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THE RELATIVISTIC PROPER-VELOCITY TRANSFORMATION GROUP

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Abstract—The Lorentz transformation group of the special theory of relativity is commonly represented in terms of observer's, or coordinate, time and coordinate relative velocities. The aim of this article is to uncover the representation of the Lorentz transformation group in terms of traveler's, or proper, time and proper relative velocities. Following a recent demonstration by M. Idemen, according to which the Lorentz transformation group is inherent in Maxwell equations, our proper velocity Lorentz transformation group may pave the way to uncover the proper time Maxwell equations.

1. INTRODUCTION

Coordinate time, or observer's time, is the time t of a moving object measured by an observer at rest. Accordingly, special relativity theory is formulated in terms of Coordinate time. Contrasting coordinate time, Proper time, or traveler's time, is the time τ of a moving object measured by a co-moving observer. Proper time is useful, for instance, in the understanding of the twin paradox [1] and the mean life time of unstable moving particles. The need to reformulate relativity physics in terms of proper time instead of coordinate time arises, for instance, in quantum mechanics [2].

The proper mean lifetime of muons between creation, in the upper atmosphere, and disintegration is $2.2\,\mu$s measured by their proper time. This proper time of the moving muon, measured by the muon own clock, is several orders of magnitude shorter than the time the muon is seen traveling through the atmosphere by Earth observers. Of course, there is no need to attach a co-moving observer to the moving muon in order to measure its proper time. Observers at rest measure the coordinate mean lifetime of the moving muon that, owing to time dilation, is observer dependent. Each observer, however, can translate his measure of the muon coordinate mean lifetime into the muon proper mean lifetime, which is an intrinsic property of the muon and hence observer independent [3].

The aim of this article is, accordingly, to uncover the translation of the Lorentz group of Einstein's special theory of relativity from its standard representation by means of coordinate time and coordinate velocities to a new representation by means of proper time and proper velocities.

M. Idemen has recently shown in [4] that the Lorentz transformation group of Einstein's special theory of relativity is inherent in Maxwell equations. One may therefore hope that the translation of the Lorentz group from its standard representation by means of coordinate time and coordinate velocities to a new representation by means of proper time and proper velocities will pave the way for the translation of Maxwell equations from their standard coordinate time representation to their proper time representation.

2. RELATIVISTIC COORDINATE VELOCITIES AND PROPER VELOCITIES AND THEIR COMPOSITION LAWS

Relativistic coordinate velocities are measured by coordinate time and, similarly, relativistic proper velocities are measured by proper time.

Let c be the vacuum speed of light, and let

$$\mathbb{R}^3_c = \left\{ \mathbf{v} \in \mathbb{R}^3 : \|\mathbf{v}\| < c \right\} \tag{1}$$

be the c-ball of all relativistically admissible velocities of material particles. It is the open ball of radius c, centered at the origin of the Euclidean three-space $\mathbb{R}^3$, consisting of all vectors $\mathbf{v}$ in $\mathbb{R}^3$ with magnitude $\|\mathbf{v}\|$ smaller than c. Einstein addition $\oplus_E$ in the c-ball of $\mathbb{R}^3_c$ is given by the Equation [5–8]

$$\mathbf{u} \oplus_E \mathbf{v} = \frac{1}{1 + \dfrac{\mathbf{u}\cdot\mathbf{v}}{c^2}} \left\{ \mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{1}{c^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u} \right\} \tag{2}$$

for all $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^3_c$, where $\mathbf{u}\cdot\mathbf{v}$ is the inner product that the ball $\mathbb{R}^3_c$ inherits from its space $\mathbb{R}^3$, and where $\gamma_{\mathbf{u}}$ is the gamma factor

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1 - \dfrac{\|\mathbf{u}\|^2}{c^2}}} \tag{3}$$

in the c-ball, $\|\mathbf{u}\|^2 = \mathbf{u}\cdot\mathbf{u}$.

Einstein velocity addition (2) satisfies the *gamma identity* [7, p. 79][8],

$$\gamma_{\mathbf{u}\oplus_E\mathbf{v}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\left(1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right) \tag{4}$$

implying

$$\frac{\gamma_{\mathbf{u}\oplus_E\mathbf{v}}}{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}} = 1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2} \tag{5}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3_c$.

In Einstein's special relativity theory we distinguish between *coordinate velocities* and *proper velocities.* Einstein's velocity addition law (2) gives the addition law of *coordinate velocities*, while the addition law of proper velocities will be uncovered below.

Coordinate time, or observer's time, is the time t of a moving object measured by an observer at rest. Proper time, or traveler's time, is the time τ of a moving object measured by a co-moving observer [9, Sec. 6.2] [10, p. 72].

The coordinate time, t, and the proper time, τ, of a uniformly moving object with relative velocity $\mathbf{v} \in \mathbb{R}^3_c$ measured by coordinate time, are related by the equation [9, Eq. (6.1)]

$$t = \gamma_{\mathbf{v}}\tau \tag{6}$$

where, by choice, $\tau = 0$ when $t = 0$.

Accordingly, the relative coordinate velocity $\mathbf{v} \in \mathbb{R}^3_c$ and the proper velocity $\mathbf{w} \in \mathbb{R}^3$ of an object measured by its coordinate time t and proper time τ, respectively, are related by the following equations, which are known since 1915 [11],

$$\begin{aligned}\mathbf{w} &= \gamma_{\mathbf{v}}\mathbf{v} \in \mathbb{R}^3_c\\ \mathbf{v} &= \beta_{\mathbf{w}}\mathbf{w} \in \mathbb{R}^3_c\end{aligned} \tag{7}$$

where $\gamma_{\mathbf{v}}$ is the gamma factor (3) and where $\beta_{\mathbf{w}}$ is the *beta factor* given by the equation

$$\beta_{\mathbf{w}} = \frac{1}{\sqrt{1+\dfrac{\|\mathbf{w}\|^2}{c^2}}} \tag{8}$$

Let ϕ: $\mathbb{R}^3 \to \mathbb{R}^3_3$ be the bijective map that the second equation in (7) suggests,

$$\phi\mathbf{w} = \beta_{\mathbf{w}}\mathbf{w} = \mathbf{v} \tag{9}$$

taking the space $\mathbb{R}^3$ of proper velocities to the ball $\mathbb{R}^3_c$ of coordinate velocities.

The inverse map ϕ^{-1}: $\mathbb{R}^3_c \to \mathbb{R}^3$, from coordinate velocities to proper velocities is given by

$$\phi^{-1}\mathbf{v} = \gamma_{\mathbf{v}}\mathbf{v} = \mathbf{w} \tag{10}$$

as shown in (7).

It follows from (9)–(10) that Einstein addition $\oplus_E$ of coordinate velocities in $\mathbb{R}^3_c$ induces the binary operation $\oplus_U$ between proper velocities in $\mathbb{R}^3$ according to the equation

$$\mathbf{w}_1 \oplus_U \mathbf{w}_2 = \phi^{-1}(\phi\mathbf{w}_1 \oplus_E \phi\mathbf{w}_2) \tag{11}$$

$\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{R}^3_c$, thus uncovering the proper velocity composition law in $\mathbb{R}^3$.

By employing a software for symbolic manipulation, like Mathematica or Maple, it can be shown straightforwardly that the binary operation $\oplus_U$, (11), in $\mathbb{R}^3$ is given explicitly by the equation [6, 7]

$$\begin{aligned} \mathbf{u} \oplus_U \mathbf{v} &= \phi^{-1}(\phi\mathbf{u} \oplus_E \phi\mathbf{v}) \\ &= \mathbf{u} + \mathbf{v} + \left\{ \frac{\beta_{\mathbf{u}}}{1+\beta_{\mathbf{u}}} \frac{\mathbf{u}\cdot\mathbf{v}}{c^2} + \frac{1-\beta_{\mathbf{v}}}{\beta_{\mathbf{v}}} \right\} \mathbf{u} \end{aligned} \tag{12}$$

where $\beta_{\mathbf{v}}$ is the beta factor, satisfying the beta identity

$$\beta_{\mathbf{u}\oplus_U\mathbf{v}} = \frac{\beta_{\mathbf{u}}\beta_{\mathbf{v}}}{1 + \beta_{\mathbf{u}}\beta_{\mathbf{v}}\dfrac{\mathbf{u}\cdot\mathbf{v}}{c^2}} \tag{13}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$.

It follows from (11) that

$$\phi(\mathbf{w}_1 \oplus_U \mathbf{w}_2) = \phi\mathbf{w}_1 \oplus_E \phi\mathbf{w}_2 \tag{14}$$

so that by the definition of ϕ in (9) we have

$$\beta_{\mathbf{w}_1\oplus_U\mathbf{w}_2}(\mathbf{w}_1 \oplus_U \mathbf{w}_2) = \beta_{\mathbf{w}_1}\mathbf{w}_1 \oplus_E \beta_{\mathbf{w}_2}\mathbf{w}_2 \tag{15}$$

for all $\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{R}^3$.

Substituting $\mathbf{w}_k = \phi^{-1}\mathbf{v}_k$, $k = 1, 2$, in (11), we obtain

$$\phi^{-1}(\mathbf{v}_1 \oplus_E \mathbf{v}_2) = \phi^{-1}\mathbf{v}_1 \oplus_U \phi^{-1}\mathbf{v}_2 \tag{16}$$

implying, by (10),

$$\gamma_{\mathbf{v}_1 \oplus_E \mathbf{v}_2}(\mathbf{v}_1 \oplus_E \mathbf{v}_2) = \gamma_{\mathbf{v}_1}\mathbf{v}_1 \oplus_U \gamma_{\mathbf{v}_2}\mathbf{v}_2 \tag{17}$$

for all $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^3_c$.

The binary operation $\oplus_U$ between relativistic proper velocities is called "Ungar addition" in [6] and "proper velocity addition" in [7]. The term "Ungar addition" was coined by Jing-Ling Chen in [12]. Each of the two identities (15) and (17) thus gives the relationship between Einstein addition $\oplus_E$ of relativistic coordinate velocities in $\mathbb{R}^3_c$ and Ungar addition $\oplus_U$ of relativistic proper velocities in $\mathbb{R}^3$. These two binary operations are isomorphic to each other in the sense of (7) (see [7, Table 6.1, p. 202]), and they give rise to two different, but equivalent, models of hyperbolic geometry [7, 13, 14].

3. THE STANDARD, COORDINATE TIME, COORDINATE VELOCITY LORENTZ TRANSFORMATION

Let $(t, \mathbf{x})^t$, where exponent t denotes transposition, be a spacetime event represented by its coordinates relative to an inertial frame. A Lorentz transformation of spacetime coordinates is a coordinate transformation that leaves the norm

$$\left\| \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \right\| = \sqrt{t^2 - \frac{\|\mathbf{x}\|^2}{c^2}} \tag{18}$$

of a spacetime event $(t, \mathbf{x})^t$ invariant. A (homogeneous, proper, orthochronous) Lorentz transformation $L(\mathbf{u})$ without rotation that links the spacetime coordinates between two inertial frames with relative velocity $\mathbf{u}$ that coincided at time $t = 0$ is given by the equation, [15, Eq. (11.19), p. 517],

$$L(\mathbf{u}) \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} \gamma_{\mathbf{u}} \left(t + \dfrac{1}{c^2} \mathbf{u} \cdot \mathbf{x} \right) \\ \gamma_{\mathbf{u}} \mathbf{u} t + \mathbf{x} + \dfrac{1}{c^2} \dfrac{\gamma^2_{\mathbf{u}}}{1 + \gamma_{\mathbf{u}}} (\mathbf{u} \cdot \mathbf{x}) \mathbf{u} \end{pmatrix} \tag{19}$$

The Lorentz transformation (19) is (i) *homogeneous* (it takes the spacetime origin of a frame of reference into itself); (ii) proper (it is continuously connected to the identity transformation); and (iii) orthochronous (it keep invariant the sign of the time part of a spacetime event).

Let $\mathbf{v} = \mathbf{x}/t \in \mathbb{R}^3_c$, so that $\mathbf{x} = \mathbf{v}t$. Then the Lorentz transformation (19) takes the form

$$\begin{aligned} L(\mathbf{u})\begin{pmatrix} t \\ \mathbf{v}t \end{pmatrix} &= \begin{pmatrix} \gamma_{\mathbf{u}}\left(t + \frac{1}{c^2}\mathbf{u}\cdot\mathbf{v}\right)t \\ \left\{\gamma_{\mathbf{u}}\mathbf{u} + \mathbf{v} + \frac{1}{c^2}\frac{\gamma_{\mathbf{u}}^2}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\right\} t \end{pmatrix} \\ &= \begin{pmatrix} \frac{\gamma_{\mathbf{u}\oplus_E\mathbf{v}}}{\gamma_{\mathbf{v}}}t = t' \\ (\mathbf{u}\oplus_E\mathbf{v})t' \end{pmatrix} \\ &= \begin{pmatrix} t' \\ (\mathbf{u}\oplus_E\mathbf{v})t' \end{pmatrix} \end{aligned} \tag{20}$$

that we obtain by a substitution from (5) and (2).

Renaming velocities by the equations $\mathbf{u} = \mathbf{v}_1$ and $\mathbf{v} = \mathbf{v}_2$, the Lorentz transformation (20) takes the form

$$L(\mathbf{v}_1)\begin{pmatrix} t \\ \mathbf{v}_2 t \end{pmatrix} = \begin{pmatrix} t' \\ (\mathbf{v}_1\oplus_E\mathbf{v}_2)t' \end{pmatrix} \tag{21}$$

Clearly, the Lorentz transformation $L(\mathbf{v}_1)$ in (21) is equivalent to the system of two basic transformations

$$\begin{aligned} \mathbf{v}_2 &\to \mathbf{v}_1\oplus_E\mathbf{v}_2 \\ t &\to t' \end{aligned} \tag{22}$$

where

$$t' = \frac{\gamma_{\mathbf{v}_1\oplus_E\mathbf{v}_2}}{\gamma_{\mathbf{v}_2}}t \tag{23}$$

4. THE PROPER TIME, PROPER VELOCITY LORENTZ TRANSFORMATION

Let us now rearrange the two basic transformations (22) in the Lorentz transformation (21) in a different way,

$$L_p(\gamma_{\mathbf{v}_1}\mathbf{v}_1)\begin{pmatrix} \frac{t}{\gamma_{\mathbf{v}_2}} \\ \gamma_{\mathbf{v}_2}\mathbf{v}_2\frac{t}{\gamma_{\mathbf{v}_2}} \end{pmatrix} = \begin{pmatrix} \frac{t'}{\gamma_{\mathbf{v}_1\oplus_E\mathbf{v}_2}} \\ \gamma_{\mathbf{v}_1\oplus_E\mathbf{v}_2}(\mathbf{v}_1\oplus_E\mathbf{v}_2)\frac{t'}{\gamma_{\mathbf{v}_1\oplus_E\mathbf{v}_2}} \end{pmatrix} \tag{24}$$

Clearly, the transformations

$$\begin{pmatrix} t \\ \mathbf{v}_2 t \end{pmatrix} \to \begin{pmatrix} t' \\ (\mathbf{v}_1\oplus_E\mathbf{v}_2)t' \end{pmatrix} \tag{25}$$

and

$$\begin{pmatrix} \dfrac{t}{\gamma_{\mathbf{v}_2}} \\ \gamma_{\mathbf{v}_2}\mathbf{v}_2\dfrac{t}{\gamma_{\mathbf{v}_2}} \end{pmatrix} \to \begin{pmatrix} \dfrac{t'}{\gamma_{\mathbf{v}_1\oplus_E\mathbf{v}_2}} \\ \gamma_{\mathbf{v}_1\oplus_E\mathbf{v}_2}(\mathbf{v}_1 \oplus_E \mathbf{v}_2)\dfrac{t'}{\gamma_{\mathbf{v}_1\oplus_E\mathbf{v}_2}} \end{pmatrix} \qquad (26)$$

in (21) and (24) are equivalent, being just different arrangements of the same two basic transformations in (22).

Following (6) we introduce into (24) the notation

$$\begin{aligned} \tau &= \frac{t}{\gamma_{\mathbf{v}_2}} \\ \tau' &= \frac{t'}{\gamma_{\mathbf{v}_1\oplus_E\mathbf{v}_2}} \end{aligned} \qquad (27)$$

noting that, by (23),

$$\tau' = \tau \qquad (28)$$

Similarly, following (7) we introduce into (24) the notation

$$\begin{aligned} \mathbf{w}_1 &= \gamma_{\mathbf{v}_1}\mathbf{v}_1 \\ \mathbf{w}_2 &= \gamma_{\mathbf{v}_2}\mathbf{v}_2 \end{aligned} \qquad (29)$$

With this notation, noting (17) and (28), (24) takes the form

$$L_p(\mathbf{w}_1)\begin{pmatrix} \tau \\ \mathbf{w}_2\tau \end{pmatrix} = \begin{pmatrix} \tau \\ (\mathbf{w}_1 \oplus_U \mathbf{w}_2)\tau \end{pmatrix} \qquad (30)$$

thus uncovering the proper time, proper velocity Lorentz transformation group (30).

The formal resemblance between the standard, coordinate velocity Lorentz transformation (21) and the proper velocity Lorentz transformation (30) is remarkable. We should emphasize that the standard, coordinate velocity Lorentz transformation $L(\mathbf{v}_1)$ in (21) and the proper velocity Lorentz transformation $L_p(\mathbf{w}_1)$ in (30) are equivalent. They are merely different arrangements of the same system of the two basic transformations in (22), as clearly seen in (25)–(26).

Rewriting (30) in terms of the notation $\mathbf{u} = \mathbf{w}_1$ and $\mathbf{v} = \mathbf{w}_2$, along with a substitution from (12), we have

$$
\begin{aligned}
&L_p(\mathbf{u})\begin{pmatrix}\tau\\ \mathbf{v}\tau\end{pmatrix}\\
&=\begin{pmatrix}\tau\\ (\mathbf{u}\oplus_U\mathbf{v})\tau\end{pmatrix}\\
&=\begin{pmatrix}\tau\\ \left(\mathbf{u}+\mathbf{v}+\left\{\dfrac{\beta_{\mathbf{u}}}{1+\beta_{\mathbf{u}}}\dfrac{\mathbf{u}\cdot\mathbf{v}}{c^2}+\dfrac{1-\beta_{\mathbf{v}}}{\beta_{\mathbf{v}}}\right\}\mathbf{u}\right)\tau\end{pmatrix}
\end{aligned}
\tag{31}
$$

Finally, if we use the notation $\mathbf{v}\tau=\mathbf{x}$ then the proper velocity Lorentz transformation (31) takes the form

$$
\begin{aligned}
&L_p(\mathbf{u})\begin{pmatrix}\tau\\ \mathbf{x}\end{pmatrix}\\
&=\begin{pmatrix}\tau\\ \mathbf{x}+\dfrac{1}{1+\sqrt{1+\mathbf{u}^2/c^2}}\dfrac{\mathbf{u}\cdot\mathbf{x}}{c^2}\mathbf{u}+\sqrt{\tau^2+\|\mathbf{x}\|^2/c^2}\mathbf{u}\end{pmatrix}
\end{aligned}
\tag{32}
$$

$\mathbf{u},\mathbf{x}\in\mathbb{R}^3,\ \tau>0$.

The proper time, proper velocity Lorentz transformation (32) is nonlinear in $\mathbf{x}$, but it possesses the important property that it keeps the proper time τ invariant. As such, it can shed new light on relativity physics. According to M. Idemen [4] the Lorentz transformation group is inherent in Maxwell equations. Hence, it is hoped that by employing techniques that Idemen developed in [4], our proper time, proper velocity Lorentz transformation group (32) can pave the way to uncover the proper time Maxwell equations. These, in turn, can reveal the electromagnetic field generated by a moving source in terms of the invariant proper time of the source.

5. CONCLUDING REMARKS

Einstein's special relativity theory is formulated in terms of observer's (coordinate) time and observer's (coordinate) velocities. In some circumstances that arise, for instance, in quantum mechanics, an equivalent reformulation of special relativity in terms of traveler's (proper) time and traveler's (proper) velocities is preferable. Accordingly, the proper velocity composition law is derived in (12) by employing the well-known bijective relationship (7) between the space $\mathbb{R}^3$ of proper velocities and the ball $\mathbb{R}^3_c$ of coordinate velocities.

The resulting proper velocity addition law (12), in turn, proves useful in the construction of the proper time, proper velocity Lorentz transformation (32) that we may call, for short, the *proper Lorentz transformation*. The proper Lorentz transformation (32) is obtained by rearranging the two ingredients of the standard Lorentz transformation (19), as shown in (25)–(26).

Hence, our way of constructing the proper Lorentz transformation clearly exhibits the equivalence between the standard Lorentz transformation, (19), and the proper Lorentz transformation, (32).

According to a recent result of M. Idemen in [4], the Lorentz transformation group is inherent in Maxwell equations. Hence we finally remark that the discovery of the proper Lorentz transformation may pave the way to uncover the proper time Maxwell equations.

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