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ELECTROMAGNETIC GAUSSIAN BEAMS AND RIEMANNIAN GEOMETRY

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Abstract—A *Gaussian beam* is an asymptotic solution to Maxwell's equations that propagate along a curve; at each time instant its energy is concentrated around one point on the curve. Such a solution is of the form

$$E = \mathrm{Re}\{e^{iP\theta(x,t)}E_0(x,t)\},$$

where E_0 is a complex vector field, $P > 0$ is a big constant, and θ is a complex second order polynomial in coordinates adapted to the curve.

In recent work by A. P. Kachalov, electromagnetic Gaussian beams have been studied in a geometric setting. Under suitable conditions on the media, a Gaussian beam is determined by Riemann-Finsler geometry depending only on the media. For example, geodesics are admissible curves for Gaussian beams and a curvature equation determines the second order terms in θ.

This work begins with a derivation of the geometric equations for Gaussian beams following the work of A. P. Kachalov. The novel feature of this work is that we characterize a class of inhomogeneous anisotropic media where the induced geometry is Riemannian. Namely, if ε, μ are simultaneously diagonalizable with eigenvalues ε_i, μ_j, the induced geometry is Riemannian if and only if $\varepsilon_i\mu_j = \varepsilon_j\mu_i$ for some $i \neq j$. What is more, if the latter condition is not met, the geometry is ill-behaved. It is neither smooth nor convex.

We also calculate Riemannian metrics for different media. In isotropic media, $g_{ij} = \varepsilon\mu\,\delta_{ij}$ and in more complicated media there are two Riemannian metrics due to different polarizations.

1. INTRODUCTION

When writing Maxwell's equations using differential forms on a 3-manifold these equations become completely differential-topological. On the other hand, the constitutive equations become completely metrical; these depend only on two Riemannian geometries. One geometry describes permittivity and another geometry describes permeability [1, 2]. This suggests that notions such as length, angle, area, and volume, that is, geometry, is related to material properties. For example, in vacuum, plane waves propagate along straight lines, but at plane boundaries, the wave bends at an angle depending on the material properties.

Unfortunately, the above two geometries have no direct geometrical or physical interpretations related to wave propagation (or, none has been found so far.) One sought feature could be that geodesics would describe the path traversed by a ray of light. However, since such a path depends on the polarization of the wave, and since the above metrics do not take polarization into account, this is not the case. In view of the above, it is motivated to study the relation between media and the geometry determining propagation of electromagnetic waves. This is the topic of the present work.

We assume that the media is anisotropic, non-homogeneous, smooth, time-, and frequency-independent, and we assume that the media matrices are simultaneously diagonalizable (see Section 4). In addition, we shall work exclusively with *Gaussian beam solutions* to Maxwell's equations. These are asymptotic solutions that propagate along a curve such that at every time instant the entire energy of the solution is completely concentrated around one point on the curve. In fact, the envelope of the solution is a Gaussian bell curve, hence the name. A main property of these solutions is that their propagation and form are completely determined by a curve on the underlying manifold and three tensors on that curve. Since these solutions depend on very little information, they are very convenient to work with; the curve and the three tensors are determined by a Hamilton equation and a Riccati equation, which are both ODE's that are easy to solve numerically. Gaussian beams are also closely related to the classical Debye expansion [3], and they have been used to study the anisotropic wave equation [4–6]. In recent work by A. P. Kachalov [7–9], electromagnetic Gaussian beams have been studied in a Riemann-Finsler setting; under suitable assumptions, Gaussian beams are determined by Riemann-Finsler geometry depending only on the media. The present work relies heavily on the last three articles [7–9]. Essentially, a Finsler geometry on a manifold is a norm for tangent vectors whose unit spheres do not need to be ellipsoids (see for example [10, 11]).

The main result of this work is to characterize a class of media where Gaussian beams propagate using Riemannian geometry. This result shows that when ε and μ are simultaneously diagonizable, it is not possible to model propagation of Gaussian beams using Finsler geometries that are not Riemannian. This is not a dead end for the geometrization of electromagnetism, or for Finsler geometry in electromagnetism. For example, there are non-Riemannian Finsler geometries related plane waves in crystals [12], and there is works studying the geometry of electromagnetism on the tangent bundle [13, 14]. There is also a vast literature on Finsler geometry in physics in general [11, 12, 15]. However, the present result gives a class of media where the theory of Gaussian beams is valid. This is of both theoretical and practical use. The present work also motivates the search for geometries on the tangent and cotangent bundle of the manifold. For example, in symplectic geometry, physical objects such as caustics might be non-smooth on the base manifold although everything is smooth on the cotangent bundle. Lastly, let us emphasize that Gaussian beams and equations governing these are independent of local coordinates.

The work is organized as follows. Section 2 reviews the Hamilton-Jacobi equation, and in Section 3 we define Gaussian beams and formulate geometric equations determining their propagation. Section 4 contains the aforementioned characterization result, and Section 5 gives examples of Riemannian geometries for different media.

2. DEBYE EXPANSION IN MAXWELL'S EQUATIONS

By a manifold M we mean a Hausdorff, second countable, topological manifold with smooth transition maps. Its tangent and cotangent bundles are denoted by TM and T^*M, respectively, and the set of p-forms on M by $\Omega^p M$. Vector fields on M are denoted by $\mathfrak{X}(M)$. Let us also assume that I is an open interval (and sometimes also the identity matrix). Starting from Section 3 we will also employ Einstein's summing convention.

2.1. Maxwell's equations in differential forms

Suppose M is a 3-dimensional oriented manifold. Then the dynamical Maxwell's equations read

$$dE = -\frac{\partial B}{\partial t},$$

$$dH = \frac{\partial D}{\partial t},$$

where $E, H \in (\Omega^1 M) \times I$ and $B, D \in (\Omega^2 M) \times I$ are the smooth electromagnetic field quantities depending on time $t \in I$. Also, we assume that the constitutive equations can be written as

$$\begin{aligned} D &= *_\varepsilon E, \\ B &= *_\mu H, \end{aligned}$$

where $*_\varepsilon$ and $*_\mu$ are the Hodge star operators induced by time-independent Riemannian geometries g_ε, g_μ, respectively [1, 2].

We always assume that E, D, B, H are real. However, we also need complex forms, which we denote by $\Omega^p(M, \mathbb{C})$. These are simply p-forms whose component functions are possibly complex valued. Similarly, we define $T^*(M, \mathbb{C})$. Since the transition functions are real valued, the real part $\Re\{\cdot\}$ of a complex form is well defined. On a Riemannian manifold, the (complex) *Hodge star operator* is the linear operator

$$*\colon \Omega^p(M, \mathbb{C}) \to \Omega^{n-p}(M, \mathbb{C})$$

that maps basis elements of $\Omega^p(M, \mathbb{C})$ as

$$*(dx^{i_1} \wedge \cdots \wedge dx^{i_p}) = \frac{\sqrt{\det g}}{(n-p)!} g^{i_1 l_1} \cdots g^{i_p l_p} \epsilon_{l_1 \cdots l_p \, l_{p+1} \cdots l_n} dx^{l_{p+1}} \wedge \cdots \wedge dx^{l_n},$$

where ϵ ($\neq \varepsilon$) is the Levi-Civita permutation symbol. Also, since g_{ij} is real, the $\Re$-operator commutes with the Hodge operator, that is, $* \circ \Re = \Re \circ *$. Also, on a 3-manifold, $* = *^{-1}$ for $p = 0, 1, 2, 3$.

2.2. Debye Expansion

We next perform a *Debye expansion* [3] in Maxwell's equations. That is, we assume that the electric and magnetic fields E, H are of the form

$$E(x, t, P) = \Re\left\{ e^{iP\theta(x,t)} \sum_{k=0}^{N} \frac{E_k(x,t)}{(iP)^k} \right\}, \tag{1}$$

$$H(x, t, P) = \Re\left\{ e^{iP\theta(x,t)} \sum_{k=0}^{N} \frac{H_k(x,t)}{(iP)^k} \right\}, \quad (x,t) \in M \times I, \tag{2}$$

where $N \geq 0$, $P > 0$ is a (large) positive constant, $E_k, H_k \in \Omega^1(M, \mathbb{C}) \times I$, and $\theta \in C^\infty(M \times I)$ is a complex valued *phase function*.

In what follows we will only be interested in solving the phase function. Intuitively, one can think of θ as a function that determines how the wave propagates. For example, if E is a plane wave, then $P\theta = x \cdot p - \omega t$, so in this case θ determines the direction of propagation. A discussion on the role of P can be found in [6], and equations for E_i, H_i are studied in [9]. Let us also point out that we do not have any sources or boundary conditions. We only study how Gaussian beams propagate assuming that they have been generated in some way.

Plugging the above trials into Maxwell's equations and equating terms of equal power (in the $\frac{1}{(iP)^k}$ variable) inside $\Re\{\cdot\}$ yields conditions

$$d\theta \wedge H_0 = \frac{\partial \theta}{\partial t} *_\varepsilon E_0, \tag{3}$$

$$d\theta \wedge E_0 = -\frac{\partial \theta}{\partial t} *_\mu H_0, \tag{4}$$

and for $k = 1, \ldots, N-1$, conditions

$$\begin{aligned} -d\theta \wedge E_{k+1} - \frac{\partial \theta}{\partial t} *_\mu H_{k+1} &= *_\mu \frac{\partial H_k}{\partial t} + dE_k, \\ d\theta \wedge H_{k+1} - \frac{\partial \theta}{\partial t} *_\varepsilon E_{k+1} &= *_\varepsilon \frac{\partial E_k}{\partial t} - dH_k. \end{aligned}$$

Our next aim is to derive the Hamilton-Jacobi equation (equation 12) for the phase function. This yields a sufficient condition on θ to solve equations (3)–(4) that does not involve E_0 and H_0. The idea is to show that θ is determined by the spectrum of a linear operator (L in equation (6)) involving only the medium parameters. This is the key result which makes it possible to study propagation of Gaussian beams without solving the whole field.

With matrix notation equations (3)–(4) read

$$\begin{pmatrix} *_\varepsilon & 0 \\ 0 & *_\mu \end{pmatrix} \begin{pmatrix} 0 & d\theta \wedge I \\ -d\theta \wedge I & 0 \end{pmatrix} \begin{pmatrix} E_0 \\ H_0 \end{pmatrix} = \frac{\partial \theta}{\partial t} \begin{pmatrix} E_0 \\ H_0 \end{pmatrix} \text{ on } M \times I, \tag{5}$$

where $d\theta \wedge I$ is the operator $\alpha \mapsto d\theta \wedge \alpha$ for $\alpha \in \Omega^p(M, \mathbb{C})$.

Let us define the following family of linear operators:

$$\begin{aligned} L \colon T^*(M, \mathbb{C}) &\to \left((\Omega^1_x(M, \mathbb{C}))^2 \to (\Omega^1_x(M, \mathbb{C}))^2 \right), \\ L(\eta) &= \begin{pmatrix} *_\varepsilon & 0 \\ 0 & *_\mu \end{pmatrix} \begin{pmatrix} 0 & \eta \wedge I \\ -\eta \wedge I & 0 \end{pmatrix}, \quad \eta \in T^*_x(M, \mathbb{C}). \end{aligned} \tag{6}$$

Here we have not introduced any local coordinates. Thus $L(\eta)$ and its spectrum are well defined for each $\eta \in T^*(M, \mathbb{C})$.

Next we formulate equation (5) in local coordinates. For this purpose, let x^i be local coordinates for M on a set $U \subset \mathbb{R}^3$, and let

$\mathcal{U} \subset M$ be the corresponding chart. Furthermore, let $\mathbf{e}, \mathbf{h}: U \times I \to \mathbb{C}^3$ be complex covector fields representing E_0, H_0 in these coordinates, and let us denote by θ also the local representative $\theta: U \to \mathbb{C}$ of $\theta: M \to \mathbb{C}$. From the argument of θ it will always be clear which θ is meant. Then equation (5) restricted to $\mathcal{U} \times I$ is equivalent to

$$L'(x, \nabla\theta) \begin{pmatrix} \mathbf{e} \\ \mathbf{h} \end{pmatrix} = \frac{\partial \theta}{\partial t} \begin{pmatrix} \mathbf{e} \\ \mathbf{h} \end{pmatrix} \text{ on } U \times I, \tag{7}$$

where $\nabla\theta$ is the usual gradient in $\mathbb{R}^3$, and

$$\begin{aligned} L'(x,z) &= \chi^{-1}(x)J(z), \quad (x,z) \in U \times \mathbb{C}^3, \\ J(z) &= \begin{pmatrix} 0 & z \times I \\ -z \times I & 0 \end{pmatrix}, \quad z \in \mathbb{C}^3, \\ \chi &= \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \text{ on } U. \end{aligned}$$

Here $z \times I$ is the 3×3 matrix representing the mapping $p \mapsto z \times p$ for $p \in \mathbb{C}^3$. Also, ε, μ are the 3×3 matrices (defined on U) whose (i,j):th elements are

$$\varepsilon^{ij} = \sqrt{\det g_\varepsilon}\; g_\varepsilon^{ij}, \quad \mu^{ij} = \sqrt{\det g_\mu}\; g_\mu^{ij}. \tag{8}$$

It follows that ε and μ are symmetric positive definite matrices. In fact, these are the "usual" matrices appearing in $\mathbb{R}^3$ [19].

Let us further rewrite equation (7) in a slightly different form. For this purpose, let us introduce $M: U \times \mathbb{C}^3 \to \mathbb{C}^{6\times 6}$, and $v: U \times I \to \mathbb{C}^3$, by

$$\begin{aligned} M(x,z) &= \chi^{-1/2}(x)J(z)\chi^{-1/2}(x), \\ v &= \chi^{1/2}(x) \begin{pmatrix} \mathbf{e} \\ \mathbf{h} \end{pmatrix}, \end{aligned}$$

whence equation (7) is equivalent to

$$M(x, \nabla\theta)v = \frac{\partial\theta}{\partial t} v \quad \text{on } U \times I. \tag{9}$$

If $\eta \in T^*(\mathcal{U}, \mathbb{C})$, we denote by $M(\eta)$ the matrix M evaluated at the local coordinates for η. We also denote the *spectrum* of a matrix L or a linear operator L by $\sigma(L)$.

Lemma 2.1 (Properties of the M matrix) Let $\eta \in T^*(\mathcal{U}, \mathbb{C})$.

(i) If η is real, then $M(\eta)$ is symmetric and has real eigenvalues.

(ii) If η is real and non-zero, then $M(\eta)$ has four non-zero eigenvalues, and 0 *is an eigenvalue of multiplicity* 2*.*

(iii) 0 *is an eigenvalue of multiplicity at least* 2*, and if λ is a non-zero eigenvalue of $M(\eta)$, then $-\lambda$ is also an eigenvalue.*

(iv) The spectra of L, L' and M coincide,

$$\sigma(L(\eta)) = \sigma(L'(\eta)) = \sigma(M(\eta)),$$

and the latter two spectra do not depend on the local coordinates.

Proof. Property (i) is immediate. For Property (ii), let (x, z) be local coordinates for η. As $\chi(x)$ is invertible, rank $M(\eta) = 2$, so $\dim\ker M(\eta) = 2$. Since M is symmetric, it is diagonalizable [16]; there is an orthogonal matrix R such that $M(\eta) = R^{-1}\Lambda R$ where Λ is an diagonal matrix containing the eigenvalues of $M(\eta)$. Thus $\dim\ker\Lambda = 2$, and 0 is an eigenvalue of multiplicity (precisely) 2, and property (ii) follows. For property (iii), let $(x, z) \in U \times \mathbb{C}^3$, and let

$$A = \varepsilon^{-1/2}(x) \cdot z \times I \cdot \mu^{-1/2}(x),$$

whence

$$M(\eta) = \begin{pmatrix} 0 & A \\ A^{\mathrm{T}} & 0 \end{pmatrix}.$$

If L is an invertible 3×3 matrix, then

$$\begin{pmatrix} L & A \\ A^{\mathrm{T}} & L \end{pmatrix} = \begin{pmatrix} I & 0 \\ A^{\mathrm{T}}L^{-1} & I \end{pmatrix} \cdot \begin{pmatrix} L & 0 \\ 0 & L - A^{\mathrm{T}}L^{-1}A \end{pmatrix} \cdot \begin{pmatrix} I & L^{-1}A \\ 0 & I \end{pmatrix},$$

whence

$$\det(M(\eta) - \lambda I) = -\det(A^{\mathrm{T}}A - \lambda^2 I), \quad \lambda \in \mathbb{C} \tag{10}$$

and property (iii) follows. The first equality in the last property follows as $L'(\eta)$ is the matrix representation of $L(\eta)$ in the basis (dx^i, dx^j). The last equality follows since $\sigma(AB) = \sigma(SABS^{-1}) = \sigma(SAS)$ when A, B are invertible matrices and $B = S^2$. $\square$

Using equation (10) and the fact that roots of polynomials can continuously be parametrized by functions that are lexicographically ordered [17], one can prove existence of continuous functions

$$h_+, h_- \colon T^*(\mathcal{U}, \mathbb{C}) \to \mathbb{C},$$

such that these parametrize the spectrum of L,

$$\sigma(L(\eta)) = \pm\{0, h_+(\eta), h_-(\eta)\}, \quad \eta \in T^*(\mathcal{U}, \mathbb{C}),$$

and $h_\pm$ are strictly positive on $T^*\mathcal{U} \setminus \{0\}$. However, the $h_\pm$-functions are not uniquely determined by L. For example, if they coincide on a sphere in $\mathcal{U}$, one can glue together an alternative continuous parametrization of the spectrum. Nevertheless, these functions will be important in what follows; under suitable assumptions on the media, the functions induce Riemannian geometries and Gaussian beams propagate along geodesics of these.

In the above we have only studied the $h_\pm$-functions locally. This should be enough for almost all practical electromagnetic applications. Global questions about the phase function are considerably more involved [18].

From the definitions of the $h_\pm$-functions it follows that a sufficient condition on θ to be a solution to equation (5) is that [7]

$$\frac{\partial \theta}{\partial t} = 0, \quad \text{or} \quad \frac{\partial \theta}{\partial t} = \pm h_\pm(d\theta), \tag{11}$$

where the $\pm$-signs are independent. Essentially, $d\theta$ is the direction of propagation. Thus if $\frac{\partial \theta}{\partial t} = 0$, equations (3)–(4) state that E_0, H_0 are parallel to the direction of propagation. Typically such solutions decay exponentially. Therefore we shall not study these. The first $\pm$-sign in equation (11) is also irrelevant as reversing time exchanges this sign. In conclusion we have shown that the phase function $\theta : \mathcal{U} \times I \to \mathbb{C}$ can be solved from

$$\frac{\partial \theta}{\partial t} = h_\pm(d\theta). \tag{12}$$

This equation is known as the *Hamilton-Jacobi equation.*

3. GAUSSIAN BEAMS

Next we study the Hamilton-Jacobi equation on a curve and assume that the phase function is a second order polynomial on that curve. Such phase functions will be called Gaussian beams. We shall do all the analysis in local coordinates. However, once we have derived equations for Gaussian beams we show that everything is coordinate independent.

Let us choose one of $h_\pm$ and denote that by h. In U the Hamilton-Jacobi equation then reads

$$\frac{\partial \theta}{\partial t} + h(x, \nabla\theta) \quad = \quad 0 \quad \text{on } U \times I.$$

Here h is the (known) local representative $h : U \times \mathbb{C}^3 \to \mathbb{C}$ of $h : T^*(\mathcal{U}, \mathbb{C}) \to \mathbb{C}$, and $\theta : U \times I \to \mathbb{C}$ is the local representative of the unknown phase function $\theta : \mathcal{U} \times I \to \mathbb{C}$.

Logically, we could insert Section 4 right after Section 2; it only calculates the $h_\pm$-functions for specific media. However, for a more coherent presentation, Section 4 follows this section. Nevertheless, by Proposition 4.3, we can *assume* that h^2 restricted to real arguments is a smoothly varying positive definite quadratic form on $T^*\mathcal{U}$. That is, for some smooth positive definite 2-tensor g^{ij} on M,

$$h(\xi) \quad = \quad \sqrt{g^{ij}(x)\xi_i\xi_j}, \quad \xi \in T^*_x\mathcal{U},$$

and $g_{ij} = (g^{ij})^{-1}$ is a Riemannian inner product on $\mathcal{U}$. For the rest of this section, we assume that h has this form for real arguments, but by the previous section we also know that h is defined for complex arguments. We will see that Gaussian beams do not depend on the complex behaviour of h.

Let $\mathcal{L}\colon T^*M \to TM$ be the *Legendre transformation* given by h,

$$\mathcal{L}(\xi) = \mathcal{L}^i(\xi)\frac{\partial}{\partial x^i}, \quad \xi \in T^*_x M,$$

where $\mathcal{L}^i(\xi) = g^{ij}(x)\xi_j$. The inverse Legendre transformation is

$$\mathcal{L}^{-1}(y) = \mathcal{L}^{-1}_i(y)dx^i, \quad y \in T^*_x M$$

where $\mathcal{L}^{-1}_i(y) = g_{ij}(x)y^j$. Let Γ^i_{jk} be the *Christoffel symbols*,

$$\Gamma_{ijk} \quad = \quad \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^i}\right),$$
$$\Gamma^i_{jk} \quad = \quad g^{ir}\Gamma_{rjk}, \qquad \Gamma^i_{jk} = \Gamma^i_{kj}.$$

On overlapping coordinates $(\tilde{x}^i)$, we have transformation rules

$$\frac{\partial \tilde{x}^l}{\partial x^i}\Gamma^i_{jk} = \frac{\partial^2 \tilde{x}^l}{\partial x^j\,\partial x^k} + \frac{\partial \tilde{x}^r}{\partial x^j}\frac{\partial \tilde{x}^s}{\partial x^k}\tilde{\Gamma}^l_{rs}.$$

Let us introduce functions N^i_j on the tangent bundle defined as [10]

$$N^i_j = \Gamma^i_{jk}y^k. \tag{13}$$

We will also need the *Riemann curvature tensor*, with components

$$R^m_{ijk} = \frac{\partial \Gamma^m_{ik}}{\partial x^j} - \frac{\partial \Gamma^m_{ij}}{\partial x^k} + \Gamma^s_{ik}\Gamma^m_{js} - \Gamma^s_{ij}\Gamma^m_{ks},$$

and transformation rules $R^m_{ijk} = \frac{\partial \tilde{x}^l}{\partial x^i}\frac{\partial \tilde{x}^r}{\partial x^j}\frac{\partial \tilde{x}^s}{\partial x^k}\frac{\partial x^m}{\partial \tilde{x}^p}\tilde{R}^p_{lrs}$.

3.1. Gaussian Beams

Definition 3.1 (Gaussian beam) *Suppose* $\theta: \mathcal{U} \times I \to \mathbb{C}$ *is a smooth function,* $c: I \to \mathcal{U}$ *is a smooth curve, and* $c_U: I \to U$ *is its local representation. Furthermore, let* $\phi: I \to \mathbb{C}$, $p: I \to \mathbb{C}^3$, *and* $H: I \to \mathbb{C}^{3\times 3}$ *be coefficients in the Taylor expansion of* θ *on* c_U,

$$\begin{aligned} \phi(t) &= \theta(c_U(t), t), \\ p_j(t) &= \frac{\partial \theta}{\partial x^j}(c_U(t), t), \\ H_{jk}(t) &= \frac{\partial^2 \theta}{\partial x^j \, \partial x^k}(c_U(t), t). \end{aligned}$$

Then we say that θ *corresponds to a* Gaussian beam *at* $c(t)$ *if*

$p(t) = (p_i(t))$ *is non-zero,*

$\phi(t)$ *and* $p(t)$ *are real,*

The matrix $H(t) = (H_{ij}(t))$ *has positive definite imaginary part.*

We shall also say that an electromagnetic field as in equations (1)–(2) is a *Gaussian beam* on c if θ is a Gaussian beam on $c(t)$ for all t. Let us first motivate the name and then show that the definition does not depend on local coordinates.

If c, θ, p, H are as above, then for $(x, t) \in U \times I$,

$$\begin{aligned} \theta(x,t) &= \phi(t) + p_j(t)\, z^j + \frac{1}{2} H_{jk}(t)\, z^j z^k + o(|z|^3), \\ z &= z(x,t) = x - c_U(t) \in \mathbb{R}^3. \end{aligned} \tag{14}$$

To see this, fix $t \in I$ and let $s(z) = (c_U(t) + z, t)$. The above formula then follows by expanding $\theta \circ s$ using Taylor's theorem. Thus

$$|e^{iP\theta(x,t)}| \approx e^{-P/2\, z^T \cdot \Im H \cdot z}. \tag{15}$$

As $P > 0$ is large, this means that a Gaussian beam decreases very rapidly away from $c(t)$. This motivates the name Gaussian beam and the assumptions on ϕ, p and H. The above equation also shows that $\Im H$ represents the "shape" of the Gaussian beam.

Transformation rules for ϕ, p, H

If $(\tilde{x}^i)$ are other coordinates overlapping the (x^i)-coordinates, then the coefficients ϕ, p_i, H_{ij} transform as

$$\tilde{\phi}(t) = \phi(t),$$

$$\begin{aligned}\tilde{p}_j(t) &= \frac{\partial x^r}{\partial \tilde{x}^j} \circ c(t)\, p_r(t),\\ \tilde{H}_{jk}(t) &= \left(\frac{\partial x^r}{\partial \tilde{x}^j}\frac{\partial x^s}{\partial \tilde{x}^k}\right) \circ c(t)\, H_{rs}(t) + \frac{\partial^2 x^r}{\partial \tilde{x}^j\, \partial \tilde{x}^k} \circ c(t)\, p_r(t),\end{aligned}$$

and Definition 3.1 is well defined.

3.2. Equations for c, ϕ, p, H

Suppose $c\colon I \to \mathcal{U}$ is a smooth curve, z is as in equation (14), and

$$\theta(x,t) = \phi(t) + p_j(t)\, z^j + \frac{1}{2} H_{jk}(t)\, z^j z^k$$

is a Gaussian beam for all t. Next we derive sufficient conditions on c, p, H for θ to be an solution to the approximate Hamilton-Jacobi equation

$$\frac{\partial \theta}{\partial t}(c_U(t), t) + h(x, \nabla\theta(c_U(t), t)) = o(|z|^3) \text{ on } U \times I. \tag{16}$$

By differentiating we obtain

$$\begin{aligned}\frac{\partial \theta}{\partial t} &= \left(\frac{d\phi}{dt} - p_j \frac{dc^j}{dt}\right) + \left(\frac{dp_j}{dt} - H_{jr}\frac{dc^r}{dt}\right) z^j + \frac{1}{2}\frac{dH_{jk}}{dt}\ z^j z^k,\\ \frac{\partial \theta}{\partial x^j} &= p_j + H_{jr} z^r,\end{aligned}$$

where $c_U = (c^1, c^2, c^3)$. It follows that

$$\begin{aligned}h(x, \nabla\theta(x,t)) &= h(z + c_U(t), p_r + H_{rs}(t) z^s)\\ &= h^*(z,t),\end{aligned}$$

where h^* is defined on the last line, and expanding h^* yields

$$h(x, \nabla\theta(x,t)) = h^*(0,t) + \frac{\partial h^*}{\partial z^j}(0,t)\, z^j + \frac{1}{2}\frac{\partial^2 h^*}{\partial z^j \partial z^k}(0,t)\, z^j z^k + o(|z|^3).$$

By the chain rule,

$$\begin{aligned}h^*(0,t) &= h \circ \gamma(t),\\ \frac{\partial h^*}{\partial z^j}(0,t) &= \frac{\partial h}{\partial x^j} \circ \gamma(t) + \frac{\partial h}{\partial \xi_k} \circ \gamma(t) H_{jk},\\ \frac{\partial^2 h^*}{\partial z^j \partial z^k}(0,t) &= (BH + HB^T + HCH + D)_{jk},\end{aligned}$$

where $\gamma\colon I \to U \times \mathbb{R}^3 \setminus \{0\}$ and matrices $B, C, D\colon I \to \mathbb{C}^{3\times 3}$ are defined by

$$\begin{aligned} \gamma(t) &= (c(t), p(t)), \\ B_j^k(t) &= \frac{\partial^2 h}{\partial x^j\,\partial \xi_k} \circ \gamma(t), \\ C^{jk}(t) &= \frac{\partial^2 h}{\partial \xi_j\,\partial \xi_k} \circ \gamma(t), \\ D_{jk}(t) &= \frac{\partial^2 h}{\partial x^j\,\partial x^k} \circ \gamma(t), \end{aligned}$$

such that j is the row index and k is the column index. Let us denote by $\gamma_{\mathcal{U}}$ the curve $\gamma_{\mathcal{U}}\colon I \to T^*\mathcal{U}$ induced by γ. Then the above equations for B, C, D are also valid when we replace γ with $\gamma_{\mathcal{U}}$.

From the above we obtain the following sufficient conditions on c, ϕ, p, H to solve equation (16):

$$\frac{d\phi}{dt} - p_j \frac{dc^j}{dt} + h \circ \gamma = 0, \tag{17}$$

$$\left(\frac{dp_j}{dt} + \frac{\partial h}{\partial x^j} \circ \gamma\right) + \left(\frac{\partial h}{\partial \xi_k} \circ \gamma - \frac{dc^k}{dt}\right) H_{jk} = 0, \tag{18}$$

$$\frac{dH}{dt} + BH + HB^T + HCH + D = 0. \tag{19}$$

Hamilton equations for $\gamma = (c, p)$

Taking the imaginary part of equation (18) yields

$$\frac{dc^j}{dt} = \frac{\partial h}{\partial \xi_j} \circ \gamma \tag{20}$$

and combining this with equation (18) yields

$$\frac{dp_j}{dt} = -\frac{\partial h}{\partial x^j} \circ \gamma. \tag{21}$$

These equations are the *Hamilton equations* for the curve $\gamma\colon I \to U \times \mathbb{R}^3 \setminus \{0\}$. Let us equip $T^*\mathcal{U} \setminus \{0\}$ with the canonical symplectic structure, and let $X_h \in \mathfrak{X}(\mathcal{U})$ be the Hamiltonian vector field induced by h. By basic results from symplectic geometry (see e.g., [19, 20]), it follows that a sufficient and necessary condition for γ to solve equations (20)–(21) is that $\gamma_{\mathcal{U}}$ is an integral curve of X_h. This is the coordinate invariant construction for γ. To solve γ, we need an initial condition

$\gamma_0 \in T^*\mathcal{U} \setminus \{0\}$. Since γ_0 is real, it follows that $\gamma_\mathcal{U}$ is real, so p satisfies the condition in Definition 3.1. We can normalize the initial condition such that $h(\gamma_0) = 1$. In fact, if $\gamma_\mathcal{U}$ is a solution from an initial condition $\gamma_0 \in T^*\mathcal{U} \setminus \{0\}$, then $\lambda\gamma_\mathcal{U}$ is a solution from the initial condition $\lambda\gamma_0$ for $\lambda > 0$. With this normalization, it follows (again by basic symplectic geometry) that $h \circ \gamma = 1$ identically.

Appendix A shows that solutions to Hamilton's equations for h are in one to one correspondence with geodesics of g_{ij}. In particular, Gaussian beams propagate along geodesics.

ϕ is constant

Combining equation (17) and (20) yields $\frac{d\phi}{dt} = \left(\xi_k \frac{\partial h}{\partial \xi_k}\right) \circ \gamma - h \circ \gamma$, so ϕ is constant by Euler's theorem for homogeneous functions [19].

Riccati equation for H

It remains to analyze equation (19) which is a matrix *Riccati equation*. The below proposition shows that it is uniquely solvable, and if H satisfies the condition in Definition 3.1 at $t = 0$, then H satisfies the condition for all t.

Proposition 3.2 ([5]) *Suppose H_0 is a symmetric 3×3 matrix such that $\Im H_0$ is positive definite. Then equation (19) has a unique solution H on I such that*

(i) $H(0) = H_0$,

(ii) *H is symmetric and $\Im H$ is positive definite for all t.*

The Riccati equation is scaling invariant with respect to the normalization of γ. In fact, suppose H is a solution to equation (19) on a curve γ from an initial condition H_0. Then for $\lambda > 0$, λH is a solution to equation (19) on the curve $\lambda\gamma$ from λH_0. This follows as C is (-1)-homogeneous, B is 0-homogeneous, and D is 1-homogeneous. In view of this invariance, it suffices to study equation (19) on a curve γ satisfying $h \circ \gamma = 1$. Let us also point out that H depends only on h for real arguments.

Let us define

$$G_{ij}(t) = H_{ij}(t) - (\Gamma^r_{ij}\mathcal{L}^{-1}_r) \circ \mathcal{L} \circ \gamma(t).$$

Then

$$G(t) = G_{ij}(t)\, dx^i \otimes dx^j \Big|_{c(t)}$$

is a 2-tensor on c with complex components (see Appendix B) and $\Im G = \Im H$. In other words, by perturbing the real part of H we have made H into a 2-tensor.

To find the equation for G, let us treat G_{ij} as a matrix, and let Λ be the symmetric matrix $\Lambda_{ij} = (\Gamma^r_{ij}\mathcal{L}^{-1}_r)\circ\mathcal{L}\circ\gamma$. Then $G = H - \Lambda$, and equation (19) is equivalent to

$$\frac{dG}{dt} + \hat{B}G + G\hat{B}^T + GCG + \hat{D} = 0,$$

where

$$\begin{aligned}\hat{B} &= B + \Lambda C,\\ \hat{D} &= D + \frac{d\Lambda}{dt} + \hat{B}\Lambda + \Lambda\hat{B}^T - \Lambda C\Lambda.\end{aligned}$$

The next lemma will enable us to give a coordinate independent equation for G. It is proven in a Riemannian setting in [5], and in a Riemann-Finsler setting in [7] (see also [19]). An outline of the proof (in the present Riemannian setting) is given in Appendix C.

Lemma 3.3 *Assuming that $h\circ\gamma = 1$, quantities C, and $\hat{D}$ satisfy*

$$\begin{aligned}C^{ij} &= (g^{ij} - y^i y^j)\circ\hat{c},\\ \hat{D}_{ij} &= -R_{ij}\circ\hat{c},\end{aligned}$$

where $\hat{c}\colon I\to TM$ is the canonical lift of c, and

$$R_{ij} = R^m_{ijk}y^k\mathcal{L}^{-1}_m.$$

By equation (B1), it follows that the equation for G is

$$\Big((D_{\dot{c}}G)_{ij} + (GCG)_{ij} + \hat{D}_{ij}\Big)\,dx^i\otimes dx^j\Big|_{c(t)} = 0.$$

Since each term in the parenthesis is a 2-tensor on c, this is a coordinate independent equation for G. From the equations in local coordinate we also know that from an initial condition the equation has a unique solution.

Finally let us point out that $\Im G$ defines a positive definite inner product on $T_{c(t)}M$ for all t. This gives an second "geometry" induced by an electromagnetic media. Its study would provide an interesting topic for further study.

4. ELECTROMAGNETIC MEDIA

As in the previous section we consider a local chart $U \subset \mathbb{R}^3$ of M, and assume that $\varepsilon, \mu: U \to \mathbb{R}^{3\times 3}$ are the positive definite matrices representing the electromagnetic media on U as in equation (8). Let us further assume that ε, μ can be written as

$$\varepsilon = R^{-1}\Lambda_\varepsilon R, \quad \mu = R^{-1}\Lambda_\mu R \quad \text{on } U \tag{22}$$

for a smooth orthogonal matrix $R: U \to \mathbb{R}^{3\times 3}$, and for some smooth diagonal matrices $\Lambda_\varepsilon = \text{diag}\ (\varepsilon_1, \varepsilon_2, \varepsilon_3)$, $\Lambda_\mu = \text{diag}\ (\mu_1, \mu_2, \mu_3)$ containing the strictly positive eigenvalues of ε and μ.

Let us recall that a matrix is orthogonally diagonalizable if and only if the matrix is symmetric [16]. In consequence, ε and μ can always pointwise be diagonalized separately. Our next aim is to calculate the $h_\pm$-functions in different media. To obtain explicit, and not too complicated expressions for x these, we assume that equation (22) holds. Examples of media satisfying this condition are:

(i) Isotropic media where ε and μ depend only on two scalar functions.

(ii) ε is positive definite, and μ is isotropic.

(iii) μ is positive definite, and ε is isotropic.

(iv) μ and ε are proportional [1].

Of these, the second class is maybe of most interest, as it can be seen as a model for biological tissue. Namely, in biological tissue there is no magnetic activity, but due to muscles and bones (which have a fibred structure), ε is anisotropic.

In this section we will only study the $h_\pm$-functions for real arguments (see beginning of Section 3). Let us also point out that this section contains many algebraic computations which are most conveniently done with a computer.

4.1. The $h_\pm$-functions

If $Q \in \mathbb{R}^{3\times 3}$ is invertible and $u, v \in \mathbb{R}^3$, then

$$Q^T(u \times v) = \det Q\ (Q^{-1}u) \times (Q^{-1}v).$$

Therefore, if ξ is the mapping $z \mapsto \xi(z) = \det R\ Rz$, it follows that

$$M(x,z) = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}^{-1} \begin{pmatrix} \Lambda_\varepsilon & 0 \\ 0 & \Lambda_\mu \end{pmatrix}^{-1/2} \begin{pmatrix} 0 & \xi\times I \\ -\xi\times I & 0 \end{pmatrix} \begin{pmatrix} \Lambda_\varepsilon & 0 \\ 0 & \Lambda_\mu \end{pmatrix}^{-1/2} \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}.$$

If $(x,z) \in U \times \mathbb{R}^3$ and $\xi = \xi(z)$, then (using computer algebra) we find that $\lambda \in \sigma(M(x,z))$ if and only if

$$\lambda^2(\lambda^4 - \|S\cdot\xi\|^2\,\lambda^2 + \|F\cdot\xi\|^2\ \|N\cdot\xi\|^2) = 0,$$

where

$$\begin{aligned} S &= \operatorname{diag}\left(\sqrt{e_2^2m_3^2+e_3^2m_2^2}, \sqrt{e_1^2m_3^2+e_3^2m_1^2}, \sqrt{e_1^2m_2^2+e_2^2m_1^2}\right), \\ F &= \operatorname{diag}\,(e_2e_3, e_1e_3, e_1e_2), \\ N &= \operatorname{diag}\,(m_2m_3, m_1m_3, m_1m_2), \\ e_i &= 1/\sqrt{\varepsilon_i}, \quad m_i = 1/\sqrt{\mu_i}, \end{aligned}$$

$\sqrt{\cdot}$ is the positive square root, and $\|\cdot\|$ is the Euclidean norm. Hence

$$\sigma(M(x,z)) \setminus \{0\} = \left\{\pm\frac{1}{\sqrt{2}}\sqrt{\|S\cdot\xi\|^2 \pm \sqrt{D}}\right\}, \qquad (23)$$

where $\pm$-signs are independent and D is the 4-homogeneous polynomial

$$D(x,z) = \|S\cdot\xi\|^4 - 4\|F\cdot\xi\|^2\ \|N\cdot\xi\|^2.$$

Since $\sigma(M)$ is real, $D \geq 0$ for all $(x,z) \in U \times \mathbb{R}^3$. Our next task is to define $h_\pm$-functions from the above spectrum as in Section 2. Since $h_\pm$ should be positive, it is clear that the first $\pm$-sign should be $+$ for all $(x,z) \in U \times \mathbb{R}^3$. The choice of the second $\pm$-sign is, however, not so clear. For example, if for fixed $x \in U$, D vanishes on a small closed curve of $\|z\| = 1$, then there are two ways to define h_+ and h_- inside that curve. In the below (Lemma 4.2) we prove that D has no such zeroes; either D vanishes identically, or it vanishes at finitely many points on $\|z\| = 1$. By a topological argument (see [19]) one can then show that for fixed $x \in U$, the second $\pm$-sign can not depend on z. It is therefore motivated to define

$$h_\pm(x,z) = \frac{1}{\sqrt{2}}\sqrt{\|S\cdot\xi\|^2 \pm \sqrt{D}}, \quad (x,z) \in U \times \mathbb{R}^3.$$

It is clear that $h_\pm$ are now continuous. However, these functions are not necessarily uniquely determined by $\sigma(M(x,z))\setminus\{0\}$. For example, if D vanishes on a sphere in U, there are two ways to define the $h_\pm$-functions inside that sphere. We shall not study this somewhat problematic issue further.

Let us point out that the $h_\pm$-functions are 1-homogeneous and $h_\pm(x,z) = 0$ if and only if $z = 0$. Furthermore, the $h_\pm$-functions do not change value under reflections in the z-variable. In particular, $h_\pm$ are symmetric; $h(x,z) = h(x,-z)$. In general, the $h_\pm$-functions are non-convex and non-smooth. Therefore they do not induce Finsler geometries [10]. However, in Propositions 4.3 and 4.4 we characterize when they are Finsler geometries. This yields a negative result. Whenever the $h_\pm$-functions define Finsler geometries, the geometry is Riemannian.

4.2. The Δ_{ij}-symbols

For $i,j = 1,2,3$ let

$$\Delta_{ij} = e_i^2 m_j^2 - e_j^2 m_i^2.$$

These symbols will be central in what follows. We will show that the $h_\pm$-functions behave qualitatively differently depending on how many of these Δ_{ij}-symbols vanish. To prove this we will need two lemmas, which rely on some alternative expressions for D:

$$\begin{aligned} D &= \Delta_{23}^2\xi_1^4 + \Delta_{13}^2\xi_2^4 + \Delta_{12}^2\xi_3^4 \\ &\quad +2\Delta_{23}\Delta_{13}\xi_1^2\xi_2^2 - 2\Delta_{23}\Delta_{12}\xi_1^2\xi_3^2 + 2\Delta_{13}\Delta_{12}\xi_2^2\xi_3^2 \qquad (24)\\ &= (\Delta_{23}\xi_1^2 + \Delta_{13}\xi_2^2 + \Delta_{12}\xi_3^2)^2 - 4\Delta_{12}\Delta_{23}\xi_1^2\xi_3^2 \qquad (25)\\ &= (\Delta_{23}\xi_1^2 + \Delta_{13}\xi_2^2 + \Delta_{21}\xi_3^2)^2 - 4\Delta_{21}\Delta_{13}\xi_2^2\xi_3^2 \qquad (26)\\ &= (\Delta_{32}\xi_1^2 + \Delta_{13}\xi_2^2 + \Delta_{12}\xi_3^2)^2 - 4\Delta_{13}\Delta_{32}\xi_1^2\xi_2^2. \qquad (27) \end{aligned}$$

The first lemma shows that if two Δ_{ij}-symbols vanish, then the third symbol also vanishes. The second lemma gives all the real zeroes of D in ξ-coordinates.

Lemma 4.1 (Δ_{ij}-symbols) *Suppose $i,j,k \in \{1,2,3\}$ are distinct.*

(i) *If $\Delta_{ij} = 0$, then*

$$\operatorname{sign}\Delta_{ik} = \operatorname{sign}\Delta_{jk},$$

where sign *is the signum function;* $\operatorname{sign}(t) = 1$, $\operatorname{sign}(-t) = -1$ *for $t > 0$ and* $\operatorname{sign}(0) = 0$.

(ii) *If one Δ_{ij}-symbol is zero, then*

$$\sqrt{D} = |\Delta_{23}|\xi_1^2 + |\Delta_{13}|\xi_2^2 + |\Delta_{12}|\xi_3^2. \qquad (28)$$

(iii) *If $\Delta_{ij} > 0$ and $\Delta_{jk} > 0$, then $\Delta_{ik} > 0$.*

(iv) *If all $\Delta_{23}, \Delta_{23}, \Delta_{12}$ are non-zero, then the possible sign configurations are:*

	1	2	3	4	5	6
Δ_{23}	+	+	+	−	−	−
Δ_{13}	+	+	−	+	−	−
Δ_{12}	+	−	−	+	+	−

Proof. Property (i) follows since $\Delta_{ik} = (m_i/m_j)^2\Delta_{jk}$, Property (ii) follows using (i) and equations (25)–(27), and property (iii) follows by a direct calculation. Property (iv) follows since Property (iii) implies that sign configurations $\Delta_{23} > 0, \Delta_{31} > 0, \Delta_{12} > 0$, and $\Delta_{32} > 0, \Delta_{13} > 0, \Delta_{21} > 0$ are impossible, and all other configurations are achieved by media $e_i^2, m_i^2 \in \{1, 2\}$. □

Lemma 4.2 (Real zeroes of D)

(i) If two (or all) of the Δ_{ij}-symbols are zero, then D is identically zero.

(ii) If only $\Delta_{ij} = 0$, then the zeroes of D are the line $\{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \xi_i = 0,\ \xi_j = 0\}$.

(iii) If all Δ_{ij}-symbols are non-zero, then the zeroes of D (modulo scalings and reflections) are

(a) $\xi = (1, 0, \sqrt{\frac{\Delta_{23}}{\Delta_{12}}})$ *for sign configurations* 1 *and* 6,

(b) $\xi = (0, 1, \sqrt{-\frac{\Delta_{13}}{\Delta_{12}}})$ *for sign configurations* 2 *and* 5,

(c) $\xi = (\sqrt{-\frac{\Delta_{13}}{\Delta_{23}}}, 1, 0)$ *for sign configurations* 3 *and* 4.

Proof. The first two properties follow from the previous lemma. If sign configuration 2–5 holds, then $\Delta_{12}\Delta_{23} < 0$, and equation (25) gives two possibilities,

$$\begin{aligned} \xi_1 &= 0, \quad \Delta_{13}\xi_2^2 + \Delta_{12}\xi_3^2 = 0, \\ \xi_3 &= 0, \quad \Delta_{23}\xi_1^2 + \Delta_{13}\xi_2^2 = 0. \end{aligned}$$

Since we are seeking roots only modulo scaling, we can assume that $\xi_2 = 1$ or $\xi_2 = 0$. However, we can dismiss the latter alternative as $\xi_2 = 0$ implies that $\xi = 0$. The choice $\xi_2 = 1$ yields two possibilities

$$\xi_1 = 0, \quad \xi_2 = 1, \quad \xi_3 = \sqrt{-\frac{\Delta_{13}}{\Delta_{12}}},$$

$$\xi_1 = \sqrt{-\frac{\Delta_{13}}{\Delta_{23}}}, \quad \xi_2 = 1, \quad \xi_3 = 0,$$

where we have excluded the −-branches; these can always be recovered by reflection of the coordinates. Inspecting the signs yields roots *(b)* and *(c)*. If sign configuration 1 or 6 holds, then $\Delta_{21}\Delta_{13} < 0$, and a similar analysis to the above gives root *(a)*. □

4.3. Classification of Media

Using the Δ_{ij}-symbols we can divide media into three classes of increasing complexity: 1° all symbols vanish, 2° one symbol vanishes, or 3° none of the symbols vanish.

If two of the Δ_{ij}-symbols vanish, then the third also vanishes, and

$$h_+(x,z) = h_-(x,z) = \frac{1}{\sqrt{2}}\|S \cdot \xi\|.$$

Here it is clear that the $h_\pm$-functions are uniquely determined from the spectrum; the functions do not depend on the choice of the second $\pm$-sign in equation (23).

The equivalence of (ii) and (iii) in the next lemma was proven in [8].

Proposition 4.3 (Characterization of media I) *The following are equivalent:*

(i) *At least two of* $\Delta_{23}, \Delta_{13}, \Delta_{12}$ *vanish for all* $x \in U$.

(ii) $h_+ = h_-$ *on* $U \times \mathbb{R}^3$.

(iii) *The medium matrices satisfy*

$$\varepsilon = \rho^2 \mu \tag{29}$$

for some strictly positive smooth function $\rho: U \to \mathbb{R}$.

Proof. By the previous lemma, (i) implies (ii). On the other hand, if (ii) holds, then D is the zero polynomial, that is, by equation (24), all Δ_{ij}:s are zero. If property (iii) holds, then $m_i = \rho e_i$, and property (i) follows. The other direction follows using the definition of the Δ_{ij}-symbols. □

Proposition 4.4 (Characterization of media II) *Pointwise the following are equivalent:*

(i) *At least one of* $\Delta_{23}, \Delta_{13}, \Delta_{12}$ *vanishes.*

(ii) *The functions* $h_\pm^2$ *are positive definite quadratic forms of* ξ.

(iii) h_+, h_- *are smooth on* $\mathbb{R}^3 \setminus \{0\}$.

(iv) h_- *is convex on* $\mathbb{R}^3$.

Proof. ((i) $\Rightarrow$ (ii)) Suppose that one of $\Delta_{23}, \Delta_{13}, \Delta_{12}$ is zero. (If two of these are zero, then the third is also zero, and the claim follows.)

Then by equation (28), we have

$$\begin{aligned} h_\pm^2 &= \frac{1}{2}((e_2^2 m_3^2 + e_3^2 m_2^2 \pm |\Delta_{23}|)\xi_1^2 \\ &\quad +(e_1^2 m_3^2 + e_3^2 m_1^2 \pm |\Delta_{13}|)\xi_2^2 \\ &\quad +(e_1^2 m_2^2 + e_2^2 m_1^2 \pm |\Delta_{12}|)\xi_3^2). \end{aligned}$$

Since $e_i^2 m_j^2 + e_j^2 m_i^2 \pm |\Delta_{ij}|$ is always positive, the claim follows. For the other direction (ii) $\Rightarrow$ (i), suppose that $h_\pm^2$ are quadratic forms and all Δ_{ij} are non-zero. Then $\sqrt{D}$ must be a polynomial, say, $\alpha^2 = D$, and α must be of the form

$$\alpha = a_1\xi_1^2 + a_2\xi_2^2 + a_3\xi_3^2$$

for some real a_i. Thus

$$D = a_1^2\xi_1^4 + a_2^2\xi_2^4 + a_3^2\xi_3^4 + 2a_1a_2\xi_1^2\xi_2^2 + 2a_1a_3\xi_1^2\xi_3^2 + 2a_2a_3\xi_2^2\xi_3^2,$$

and equation (24) implies that

$$\begin{aligned} a_1a_2 &= \Delta_{23}\Delta_{13}, \\ a_1a_3 &= -\Delta_{23}\Delta_{12}, \\ a_2a_3 &= \Delta_{13}\Delta_{12}. \end{aligned}$$

This means that two of a_1a_2, a_1a_3, a_2a_3 are positive and one is negative. This is a contradiction, so at least one Δ_{ij} must vanish. It is clear that (ii) $\Rightarrow$ (iii). To prove (iii) $\Rightarrow$ (i), suppose that $h_\pm$ are smooth on $\mathbb{R}^3 \setminus \{0\}$ and all Δ_{ij} are non-zero. Let us furthermore assume that sign configuration 1 or 6 holds. (The analysis for the other cases is completely analogous.) In view of Lemma 4.2, let

$$\gamma(t) = (1, t, \sqrt{\frac{\Delta_{23}}{\Delta_{12}}}), \quad t \in \mathbb{R},$$

whence

$$\sqrt{D \circ \gamma} = \sqrt{t^2\Delta_{13}(4\Delta_{23} + \Delta_{13}t^2)}.$$

Since Δ_{13}, Δ_{23} are non-zero, $\sqrt{4\Delta_{23}\Delta_{13} + \Delta_{13}^2 t^2} > 0$ is smooth, and $\sqrt{t^2} = |t|$ is smooth; a contradiction. The implication (ii) $\Rightarrow$ (iv) is clear. To complete the proof, let us show that (iv) $\Rightarrow$ (i). Suppose h_- is convex and all Δ_{ij} are non-zero. Then we can introduce new coordinates $\eta = \eta(\xi) = \eta(z)$,

$$\xi_1 = \frac{1}{\sqrt{|\Delta_{23}|}}\eta_1, \quad \xi_2 = \frac{1}{\sqrt{|\Delta_{13}|}}\eta_2, \quad \xi_3 = \frac{1}{\sqrt{|\Delta_{12}|}}\eta_3.$$

Then

$$h_{\pm}(x,z) = \frac{1}{\sqrt{2}}\sqrt{\alpha_{23}\eta_1^2 + \alpha_{13}\eta_2^2 + \alpha_{12}\eta_3^2 \pm \sqrt{D}},$$

where

$$\begin{aligned} D &= \eta_1^4 + \eta_2^4 + \eta_3^4 + 2\sigma_{23}\sigma_{13}\eta_1^2\eta_2^2 - 2\sigma_{23}\sigma_{12}\eta_1^2\eta_3^2 + 2\sigma_{13}\sigma_{12}\eta_2^2\eta_3^2, \\ \alpha_{ij} &= \frac{e_i^2 m_j^2 + e_j^2 m_i^2}{|\Delta_{ij}|} = \frac{e_i^2 m_j^2 + e_j^2 m_i^2}{|e_i^2 m_j^2 - e_j^2 m_i^2|}, \quad \sigma_{ij} = \operatorname{sign}\Delta_{ij}. \end{aligned}$$

Let us first assume that sign configuration 1 or 6 holds. Then $\eta = (1,0,1)$ is a root for D, so let

$$a = (1,0,1+t), \quad b = (1,0,1-t), \quad t \in (0,1)$$

in η-coordinates. As h_- is convex, $h_-(\frac{a+b}{2}) \le \frac{1}{2}h_-(a) + \frac{1}{2}h_-(b)$, so

$$\begin{aligned} 2\sqrt{\alpha_{23}+\alpha_{12}} \;\le\; & \sqrt{\alpha_{23}+\alpha_{12}(1-t)^2-(2-t)t} \\ & +\sqrt{\alpha_{23}+\alpha_{12}(1+t)^2-(2+t)t}, \quad t\in(0,1), \end{aligned}$$

a contradiction with Lemma 4.5. If sign configuration 2 or 5 holds, then $\eta = (0,1,1)$ is a root for D, and considering

$$a = (0,1,1+t), \quad b = (0,1,1-t), \quad t \in (0,1)$$

yields the result. If sign configuration 3 or 4 holds, then $\eta = (1,1,0)$ is a root for D, and considering

$$a = (1+t,1,0), \quad b = (1-t,1,0), \quad t \in (0,1)$$

yields the result. □

Lemma 4.5 (An inequality for the square root) *Suppose* $\alpha, \gamma > 0$. *Then there exists a* $t \in (0,1)$ *such that*

$$\sqrt{\alpha+\gamma(1-t)^2-(2-t)t} + \sqrt{\alpha+\gamma(1+t)^2-(2+t)t} < 2\sqrt{\alpha+\gamma}.$$

Proof. The claim follows by squaring the contrapositive inequality twice with computer algebra. □

5. GEOMETRIES INDUCED BY MEDIA

Example 5.1 (All Δ_{ij}-symbols are zero) If $\varepsilon = \rho^2\mu$, then $m_i = \rho e_i$ and

$$h_{\pm}(x,z) = \rho\|\operatorname{diag}\,(e_2e_3, e_1e_3, e_1e_2)\cdot\xi\|.$$

In particular, if the medium is isotropic, that is, $\varepsilon_1 = \varepsilon_2 = \varepsilon_3$, $\mu_1 = \mu_2 = \mu_3$, then

$$h_+(x,z) = h_-(x,z) = \frac{1}{\sqrt{\varepsilon_1(x)\mu_1(x)}}\|z\|.$$

Here we can recognize the "phase velocity" $\frac{1}{\sqrt{\varepsilon_1(x)\mu_1(x)}}$, which is now a function of $x \in U$. □

Example 5.2 (One Δ_{ij}-symbol is zero) Suppose $\Delta_{23} = 0$, and $\Delta_{12}, \Delta_{13} > 0$. Then

$$\begin{aligned} h_+(x,z) &= \|\text{diag }(e_2m_3, e_1m_3, e_1m_2)\cdot\xi\|, \\ h_-(x,z) &= \|\text{diag }(e_3m_2, e_3m_1, e_2m_1)\cdot\xi\|. \end{aligned}$$

In particular, if $\Lambda_\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_2)$, $\Lambda_\mu = (\mu_1, \mu_2, \mu_2)$, then

$$\begin{aligned} h_+(x,z) &= m_2\|\text{diag }(e_2, e_1, e_1)\cdot\xi\|, \\ h_-(x,z) &= e_2\|\text{diag }(m_2, m_1, m_1)\cdot\xi\|. \end{aligned}$$

These expressions are somewhat surprising; the order of e_1 and e_2 are reversed. Another interesting special case is $\Lambda_\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_2)$, $\Lambda_\mu = (\mu_1, \mu_1, \mu_1)$, whence

$$\begin{aligned} h_+(x,z) &= m_1\|\text{diag }(e_2, e_1, e_1)\cdot\xi\|, \\ h_-(x,z) &= m_1e_2\|z\|. \end{aligned}$$

In other words, if there is no magnetic anisotropy, then h_- will be a scaled Euclidean norm. □

Example 5.3 (All Δ_{ij}-symbols are non-zero) If all Δ_{ij}-symbols are non-zero, then the $h_\pm$-functions are non-convex and non-smooth.

In Figure 1, $h_\pm$ are plotted in media $m_i = 1, e_1 = 3, e_2 = 2, e_1 = 1$, whence all $\Delta_{ij} > 0$. The first figure shows that both functions have a $|\cdot|$-singularity in the $y = 0$ plane. From the figures we also see that h_- is not convex. □

Example 5.4 (Induced Riemannian geometries) In isotropic media the induced Riemann geometry is

$$g_{\pm,ij}(x) = \varepsilon_1(x)\mu_1(x)\delta_{ij}.$$

and in the media of Example 5.2,

$$\begin{aligned} g_{+,ij}(x,z) &= (R^{-1}\cdot\text{diag }(\varepsilon_2\mu_3, \varepsilon_1\mu_3, \varepsilon_1\mu_2)\cdot R)_{ij}, \\ g_{-,ij}(x,z) &= (R^{-1}\cdot\text{diag }(\varepsilon_3\mu_2, \varepsilon_3\mu_1, \varepsilon_2\mu_1)\cdot R)_{ij}. \end{aligned}$$

□

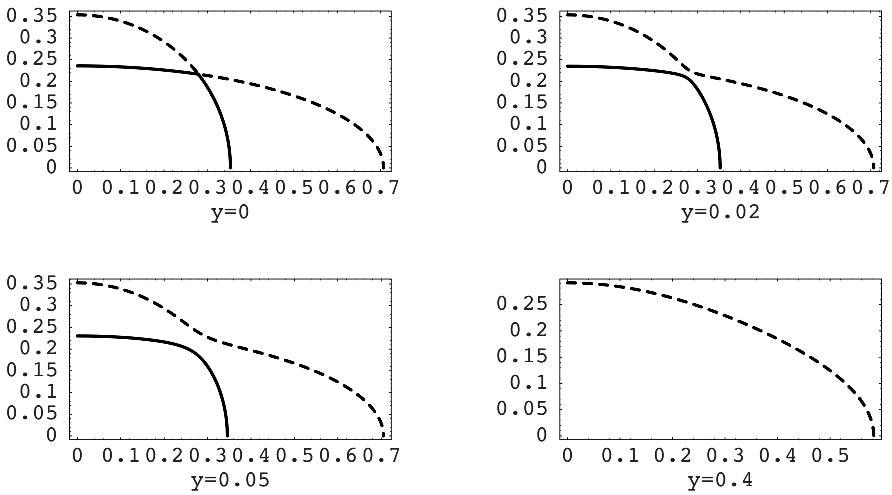

Figure 1. Cross sections of unit spheres of h_+ (solid line) and h_- (dashed line) in the $(x > 0,\ y > 0)$-quadrant for different values of y.

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APPENDIX A. HAMILTON EQUATIONS AND GEODESICS

Let us first derive some identities we shall need in Appendix A and C.

Using $\frac{\partial g_{ij}}{\partial x^k} = \Gamma_{ijk} + \Gamma_{jik}$ and the identity $(A^{-1})' = -A^{-1}A'A^{-1}$

for the derivative of an inverse matrix we obtain:

$$\frac{\partial \mathcal{L}^{-1}_j}{\partial x^k} = (\Gamma_{ijk} + \Gamma_{jik}) y^i, \tag{A1}$$
$$\frac{\partial \mathcal{L}^j}{\partial x^k} = -g^{jm}(\Gamma_{lmk} + \Gamma_{mlk})\mathcal{L}^l, \tag{A2}$$
$$\frac{\partial \mathcal{L}^j}{\partial \xi_k} = g^{jk}, \qquad \frac{\partial \mathcal{L}^{-1}_j}{\partial y^k} = g_{jk}, \tag{A3}$$
$$\frac{\partial h^2}{\partial x^k} = -2\Gamma_{mnk}\mathcal{L}^m\mathcal{L}^n, \tag{A4}$$
$$\frac{\partial h}{\partial x^k} = -\frac{1}{h}\Gamma_{mnk}\mathcal{L}^m\mathcal{L}^n, \qquad \xi \neq 0, \tag{A5}$$
$$\frac{\partial h}{\partial \xi^k} = \frac{1}{h}\mathcal{L}^k, \qquad \xi \neq 0. \tag{A6}$$

The last identity implies that if $\gamma = (c, p)$ is a solution to Hamilton's equations with $h \circ \gamma = 1$, then

$$\frac{dc^i}{dt} = \mathcal{L}^i \circ \gamma, \quad \text{or} \quad \hat{c} = \mathcal{L} \circ \gamma. \tag{A7}$$

Proposition A.5 *If $\gamma = (c, p)$ is a solution to Hamilton's equations such that $h \circ \gamma = 1$, then c is a* pathlength parametrized geodesic*, i.e.,*

$$\frac{d^2c^i}{dt^2} + \Gamma^i_{jk} \circ c \frac{dc^j}{dt}\frac{dc^k}{dt} = 0, \tag{A8}$$
$$g_{ij} \circ c \frac{dc^i}{dt}\frac{dc^j}{dt} = 1. \tag{A9}$$

Conversely, if c is a pathlength parametrized geodesic, then $\gamma = \mathcal{L}^{-1} \circ \hat{c}$ solves Hamilton's equations and $h \circ \gamma = 1$.

Proof. In the first claim, $g_{ij}\frac{dc^i}{dt}\frac{dc^k}{dt} = 1$ follows from equation (A7), and equation (A8) follows by differentiating $\frac{dc^i}{dt} = \mathcal{L}^i \circ \gamma$. For the other claim, let $p_i = \mathcal{L}^{-1}_i \circ \hat{c} = g_{il} \circ c \frac{dc^l}{dt}$. Then $\gamma = (c, p)$, and $h \circ \gamma = 1$. Equation (20) follows by expanding its right hand side, and equation (21) follows since both sides expand to $\Gamma_{ijk}\frac{dc^i}{dt}\frac{dc^k}{dt}$. □

APPENDIX B. COMPLEX TENSORS ON A CURVE

Let $\mathcal{E}^k(M, \mathbb{C})$ $(k = 1, 2, \ldots)$ be the space of *complex k-tensors on M*. That is, an element in $\mathcal{E}^k(M, \mathbb{C})$ is a smooth mapping taking a point $x \in M$ into a complex k-tensor in $T_x^*(M,\mathbb{C}) \otimes \cdots \otimes T_x^*(M,\mathbb{C})$. If $c: I \to M$ is a smooth curve on M, then by a *smooth k-tensor ($k = 1, 2, \ldots$) on c* we mean a smooth mapping $\alpha: I \to \mathcal{E}^k(M,\mathbb{C})$ such that $\pi \circ \alpha = c$ on I, where π is the canonical projection $\pi: \mathcal{E}^k(M,\mathbb{C}) \to M$. Then locally there are smooth functions $\alpha_{i_1 \ldots i_k}$ such that

$$\alpha(t) = \alpha_{i_1\ldots i_k}(t)\; dx^{i_1} \otimes \cdots \otimes dx^{i_k}\Big|_{c(t)},$$

where (x^i) are local coordinates near $c(t)$. A 0-tensor on c is just a function $I \to \mathbb{C}$.

Suppose M is a Riemannian manifold and N^i_j is defined as in equation (13). Then the *covariant derivative* $D_{\dot c}$ for tensors on a curve c is the linear mapping that takes k-tensors on c into k-tensors on c defined as follows. For a function $f: I \to \mathbb{C}$, $D_{\dot c} f = \frac{df}{dt}$, and for a 1-tensor $\alpha = \alpha_i dx^i$ on c,

$$D_{\dot c}\alpha \;=\; \left(\frac{d\alpha_i}{dt} - \alpha_r N^r_i \circ \hat c\right) dx^i|_c.$$

What is more, if α, β are smooth p- and q-tensors on c, then

$$D_{\dot c}(\alpha \otimes \beta) = (D_{\dot c}\alpha) \otimes \beta + \alpha \otimes (D_{\dot c}\beta).$$

From the transformation properties of Γ^i_{jk}, it follows that $D_{\dot c}$ is well defined.

For example, if $G(t) = G_{ij}(t)\; dx^i \otimes dx^j|_{c(t)}$ is a 2-tensor, then

$$D_{\dot c} G \;=\; \left(\frac{dG_{ij}}{dt} - G_{ir}N^r_j \circ \hat c - N^r_i \circ \hat c\; G_{rj}\right)\; dx^i \otimes dx^j\Big|_c. \quad \text{(B1)}$$

APPENDIX C. PROOF OF LEMMA 3.3

In the below proof we shall make heavy use of identities (A1)–(A6) and the assumption $h \circ \gamma = 1$. Let us also simplify the notation and write γ when we mean $\gamma_{\mathcal{U}}$. Let us also assume that in local coordinates $\gamma = (c, p)$. For C^{ij} we then have

$$\begin{aligned} C^{ij} &= \frac{\partial}{\partial \xi^i}\left(\frac{1}{h}\mathcal{L}^j\right) \circ \gamma \\ &= \left(g^{ij} - y^i y^j\right) \circ \hat c, \end{aligned}$$

and the first claim follows. By the definition R_{ij}, we have

$$R_{ij} \;=\; \left(\frac{\partial N^m_i}{\partial x^j} - \frac{\partial \Gamma^m_{ij}}{\partial x^s} y^s + N^s_i \Gamma^m_{sj} - N^m_s \Gamma^s_{ij}\right) \mathcal{L}^{-1}_m.$$

Since

$$B_i^j = \left(-N_i^j - g^{ja}\Gamma_{ai}^b\mathcal{L}_b^{-1} + N_i^a\mathcal{L}_a^{-1}y^j\right) \circ \hat{c}$$

it follows that

$$\hat{B}_i^j = -N_i^j \circ \hat{c}.$$

Next we expand each term in $\hat{D}$. For D_{ij} we obtain

$$\begin{aligned} D_{ij} &= -\frac{\partial}{\partial x^j}\left(\frac{1}{h}\Gamma_{ni}^m\xi_m\mathcal{L}^n\right) \circ \gamma \\ &= -\left[\left(N_i^n N_j^m \mathcal{L}_n^{-1} + \frac{\partial N_i^m}{\partial x^j} - N_j^s\Gamma_{is}^m - g^{nr}\Gamma_{in}^s\Gamma_{jr}^m\mathcal{L}_s^{-1}\right)\mathcal{L}_m^{-1}\right] \circ \hat{c}. \end{aligned}$$

Using that c is a geodesic yields

$$\frac{d\Lambda_{ij}}{dt} = \left[\left(\frac{\partial \Gamma_{ij}^m}{\partial x^s}y^s + N_s^m\Gamma_{ij}^s\right)\mathcal{L}_m^{-1}\right] \circ \hat{c},$$

and the last three terms in $\hat{D}$ are

$$\begin{aligned} (\hat{B}\Lambda + \Lambda\hat{B}^T)_{ij} &= \left[(-N_i^s\Gamma_{sj}^m - N_j^s\Gamma_{is}^m)\mathcal{L}_m^{-1}\right] \circ \hat{c}, \\ (\Lambda C\Lambda)_{ij} &= \left[\left(-N_i^n N_j^m\mathcal{L}_n^{-1} + g^{nr}\Gamma_{in}^s\Gamma_{jr}^m\mathcal{L}_s^{-1}\right)\mathcal{L}_m^{-1}\right] \circ \hat{c}. \end{aligned}$$

Substituting all these expressions into $\hat{D}$ yields the result. □

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