: application in high speed integrated circuits," *IEICE Trans. Electron.*, Vol. Es2-c, No. 12, 2248–2254, Dec. 1999.
10. Paul, C. R., *Analysis of Multiconductor Transmission Lines*, John Wiley and Sons Inc., 1994.
11. Khalaj Amirhosseini, M. and A. Cheldavi, "The use of grounded lines in non-uniform interconnects," *J. Electromagn. Waves and Appl.*, 637–647, May 2004.
12. Lu, K., "An efficient method for analysis of arbitrary nonuniform transmission lines," *IEEE Trans. Micro. Theory and Tech.*, 9–14, Jan. 1997.
13. Oufi, E. A., A. S. AlFuhaid, and M. M. Saied, "Transient analysis of lossless single-phase nonuniform transmission lines," *IEEE Trans. Power Delivery*, 1694–1700, Jul. 1994.
14. Curtins, H. and A. V. Shah, "Step response of lossless nonuniform transmission lines with power-law characteristic impedance function," *IEEE Trans. Micro. Theory and Tech.*, 1210–1212, Nov. 1985.
15. Cheldavi, A., M. Kamarei, and S. Safavi Naeini, "Analysis

of coupled transmission lines with power-law characteristic impedance," *IEEE Trans. Electromagn. Compat.*, 308–312, Aug. 2000.

16. Kobayashi, K., Y. Nemoto, and R. Sato, "Equivalent circuits of binomial form nonuniform coupled transmission lines," *IEEE Trans. Mic. Theory and Tech.*, 817–824, Aug. 1981.
17. Cheldavi, A., "Exact analysis of coupled nonuniform transmission lines with exponential power law characteristic impedance," *IEEE Trans. Mic. Theory and Tech.*, 197–199, Jan. 2001.
18. Cheldavi, A., "Analysis of coupled Hermite transmission lines," *IEE Proc. Mic., Ant. and Prop.*, 279–284, Aug. 2003.