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A HYBRID METHOD FOR THE SOLUTION OF PLANE WAVE DIFFRACTION BY AN IMPEDANCE LOADED PARALLEL PLATE WAVEGUIDE

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Abstract—The diffraction of E-polarized plane waves by an impedance loaded parallel plate waveguide formed by a two-part impedance plane and a parallel half plane with different face impedances is investigated rigorously by using the Fourier transform technique in conjunction with the Mode Matching Method. This mixed method of formulation gives rise to a scalar Modified Wiener-Hopf equation, the solution of which contains infinitely many constants satisfying an infinite system of linear algebraic equations. A numerical solution of this system is obtained for various values of the surface impedances and waveguide height.

1. INTRODUCTION

In the present work the diffraction of plane electromagnetic waves by an impedance loaded parallel plate waveguide formed by a two-part impedance plane and a parallel half plane with different face impedances is analyzed. This problem is a generalization of a previous work by the authors [1] who considered the same geometry in the case where the half plane is perfectly conducting. In [1] the related boundary value problem is formulated as a matrix Wiener-Hopf equation which is uncoupled by the introduction of infinite sum of poles. The exact solution is then obtained in terms of the coefficients of the poles, where these coefficients are shown to satisfy infinite system of linear algebraic equations. When the half plane has non vanishing surface impedances, the resulting matrix Wiener-Hopf equation becomes intractable. To overcome this difficulty one recourses of a hybrid formulation consisting of employing the Fourier transform technique in conjunction with the mode matching method (see for example [2–5]). By expanding the total field into a series of normal modes in the waveguide region and using the Fourier transform elsewhere, we get a scalar modified Wiener-Hopf equation of the second kind. The solution involves a set of infinitely many expansion coefficients satisfying infinite system of linear algebraic equations. Numerical solution to this system is obtained for different values of the surface impedances and height of the waveguide. In the case where the impedance of the semi-infinite plane vanishes, the results are compared numerically with those obtained in [1] by solving a matrix Wiener-Hopf equation, and it is shown than both results coincide exactly

2. FORMULATION OF THE PROBLEM

Let a time harmonic plane wave

$$E_x^i = 0, \quad E_y^i = 0, \quad E_z^i = u^i = e^{-ik(x\cos\phi_0 + y\sin\phi_0)} \tag{1a}$$

with time factor $e^{-i\omega t}$ and wave number k, illuminates the parallel plate waveguide formed by a two-part impedance plane defined by

$$\{(x, y, z)|\,, x \in (-\infty, \infty)\,, y = 0, z \in (-\infty, \infty)\}\,,$$

whose left and right parts are characterized by the surface impedances $Z_1 = \eta_1 Z_0$ and $Z_2 = \eta_2 Z_0$, respectively, with Z_0 being the characteristic impedance of the surrounding medium and a parallel half-plane with different face impedances located at $\{x < 0, y = b, z \in (-\infty, \infty)\}$. The surface impedances of the upper and lower faces of the half-plane are assumed to be $Z_3 = \eta_3 Z_0$ and $Z_4 = \eta_4 Z_0$ respectively (see Figure 1). This problem is a generalization of the one considered in [1] where the half plane is assumed to be perfectly conducting.

The method of formulation adopted in this work involves an appropriate definition of the total field such that in the waveguide region the field component may be expressed in terms of normal modes and the Fourier transform technique can be applied elsewhere. Notice that the problem considered in [1] can also be reduced to the solution of a scalar Modified Wiener-Hopf equation by using this hybrid method of Modal expansion and the Fourier transform technique.

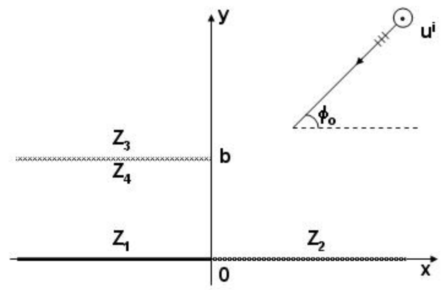

Figure 1. Impedance loaded parallel plate waveguide.

2.1. Reduction to a Modified Wiener-Hopf Equation

The total field can be expressed as

$$u^T(x,y) = \begin{cases} u^i(x,y) + u^r(x,y) + u_1(x,y) & , \quad y > b \\ u_2^{(1)}(x,y)\,\mathcal{H}(-x) + u_2^{(2)}(x,y)\,\mathcal{H}(x) & , \quad 0 < y < b \end{cases} \tag{1b}$$

where $u^i(x,y)$ is given by (1a), while $u^r(x,y)$ stands for the field that would be reflected if the whole plane $y = b$ were an impedance plane with relative surface impedance η_3

$$u^r(x,y) = -\frac{1-\eta_3\sin\phi_0}{1+\eta_3\sin\phi_0} e^{-ik[x\cos\phi_0-(y-2b)\sin\phi_0]} \tag{2}$$

In (1b), $\mathcal{H}(x)$ is the Heaviside unit step function and ϕ_0 is the angle of incidence, respectively. For the sake of analytical convenience we shall assume that k has a small imaginary part. The lossless case can then be obtained by making $\Im m(k) \to 0$ at the end of the analysis.

$u_1$and $u_2^{(j)}$ $(j = 1, 2)$, which satisfy the Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)\left[\begin{array}{c} u_1(x,y) \\ u_2^{(j)}(x,y) \end{array}\right] = 0 \quad , \quad j = 1, 2 \tag{3}$$

are to be determined with the aid of the following boundary and continuity relations:

$$\left(1 + \frac{\eta_3}{ik}\frac{\partial}{\partial y}\right) u_1(x,b) = 0, \qquad x < 0 \tag{4a}$$

$$\left(1 - \frac{\eta_4}{ik}\frac{\partial}{\partial y}\right) u_2^{(1)}(x,b) = 0, \qquad x < 0 \tag{4b}$$

$$\left(1 + \frac{\eta_1}{ik}\frac{\partial}{\partial y}\right) u_2^{(1)}(x,0) = 0, \qquad x < 0 \tag{4c}$$

$$\left(1+\frac{\eta_2}{ik}\frac{\partial}{\partial y}\right)u_2^{(2)}(x,0) = 0, \qquad x>0 \tag{4d}$$

$$u_1(x,b)+u^i(x,b)+u^r(x,b)-u_2^{(2)}(x,b) = 0, \quad x>0 \tag{4e}$$

$$\frac{\partial}{\partial y}u_1(x,b)+\frac{\partial}{\partial y}u^i(x,b)+\frac{\partial}{\partial y}u^r(x,b)-\frac{\partial}{\partial y}u_2^{(2)}(x,b) = 0, \quad x>0 \tag{4f}$$

$$u_2^{(1)}(0,y)-u_2^{(2)}(0,y) = 0, \qquad 0<y<b \tag{4g}$$

$$\frac{\partial}{\partial x}u_2^{(1)}(0,y)-\frac{\partial}{\partial x}u_2^{(2)}(0,y) = 0, \qquad 0<y<b. \tag{4h}$$

To ensure the uniqueness of the mixed boundary-value problem defined by the Helmholtz equation and the conditions (4a)–(4n), one has to take into account the radiation and edge conditions as well which are

$$\sqrt{\rho}\left[\frac{\partial u}{\partial\rho}-iku\right]\to 0 \ \ , \ \rho=\sqrt{x^2+y^2}\to\infty \tag{5a}$$

and

$$\begin{aligned} u_T(x,y) &= \mathcal{O}\left(|x|^{1/2}\right) \\ \frac{\partial}{\partial y}u_T(x,y) &= \mathcal{O}\left(|x|^{-1/2}\right), \qquad |x|\to 0 \end{aligned} \tag{5b}$$

respectively.

Since $u_1(x,y)$ satisfies the Helmholtz equation in the range $x\in(-\infty,\infty)$, $y>b$, its Fourier transform with respect to x gives

$$\left[\frac{d^2}{dy^2}+\left(k^2-\alpha^2\right)\right]F(\alpha,y)=0 \tag{6a}$$

with

$$F(\alpha,y)=F_-(\alpha,y)+F_+(\alpha,y) \tag{6b}$$

where

$$F_\pm(\alpha,y)=\pm\int_0^{\pm\infty}u_1(x,y)\ e^{i\alpha x}dx. \tag{6c}$$

By taking into account the following asymptotic behaviors of $u_1(x,y)$ for $x\to\pm\infty$

$$\begin{aligned} u_1(x,y) &= \mathcal{O}\left(e^{-ikx}/\sqrt{-x}\right) \quad , \quad x\to-\infty \\ u_1(x,y) &= \mathcal{O}\left(e^{-ikx\cos\phi_0}\right) \quad , \quad x\to\infty \end{aligned} \tag{7}$$

one can show that $F_+(\alpha, y)$ and $F_-(\alpha, y)$ are regular functions of α in the half-planes $\Im m(\alpha) > \Im m(k\cos\phi_0)$ and $\Im m(\alpha) < \Im m(k)$, respectively.

The general solution of (6a) satisfying the radiation condition for $y \to \infty$ reads

$$F_-(\alpha, y) + F_+(\alpha, y) = A(\alpha)\; e^{iK(\alpha)(y-b)} \tag{8a}$$

with $A(\alpha)$ being the unknown spectral coefficient and

$$K(\alpha) = \sqrt{k^2 - \alpha^2}. \tag{8b}$$

The square-root function is defined in the complex α-plane, cut along $\alpha = k$ to $\alpha = k + i\infty$ and $\alpha = -k$ to $\alpha = -k - i\infty$, such that $K(0) = k$.

In the Fourier transform domain (4a) yields

$$F_-(\alpha, b) + \frac{\eta_3}{ik}\dot{F}_-(\alpha, b) = 0. \tag{9}$$

where the dot represents first degree derivative with respect to y. By using (8a) and its derivative with respect to y, one can write

$$F_+(\alpha, b) + \frac{\eta_3}{ik}\dot{F}_+(\alpha, b) = \left[1 + \frac{\eta_3}{k}K(\alpha)\right]A(\alpha) \tag{10a}$$

and

$$\dot{F}_+(\alpha, b) = iK(\alpha)\,\frac{P_+(\alpha)}{\left[1 + \frac{\eta_3}{k}K(\alpha)\right]} - \dot{F}_-(\alpha, b), \tag{10b}$$

respectively where

$$P_+(\alpha) = F_+(\alpha, b) + \frac{\eta_3}{ik}\dot{F}_+(\alpha, b). \tag{10c}$$

On the other hand, $u_2^{(2)}(x, y)$ satisfies the Helmholtz equation in the range $x \in (0, \infty)$, $0 < y < b$ whose half range Fourier transform with respect to x gives

$$\left[\frac{d^2}{dy^2} + K^2(\alpha)\right]G_+(\alpha, y) = f(y) - i\alpha g(y) \tag{11a}$$

where $G_+(\alpha, y)$ is the Fourier transform of the field $u_2^{(2)}(x, y)$ in the specified region defined by

$$G_+(\alpha, y) = \int_0^\infty u_2^{(2)}(x, y)\, e^{i\alpha x} dx \tag{11b}$$

and $f(y)$ and $g(y)$ stand for

$$f(y) = \frac{\partial}{\partial x} u_2^{(2)}(0,y) \tag{11c}$$

$$g(y) = u_2^{(2)}(0,y). \tag{11d}$$

The general solution of the equation (11a) is

$$G_+(\alpha,y) = B(\alpha)\sin Ky + C(\alpha)\cos Ky + \frac{1}{K}\int_0^y [f(t) - i\alpha g(t)]\sin[K(y-t)]\,dt. \tag{12a}$$

Here $B(\alpha)$ and $C(\alpha)$ are the unknown spectral coefficients. When the boundary condition (4d) is taken into account

$$C(\alpha) = -\frac{\eta_2}{ik} K(\alpha) B(\alpha) \tag{12b}$$

relation is determined for these coefficients. If this relation is substituted in (12a), $G_+(\alpha,y)$ becomes

$$G_+(\alpha,y) = \left[\sin Ky - \frac{\eta_2}{ik} K\cos Ky\right] B(\alpha) + \frac{1}{K}\int_0^y [f(t) - i\alpha g(t)]\sin[K(y-t)]\,dt. \tag{12c}$$

$B(\alpha)$ can be expressed in terms of $F_+(\alpha,b)$ by using the continuity relation (4e) which yields

$$B(\alpha) = \frac{1}{K(\alpha)M(\alpha)}\left\{P_+(\alpha) - \int_0^b [f(t) - i\alpha g(t)]\left[\frac{\sin K(b-t)}{K} + \frac{\eta_3}{ik}\cos K(b-t)\right]dt\right\} \tag{12d}$$

with

$$M(\alpha) = \left(\frac{\eta_3 - \eta_2}{ik}\right)\cos Kb + \left(1 - \frac{\eta_2\eta_3}{k^2}K^2\right)\frac{\sin Kb}{K}. \tag{12e}$$

By using (12d) in (12a)

$$G_+(\alpha,y) = \frac{\left[\sin Ky - \frac{\eta_2}{ik}K\cos Ky\right]}{K(\alpha)M(\alpha)}\{P_+(\alpha)$$

$$-\int_0^b [f(t) - i\alpha g(t)] \left[\frac{\sin K(b-t)}{K} + \frac{\eta_3}{ik}\cos K(b-t)\right] dt \Bigg\}$$

$$+\frac{1}{K}\int_0^y [f(t) - i\alpha g(t)] \sin[K(y-t)]\, dt \tag{13}$$

is found. Although the left-hand side of the above equation is regular in the upper half-plane $\Im m(\alpha) > \Im m(k\cos\phi_0)$, the regularity of the right-hand side is violated by the presence of the simple poles occurring at the zeros of $M(\alpha)$, namely at $\alpha = \pm\alpha_m$ satisfying

$$M(\pm\alpha_m) = 0 \quad , \quad \Im m(\alpha_m) > \Im m(k) \quad , \quad m = 1, 2, \ldots. \tag{14}$$

These poles can be eliminated by imposing that their residues are zero. This gives

$$P_+(\alpha_m) = Q_m^2 \left[\frac{\eta_3}{ik} K_m \sin K_m b - \cos K_m b\right] (f_m - i\alpha_m g_m) \tag{15a}$$

where f_m and g_m are specified by

$$\begin{bmatrix} f_m \\ g_m \end{bmatrix} = \frac{1}{Q_m^2}\int_0^b \begin{bmatrix} f(y) \\ g(y) \end{bmatrix} \left[\frac{\sin K_m y}{K_m} - \frac{\eta_2}{ik}\cos K_m y\right] dy. \tag{15b}$$

with

$$K_m = \sqrt{k^2 - \alpha_m^2} \tag{15c}$$

and

$$Q_m^2 = -\frac{K_m \left(\dfrac{\cos K_m b}{K_m} + \dfrac{\eta_2}{ik}\sin K_m b\right) M'(\alpha_m)}{2\alpha_m}, \tag{15d}$$

where the prime sign on the above equation denotes first degree derivative with respect to α.

Hence, taking into account equation (10b) and the Fourier transform of the continuity relation given by (4f) together

$$ik\chi(\eta_3, \alpha) P_+(\alpha) - \dot{F}_-(\alpha, b)$$

$$= -\frac{2k\sin\phi_o}{(\eta_3\sin\phi_o + 1)} \frac{e^{-ikb\sin\phi_o}}{(\alpha - k\cos\phi_o)} + \frac{\left[\cos Kb + \dfrac{\eta_2}{ik} K\sin Kb\right]}{M(\alpha)}$$

$$\times \left\{ P_+(\alpha) - \int_0^b [f(t) - i\alpha g(t)] \left[\frac{\sin K(b-t)}{K} + \frac{\eta_3}{ik}\cos K(b-t)\right] dt \right\}$$

$$+\int_0^b \left[f(t) - i\alpha g(t)\right] \cos\left[K(b-t)\right] dt \tag{16}$$

is obtained where

$$\chi(\eta_3, \alpha) = \frac{K(\alpha)}{\eta_3 K(\alpha) + k}. \tag{17}$$

Equation (16) can be arranged as

$$\frac{\chi(\eta_3,\alpha)}{\chi(\eta_2,\alpha)} \frac{P_+(\alpha)}{N(\alpha)} + \dot{F}_-(\alpha, b) = \frac{2k\sin\phi_o}{(\eta_3\sin\phi_o + 1)} \frac{e^{-ikb\sin\phi_o}}{(\alpha - k\cos\phi_o)}$$
$$-\frac{1}{M(\alpha)} \int_0^b \left[f(t) - i\alpha g(t)\right] \left[\frac{\sin Kt}{K} - \frac{\eta_2}{ik}\cos Kt\right] dt \tag{18}$$

with

$$N(\alpha) = M(\alpha) e^{iKb}. \tag{19}$$

Incorporating the series expansions for the functions $f(y)$ and $g(y)$

$$\begin{bmatrix} f(y) \\ g(y) \end{bmatrix} = \sum_{m=1}^{\infty} \begin{bmatrix} f_m \\ g_m \end{bmatrix} \left[\frac{\sin K_m y}{K_m} - \frac{\eta_2}{ik}\cos K_m y\right], \tag{20}$$

equation (18) becomes

$$\frac{\chi(\eta_3,\alpha)}{\chi(\eta_2,\alpha)} \frac{P_+(\alpha)}{N(\alpha)} + \dot{F}_-(\alpha, b) = \frac{2k\sin\phi_o}{(\eta_3\sin\phi_o + 1)} \frac{e^{-ikb\sin\phi_o}}{(\alpha - k\cos\phi_o)}$$
$$+\sum_{m=1}^{\infty} (f_m - i\alpha g_m) \frac{K_m}{(\alpha^2 - \alpha_m^2)} \left(\frac{\cos K_m b}{K_m} + \frac{\eta_2}{ik}\sin K_m b\right). \tag{21}$$

This is nothing but the scalar modified Wiener-Hopf equation to be solved. The kernel functions in (21), i.e. the functions $N(\alpha)$ and $\chi(\eta_j, \alpha)$ can be factorized by using known expressions. Thus, the factors of $N(\alpha)$ are [6]

$$N_+(\alpha) = \left[\left(\frac{\eta_3 - \eta_2}{ik}\right)\cos kb + (1 - \eta_2\eta_3)\frac{\sin kb}{k}\right]^{1/2} e^{\left[\frac{Kb}{\pi}\ln\frac{\alpha + iK}{k}\right]}$$
$$\times e^{\left[\frac{i\alpha b}{\pi}\left(1 - \mathcal{C} + \ln\frac{2\pi}{kb} + i\frac{\pi}{2}\right)\right]} \prod_{m=1}^{\infty}\left(1 + \frac{\alpha}{\alpha_m}\right) e^{\left(\frac{i\alpha b}{m\pi}\right)} \tag{22a}$$

and

$$N_-(\alpha) = N_+(-\alpha). \tag{22b}$$

In the equation (22a) $\mathcal{C}$ is the well-known Euler constant which is $0,577215...$ As to the factors of $\chi(\eta_j, \alpha)$ they can be written in terms of the Maliuzhinets functions [7]:

$$\chi_-(\eta_j, k\cos\phi) = \frac{4}{\sqrt{\eta_j}} \sin\frac{\phi}{2} \left\{ \frac{\mathcal{M}_\pi(3\pi/2-\phi-\theta)\,\mathcal{M}_\pi(\pi/2-\phi+\theta)}{\mathcal{M}_\pi^2(\pi/2)} \right\}^2$$
$$\times \left\{ \left[1+\sqrt{2}\cos\left(\frac{3\pi/2-\phi-\theta}{2}\right)\right] \left[1+\sqrt{2}\cos\left(\frac{\pi/2-\phi+\theta}{2}\right)\right] \right\}^{-1} \quad (23a)$$

and

$$\chi_+(\eta_j, k\cos\phi) = \chi_-(\eta_j, -k\cos\phi) \quad (23b)$$

with $\mathcal{M}_\pi(z)$ and θ are defined by

$$\sin\theta = \frac{1}{\eta_j} \quad (23c)$$

and

$$\mathcal{M}_\pi(z) = \exp\left\{ -\frac{1}{8\pi} \int_0^z \frac{\pi\sin u - 2\sqrt{2}\pi\sin(u/2) + 2u}{\cos u} du \right\}. \quad (23d)$$

Hence, multiplying the Wiener-Hopf equation term by term with

$$\frac{\chi_-(\eta_2, \alpha)}{\chi_-(\eta_3, \alpha)} N_-(\alpha)$$

gives

$$\frac{\chi_+(\eta_3, \alpha)}{\chi_+(\eta_2, \alpha)} \frac{P_+(\alpha)}{N_+(\alpha)} + \frac{\chi_-(\eta_2, \alpha)}{\chi_-(\eta_3, \alpha)} N_-(\alpha)\,\dot{F}_-(\alpha, b)$$
$$= \frac{2k\sin\phi_o}{(\eta_3\sin\phi_o+1)} \frac{e^{-ikb\sin\phi_o}}{(\alpha-k\cos\phi_o)} \frac{\chi_-(\eta_2, \alpha)}{\chi_-(\eta_3, \alpha)} N_-(\alpha)$$
$$+\frac{\chi_-(\eta_2, \alpha)}{\chi_-(\eta_3, \alpha)} N_-(\alpha) \sum_{m=1}^{\infty} (f_m - i\alpha g_m) \frac{K_m}{(\alpha^2-\alpha_m^2)} \left(\frac{\cos K_m b}{K_m} + \frac{\eta_2}{ik}\sin K_m b \right). \quad (24)$$

The terms at the right-hand side of the equation must be decomposed with the help of Cauchy formula which leads

$$\frac{2k\sin\phi_o}{(\eta_3\sin\phi_o+1)}\frac{e^{-ikb\sin\phi_o}}{(\alpha-k\cos\phi_o)}\frac{\chi_-(\eta_2,\alpha)}{\chi_-(\eta_3,\alpha)}N_-(\alpha)$$
$$=\frac{2k\sin\phi_o e^{-ikb\sin\phi_o}}{(\eta_3\sin\phi_o+1)(\alpha-k\cos\phi_o)}$$
$$\times\left[\frac{\chi_-(\eta_2,\alpha)}{\chi_-(\eta_3,\alpha)}N_-(\alpha)-\frac{\chi_-(\eta_2,k\cos\phi_o)N_-(k\cos\phi_o)}{\chi_-(\eta_3,k\cos\phi_o)}\right]$$
$$+\frac{2k\sin\phi_o e^{-ikb\sin\phi_o}}{(\eta_3\sin\phi_o+1)(\alpha-k\cos\phi_o)}\frac{\chi_-(\eta_2,k\cos\phi_o)N_-(k\cos\phi_o)}{\chi_-(\eta_3,k\cos\phi_o)} \quad (25a)$$

and

$$\frac{\chi_-(\eta_2,\alpha)}{\chi_-(\eta_3,\alpha)}N_-(\alpha)\sum_{m=1}^{\infty}(f_m-i\alpha g_m)\frac{K_m}{(\alpha^2-\alpha_m^2)}\left(\frac{\cos K_m b}{K_m}+\frac{\eta_2}{ik}\sin K_m b\right)$$
$$=\sum_{m=1}^{\infty}\frac{K_m}{(\alpha+\alpha_m)}\left(\frac{\cos K_m b}{K_m}+\frac{\eta_2}{ik}\sin K_m b\right)$$
$$\times\left[\frac{(f_m-i\alpha g_m)}{(\alpha-\alpha_m)}\frac{\chi_-(\eta_2,\alpha)}{\chi_-(\eta_3,\alpha)}N_-(\alpha)+\frac{(f_m+i\alpha_m g_m)}{2\alpha_m}\frac{\chi_+(\eta_2,\alpha_m)N_+(\alpha_m)}{\chi_+(\eta_3,\alpha_m)}\right]$$
$$-\sum_{m=1}^{\infty}\frac{(f_m+i\alpha_m g_m)}{2\alpha_m(\alpha+\alpha_m)}K_m\left(\frac{\cos K_m b}{K_m}+\frac{\eta_2}{ik}\sin K_m b\right)\frac{\chi_+(\eta_2,\alpha_m)N_+(\alpha_m)}{\chi_+(\eta_3,\alpha_m)}. \quad (25b)$$

Collecting the terms which are regular in the upper half-plane at the left-hand side and those regular in the lower half-plane at the right-hand side, the equation becomes

$$\frac{\chi_+(\eta_3,\alpha)}{\chi_+(\eta_2,\alpha)}\frac{P_+(\alpha)}{N_+(\alpha)}-\frac{2k\sin\phi_o e^{-ikb\sin\phi_o}}{(\eta_3\sin\phi_o+1)(\alpha-k\cos\phi_o)}\frac{\chi_-(\eta_2,k\cos\phi_o)N_-(k\cos\phi_o)}{\chi_-(\eta_3,k\cos\phi_o)}$$
$$+\sum_{m=1}^{\infty}\frac{(f_m+i\alpha_m g_m)}{2\alpha_m(\alpha+\alpha_m)}K_m\left(\frac{\cos K_m b}{K_m}+\frac{\eta_2}{ik}\sin K_m b\right)\frac{\chi_+(\eta_2,\alpha_m)N_+(\alpha_m)}{\chi_+(\eta_3,\alpha_m)}$$
$$=-\frac{\chi_-(\eta_2,\alpha)}{\chi_-(\eta_3,\alpha)}N_-(\alpha)\dot{F}_-(\alpha,b)+\frac{2k\sin\phi_o e^{-ikb\sin\phi_o}}{(\eta_3\sin\phi_o+1)(\alpha-k\cos\phi_o)}$$
$$\times\left[\frac{\chi_-(\eta_2,\alpha)}{\chi_-(\eta_3,\alpha)}N_-(\alpha)-\frac{\chi_-(\eta_2,k\cos\phi_o)N_-(k\cos\phi_o)}{\chi_-(\eta_3,k\cos\phi_o)}\right]$$
$$+\sum_{m=1}^{\infty}\frac{K_m}{(\alpha+\alpha_m)}\left(\frac{\cos K_m b}{K_m}+\frac{\eta_2}{ik}\sin K_m b\right)$$

$$\times\left[\frac{(f_m-i\alpha g_m)}{(\alpha-\alpha_m)}\frac{\chi_-(\eta_2,\alpha)}{\chi_-(\eta_3,\alpha)}N_-(\alpha)+\frac{(f_m+i\alpha_m g_m)}{2\alpha_m}\frac{\chi_+(\eta_2,\alpha_m)N_+(\alpha_m)}{\chi_+(\eta_3,\alpha_m)}\right]. \tag{26}$$

Taking into account the analytical continuation principle followed by Liouville's theorem, the solution of the Wiener-Hopf equation yields

$$\begin{aligned}\frac{\chi_+(\eta_3,\alpha)}{\chi_+(\eta_2,\alpha)}\frac{P_+(\alpha)}{N_+(\alpha)}&=\frac{2k\sin\phi_o e^{-ikb\sin\phi_o}}{(\eta_3\sin\phi_o+1)(\alpha-k\cos\phi_o)}\\&\quad\times\frac{\chi_-(\eta_2,k\cos\phi_o)N_-(k\cos\phi_o)}{\chi_-(\eta_3,k\cos\phi_o)}-\sum_{m=1}^{\infty}\frac{(f_m+i\alpha_m g_m)}{2\alpha_m(\alpha+\alpha_m)}\\&\quad\times K_m\left(\frac{\cos K_m b}{K_m}+\frac{\eta_2}{ik}\sin K_m b\right)\frac{\chi_+(\eta_2,\alpha_m)N_+(\alpha_m)}{\chi_+(\eta_3,\alpha_m)}\end{aligned} \tag{27}$$

2.2. Determination of the Unknown Coefficients

The important difference of this kind of formulation from the one applied in [1], was the simultaneous use of Mode-Matching technique with the Fourier transform. The Mode-Matching technique allows us to express the field component defined in the waveguide region in terms of normal modes as

$$u_2^{(1)}(x,y)=\sum_{n=1}^{\infty}a_n e^{-i\beta_n x}\left(\frac{\sin\xi_n y}{\xi_n}-\frac{\eta_1}{ik}\cos\xi_n y\right). \tag{28}$$

Here β_n's and ξ_n's are solved through

$$\frac{(\eta_1+\eta_4)}{ik}\cos\xi_n b-\left(1+\frac{\eta_1\eta_4}{k^2}\xi_n^2\right)\frac{\sin\xi_n b}{\xi_n}=0,\quad n=1,2,... \tag{29a}$$

and

$$\beta_n=\sqrt{k^2-\xi_n^2},\quad \Im m(\beta_n)>\Im m(k),\quad n=1,2,.... \tag{29b}$$

If the continuity relations (4g) and (4h) are used simultaneously

$$f(y)-i\alpha g(y)=\frac{\partial}{\partial x}u_2^{(1)}(0,y)-i\alpha u_2^{(1)}(0,y) \tag{30}$$

relation is determined which is

$$\begin{aligned}\sum_{m=1}^{\infty}(f_m-i\alpha g_m)\left(\frac{\sin K_m y}{K_m}-\frac{\eta_2}{ik}\cos K_m y\right)=\\-i\sum_{n=1}^{\infty}a_n(\alpha+\beta_n)\left(\frac{\sin\xi_n y}{\xi_n}-\frac{\eta_1}{ik}\cos\xi_n y\right).\end{aligned} \tag{31}$$

Multiplying both sides by

$$\frac{\sin K_j y}{K_j} - \frac{\eta_2}{ik}\cos K_j y$$

integrating over $0 < y < b$ and taking $j \to m$ results with

$$(f_m - i\alpha g_m) = -\frac{i}{Q_m^2}\sum_{n=1}^{\infty} a_n (\alpha + \beta_n) \triangle_{nm}, \tag{32}$$

where $\triangle_{nm}$ is

$$\begin{aligned}\left(\xi_n^2 - K_m^2\right)\triangle_{nm} \;=\; & \frac{(\eta_1 - \eta_2)}{ik} + \frac{(\eta_3 + \eta_4)}{ik} K_m \xi_n \left(\frac{\cos \xi_n b}{\xi_n} + \frac{\eta_1}{ik}\sin \xi_n b\right) \\ & \times \left(\frac{\cos K_m b}{K_m} + \frac{\eta_2}{ik}\sin K_m b\right).\end{aligned} \tag{33}$$

This result can be substituted in the equations (15a) and (27) giving

$$\sum_{n=1}^{\infty} A_n(\alpha_j)\, a_n = B(\alpha_j), \tag{34}$$

with

$$\begin{aligned}A_n(\alpha_j) \;=\; & -i\left[\frac{\eta_3}{ik} K_j \sin K_j b - \cos K_j b\right](\alpha_j + \beta_n)\triangle_{nj} \\ & -iN_+(\alpha_j)\frac{\chi_+(\eta_2, \alpha_j)}{\chi_+(\eta_3, \alpha_j)}\sum_{m=1}^{\infty}\frac{K_m(\beta_n - \alpha_m)\triangle_{nm}}{2\alpha_m Q_m^2(\alpha_j + \alpha_m)} \\ & \times\left(\frac{\cos K_m b}{K_m} + \frac{\eta_2}{ik}\sin K_m b\right)\frac{\chi_+(\eta_2, \alpha_m) N_+(\alpha_m)}{\chi_+(\eta_3, \alpha_m)},\end{aligned} \tag{35}$$

and

$$\begin{aligned}B(\alpha_j) \;=\; & \frac{2k\sin\phi_o e^{-ikb\sin\phi_o}}{(\eta_3 \sin\phi_o + 1)}\frac{\chi_-(\eta_2, k\cos\phi_o) N_-(k\cos\phi_o)}{\chi_-(\eta_3, k\cos\phi_o)} \\ & \times\frac{N_+(\alpha_j)}{(\alpha_j - k\cos\phi_o)}\frac{\chi_+(\eta_2, \alpha_j)}{\chi_+(\eta_3, \alpha_j)}.\end{aligned} \tag{36}$$

The infinite system of algebraic equations in (34) will be solved numerically. To this end we truncate the infinite series and the infinite system of algebraic equations after the first N terms. Figure 2 shows the variation of the modulus of the diffracted field against the truncation number N. It is seen that the amplitude of the diffracted field becomes insensitive to the increase of the truncation number after $N = 30$.

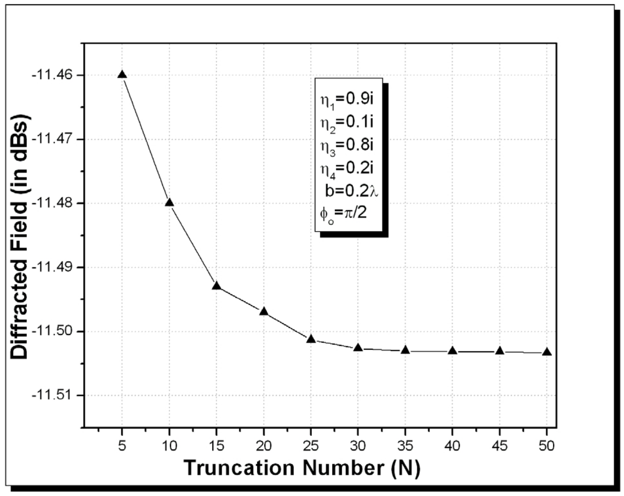

Figure 2. Modulus of the diffracted field versus the truncation number N.

3. THE DIFFRACTED FIELD AND COMPUTATIONAL RESULTS

The diffracted field $u_1(x, y)$ can be determined through

$$u_1(x,y) = \frac{1}{2\pi}\int_L \frac{P_+(\alpha)}{[1+\eta_3 K(\alpha)/k]} e^{iK(\alpha)(y-b)} e^{-i\alpha x} d\alpha. \tag{37}$$

Applying the change of variables $\alpha = -k\cos t$, $x = \rho\cos\phi$ and $y = \rho\sin\phi$, the above integral becomes

$$u_1(x,y) = \frac{1}{2\pi}\int_{-} \frac{P_+(-k\cos t)}{(\eta_3\sin t+1)} e^{-ikb\sin t} e^{ik\rho\cos(t-\phi)} k\sin t dt. \tag{38}$$

This integral can be solved asymptotically via the saddle point technique. Here the saddle point occurs at $t = \phi$ whose contribution is

$$u_1(\rho,\phi) = \frac{e^{ik\rho}}{\sqrt{k\rho}} \frac{e^{-i\pi/4}}{\sqrt{2\pi}} \frac{k\sin\phi}{(\eta_3\sin\phi+1)} e^{-ikb\sin\phi} P_+(-k\cos\phi). \tag{39}$$

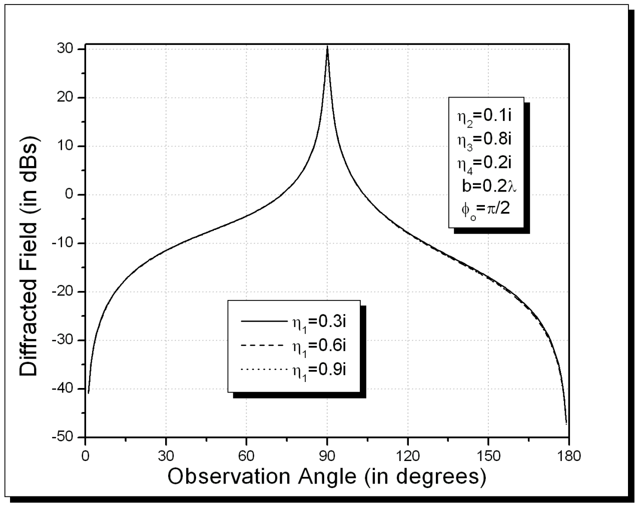

Figure 3. The variation of the diffracted field with respect to η_1.

Taking into account equation (27), the diffracted field can be cast into the following form:

$$u_1(\rho,\phi) = -\frac{e^{ik\rho}}{\sqrt{k\rho}}\frac{e^{-i\pi/4}}{\sqrt{2\pi}}\frac{k\sin\phi}{(\eta_3\sin\phi+1)}e^{-ikb\sin\phi}\frac{\chi_-(\eta_2,k\cos\phi)}{\chi_-(\eta_3,k\cos\phi)}N_-(k\cos\phi)$$
$$\times\left\{\frac{2\sin\phi_o e^{-ikb\sin\phi_o}}{(\eta_3\sin\phi_o+1)(\cos\phi+\cos\phi_o)}\frac{\chi_-(\eta_2,k\cos\phi_o)}{\chi_-(\eta_3,k\cos\phi_o)}N_-(k\cos\phi_o)\right.$$
$$\left.+\sum_{m=1}^{\infty}\frac{(f_m+i\alpha_m g_m)}{2\alpha_m(\alpha_m-k\cos\phi)}K_m\left(\frac{\cos K_m b}{K_m}+\frac{\eta_2}{ik}\sin K_m b\right)\frac{\chi_+(\eta_2,\alpha_m)N_+(\alpha_m)}{\chi_+(\eta_3,\alpha_m)}\right\}. \tag{40}$$

Now, some graphical results showing the effects of various geometrical and physical parameters on the diffraction phenomenon are presented.

From the graphical results displayed so far one can conclude that the diffracted field is insensitive to the the variations of η_1 and η_4, as expected. The diffracted field amplitude is affected notably when η_3 is capacitive and a decrease in the amplitude of the diffracted field is observed when $|\eta_3|$ increases.

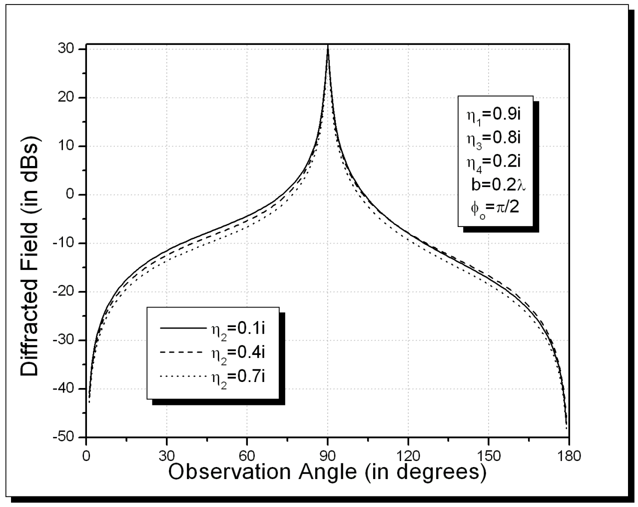

Figure 4a. The variation of the diffracted field with respect to η_2.

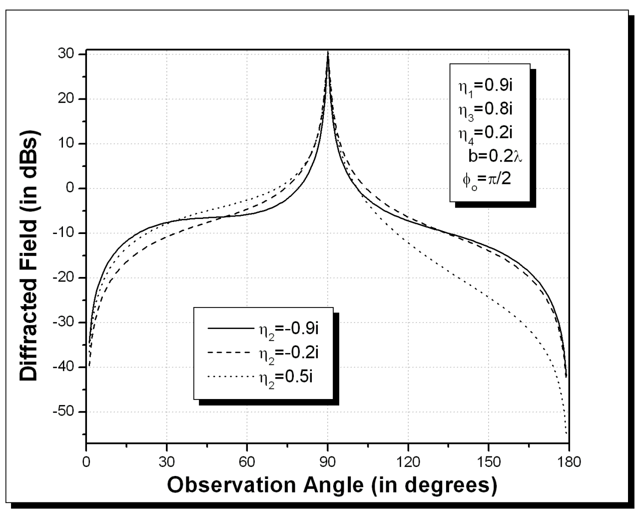

Figure 4b. The variation of the diffracted field with respect to η_2.

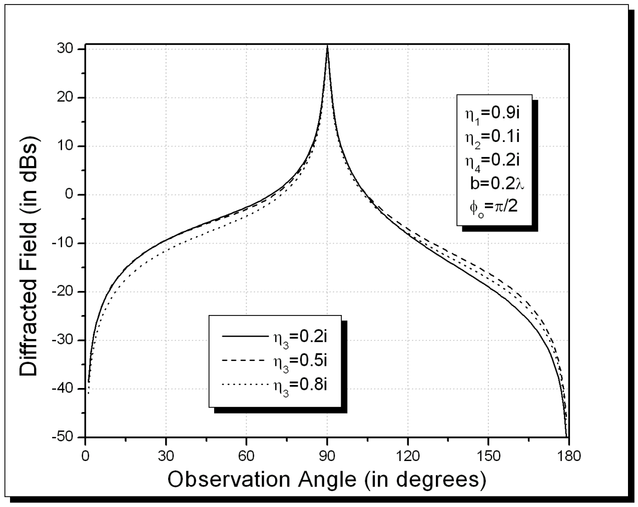

Figure 5a. The variation of the diffracted field with respect to η_3.

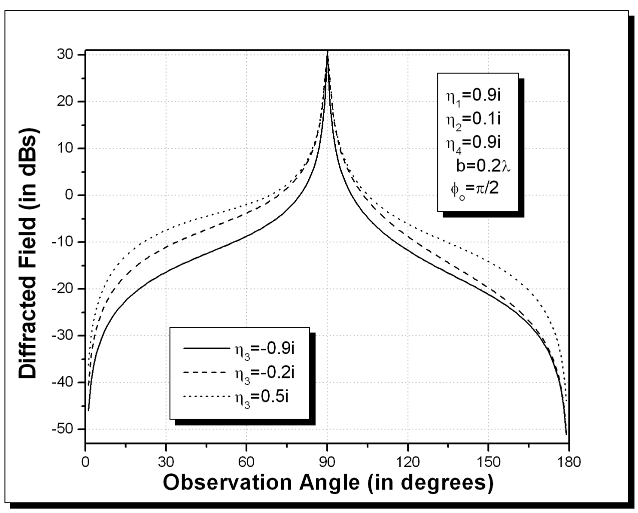

Figure 5b. The variation of the diffracted field with respect to η_3.

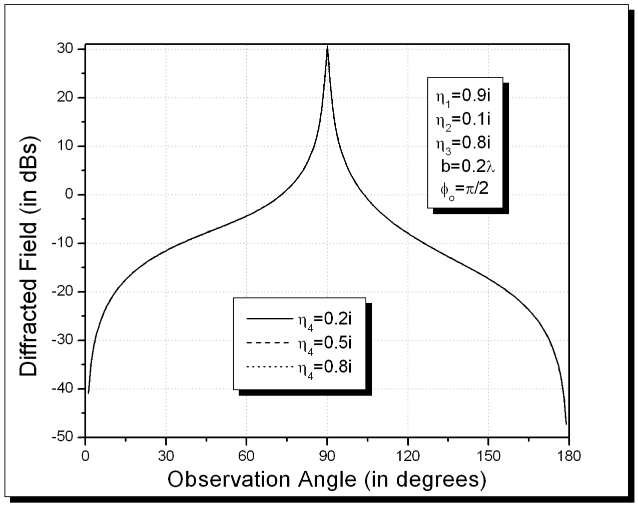

Figure 6. The variation of the diffracted field with respect to η_4.

4. CONCLUDING REMARKS

In this work, the diffraction of E-polarized plane waves by an impedance loaded parallel plate waveguide is investigated rigorously by using a mixed method of formulation consisting of employing the mode matching method in conjunction with the Fourier transform technique.

For the special case when b tends to zero, the diffracted field given in (40) becomes

$$u_1 \sim e^{i\pi/4}\sqrt{\frac{2}{\pi}}\frac{(\eta_3-\eta_2)}{(1+\eta_3\sin\phi_0)\,(1+\eta_3\sin\phi)}\frac{\sin\phi_0\sin\phi}{(\cos\phi_0+\cos\phi)}$$
$$\frac{\chi^-(\eta_2,\cos\phi_0)\,\chi^-(\eta_2,\cos\phi)}{\chi^-(\eta_3,\cos\phi_0)\,\chi^-(\eta_3,\cos\phi)}\frac{e^{ik\rho}}{\sqrt{k\rho}},$$

which is the well-known solution for the two-part impedance plane diffraction problem whose parts $x < 0$ and $x > 0$ are characterized by the relative surface impedances η_3 and η_2, respectively.

Figure 7 is a comparison between matrix Wiener-Hopf solution derived in [1] and the hybrid method used in this work, when the half plane is perfectly conducting ($Z_3 = Z_4 = 0$). It is shown that the two curves coincide exactly.

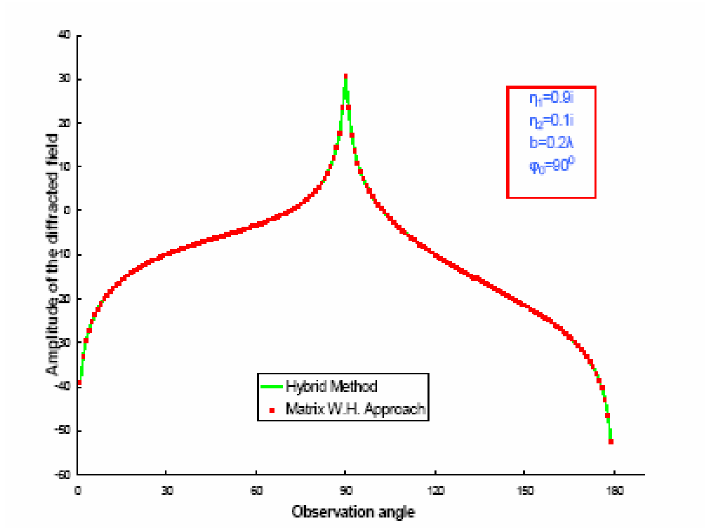

Figure 7. The comparison of hybrid and direct Fourier transform formulations.

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