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# WAVE PROPAGATION AND DISPERSION CHARACTERISTICS FOR A NONRECIPROCAL ELECTRICALLY GYROTROPIC MEDIUM

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**Abstract**—The general dispersion relation for a nonreciprocal electrically gyrotropic or a gyroelectric medium is derived in two distinct forms by using three different methods. One of them is a new method which can be used when the stratification of the layers is in the $z$-direction. The wave numbers corresponding to each dispersion relation are obtained in closed form. It is shown that there exist two types of waves, type I and type II, in a gyroelectric medium. The wave propagation is investigated and the polarization of the waves, resonances and cut off conditions are obtained for the principle waves. The general wave propagation regions are identified using resonances and cut off conditions. These regions are then used to construct the Clemmow-Mually-Allis (CMA) diagram. The conditions showing the frequency bands for which wave can propagate in each region are tabulated for the first time. The results presented in this paper can be used in the development of nonreciprocal devices and in ionospheric problems including radiation and scattering applications.

## 1. INTRODUCTION

Nonreciprocal anisotropic or gyrotropic media have been an active research topic because of the existence of natural gyrotropic-anisotropic crystals and easy realization of artificial composites, e.g., magnetically biased plasma or ferrite. This led to realization of microwave devices such as circulators, isolators, resonators, and optical devices such as modulators, switches, phase shifters using gyrotropic materials, which give nonreciprocal effects. Hence this makes it inevitable to study extensively wave propagation and dispersion characteristics in a gyrotropic medium to develop nonreciprocal devices at millimeter and submillimeter wavelength ranges. The lack of any published complete analysis for a nonreciprocal electrically gyrotropic medium in the literature has motivated this work.

An effect of nonreciprocity is first reported in a waveguide which is loaded with thin polarized germanium plate in a transverse magnetic field by H. E. M. Barlow and R. Koike [1] as early as 1963. A variety of gyroelectric waveguiding structures have been addressed over the past three decades [2–4] and experimental nonreciprocal devices operating at liquid nitrogen and room temperature have been designed and tested for different microwave frequencies. The recent developments include the work done by R. Sloan, C. K. Young and L. E. Davis [5], and V. H. Mok and L. E. Davis [6]. In [5], a design of an electrically gyrotropic junction circulator using solid-fed gallium arsenide (GaAs) is reported. It operates at 77°K from 50 GHz to 125 GHz with 20 dB isolation bandwidth of 90% at a center frequency of 87 GHz and an insertion loss of 1 dB. In [6] the nonreciprocal wave propagation in three multilayer gyrotropic thin film semiconductor waveguides comprising S-I GaAs/AlAs/ n- GaAs/AlGaAs in a static magnetic field of 0.15 T over the frequency range of 0–200 GHz is analyzed. The wave propagation in a nonreciprocal medium such as cold plasma is best understood by using the *Clemmow-Mually-Allis* (CMA) diagram. The analysis of the CMA diagram is given in [7, 8].

In this paper, the general dispersion relations for an electrically gyrotropic medium are derived using three different methods. The dispersion relations are then put in two distinct forms that can be used based on the problem geometry, i.e., the direction of the stratification of layers. The detailed analysis of wave propagation and the dispersion characteristics for an electrically gyrotropic medium is given. The analysis of the polarization of the waves is simplified and given for the principle waves that exist in a gyroelectric medium. From dispersion relations, we show that there exist two types of waves, type I and type II, in an electrically gyrotropic medium. The wave numbers corresponding to type I and type II waves are used to obtain dispersion relations in closed forms. The *resonance* and *cut off* conditions are investigated for the *principle waves.* To analyze wave propagation and dispersion characteristics for an electrically gyrotropic medium such as cold plasma, *CMA* diagram is constructed using the results obtained. The frequency bands for which wave can propagate in each region are given on this diagram. The findings are then tabulated to show the conditions and the types of the waves that can propagate in each region. Our results can be used in development of nonreciprocal devices and in ionospheric applications such as calculation of radiation fields in an electrically gyrotropic medium.

## 2. DISPERSION RELATIONS

An electrically gyrotropic or a gyroelectric medium is a medium whose relative permittivity and permeability tensors are in the following dyadic form:

$$\overline{\overline{\varepsilon}} = \varepsilon_1 \left( \overline{\overline{I}} - \hat{b}_0 \hat{b}_0 \right) + i\varepsilon_2 \left( \hat{b}_0 \times \overline{\overline{I}} \right) + \varepsilon_3 \hat{b}_0 \hat{b}_0 \tag{1}$$

$$\overline{\overline{\mu}} = \mu \overline{\overline{I}} \tag{2}$$

where $\hat{b}_0$ shows the direction of the applied dc magnetic field $\overline{B}_0$. It is assumed that the fields are time-harmonic with $e^{-i\omega t}$ dependence. When $\overline{B}_0 \equiv \hat{b}_0 B_0 = \hat{z} B_0$, i.e., $\hat{b}_0 = \hat{z} = (0, 0, 1)$, the relative permittivity tensor $\overline{\overline{\varepsilon}}$ takes the following form in matrix notation

$$\overline{\overline{\varepsilon}} = \begin{bmatrix} \varepsilon_1 & -i\varepsilon_2 & 0 \\ i\varepsilon_2 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} \tag{3}$$

The geometry of this structure is shown in Figure 1. The angle between the wave vector $\overline{k}$ and the applied magnetic field $B_0$, is denoted as $\theta$. The wave vector $\overline{k}$ is defined as

$$\overline{k} = \overline{k}_\rho + \hat{z} k_z \tag{4}$$

where

$$\overline{k}_\rho = \hat{x} k_x + \hat{y} k_y \tag{5}$$

then,

$$k^2 = k_x^2 + k_y^2 + k_z^2$$

One example of such a medium is a magnetized plasma where $\mu = 1$ in (2). For a *cold* plasma which is collisionless, the gyrotropic medium is lossless. Hence $\varepsilon_1, \varepsilon_2$, and $\varepsilon_3$ become all real quantities and $\overline{\overline{\varepsilon}}$ is Hermitian as shown in (3) where

$$\varepsilon_1 = 1 - \frac{\omega_p^2}{\omega^2 - {\omega_b}^2} \quad \varepsilon_2 = -\frac{\omega_b \omega_p^2}{\omega \left( \omega^2 - \omega_b^2 \right)} \quad \varepsilon_3 = 1 - \frac{\omega_p^2}{\omega^2} \tag{6a}$$

and

$$\omega_b = \frac{eB_0}{m} \qquad \omega_p = \left( \frac{N_0 e^2}{m \varepsilon_0} \right)^{1/2} \tag{6b}$$

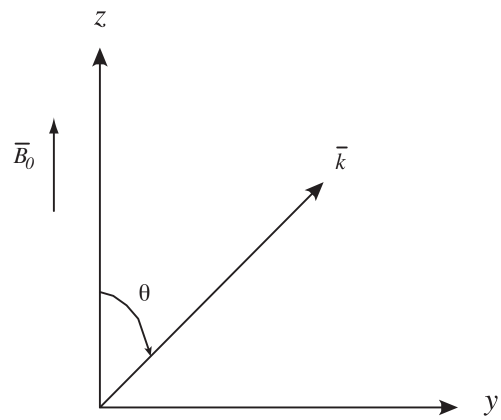


**Figure 1.** Wave propagation in a gyroelectric medium with an arbitrary direction of $\overline{k}$ and applied external magnetic field $\overline{B}_0$.

$\omega_b$ is called the gyrofrequency or cyclotron frequency and $\omega_p$ is called the plasma frequency [9]. $N_0$ shows the number of free electrons per unit volume, and $m$ represents the mass of each electron with charge $-e$. In the following sections throughout the analysis, we assumed that the electrically gyrotropic medium under consideration is a cold plasma for practical considerations.

The Maxwell's equations in the source free region, i.e., $\overline{J} = 0$, $\overline{M} = 0$ (with $e^{-i\omega t}$ dependence) are:

$$\nabla \times \overline{E} = i\omega\mu_0\overline{H} \tag{7}$$
$$\nabla \times \overline{H} = -i\omega\varepsilon_0\overline{\overline{\varepsilon}} \cdot \overline{E} \tag{8}$$

The constitutive relations for a homogeneous lossless gyroelectric medium can be written as,

$$\overline{D} = \varepsilon_0\overline{\overline{\varepsilon}} \cdot \overline{E} \tag{9}$$
$$\overline{B} = \mu_0\overline{H} \tag{10}$$

The dielectric tensor $\overline{\overline{\varepsilon}}$ is given by (3). Assuming the plane wave solution, e.g., $\overline{E} = \overline{E}_0 e^{i\overline{k}\cdot\overline{r}}$, we can rewrite Maxwell's equations (7)–(8) in $k$-*domain* as follows.

$$\overline{k} \times \overline{E}_0 = \omega\mu_0\overline{H}_0 \tag{11}$$
$$\overline{k} \times \overline{H}_0 = -\omega\overline{D}_0 = -\omega\varepsilon_0\overline{\overline{\varepsilon}} \cdot \overline{E}_0 \tag{12}$$

Eliminating $\overline{H}_0$ from (11) and (12) and solving for $\overline{E}_0$ gives

$$\left[k_0{}^2\overline{\overline{\varepsilon}} - k^2\overline{\overline{I}} + \overline{k}\,\overline{k}\right] \cdot \overline{E}_0 = 0 \tag{13}$$

where $k^2 = \overline{k} \cdot \overline{k}$. We define

$$\overline{\overline{W}}_E = \left[k_0^2\overline{\overline{\varepsilon}} - k^2\overline{\overline{I}} + \overline{k}\,\overline{k}\right] \tag{14}$$

as the *electric wave matrix*. Hence (13) can be expressed as

$$\overline{\overline{W}}_E \cdot \overline{E}_0 = 0 \tag{15}$$

A non-zero solution $\overline{E}_0$ exists only if the determinant of the electric wave matrix is zero, i.e.,

$$\left|\overline{\overline{W}}_E\right| = \left|k_0^2\overline{\overline{\varepsilon}} - k^2\overline{\overline{I}} + \overline{k}\,\overline{k}\right| = 0 \tag{16}$$

Equation (16) defines the general *dispersion relation* for an electrically gyrotropic or a gyroelectric medium. Equation (16) has two roots in $k^2$. The two roots $k_I$ and $k_{II}$ represent the wave numbers for the *type I* and the *type II waves*.

In the following subsections, we will express the dispersion relation given in (16) in different forms using three separate methods. In the first method, we will represent the dispersion relation in terms of the wave numbers $k_{I,II}$, which are functions of $\theta$ using tensor analysis. In the second method, we represent the dispersion relation in terms of $k_{zI,zII}$ which are functions of $k_\rho$, the transverse component of the wave vector. In the last method, the dispersion relation is represented by the wave numbers $k_I$ and $k_{II}$ in terms of $\theta$ using field vectors with simple matrix algebra.

### 2.1. Dispersion Relation in Terms of $k$ Using Tensor Analysis — Method I

The dispersion relation for an electrically gyrotropic medium given in (16) can be written as

$$\left|\left(k_0^2\overline{\overline{\varepsilon}} - k^2\overline{\overline{I}}\right) + \overline{k}\,\overline{k}\right| = \left|k_0^2\overline{\overline{\varepsilon}} - k^2\overline{\overline{I}}\right| + \overline{k} \cdot adj\left(k_0^2\overline{\overline{\varepsilon}} - k^2\overline{\overline{I}}\right) \cdot \overline{k} = 0 \tag{17}$$

or

$$\left|\left(k_0^2\overline{\overline{\varepsilon}} - k^2\overline{\overline{I}}\right) + \overline{k}\,\overline{k}\right| = \hat{k} \cdot \left(k_0^2\overline{\overline{\varepsilon}}\right) \cdot adj\left(k_0^2\overline{\overline{\varepsilon}} - k^2\overline{\overline{I}}\right) \cdot \hat{k} = 0 \tag{18}$$

where $\hat{k} = \frac{\overline{k}}{k}$. It can be shown that

$$adj\left(k_0^2\overline{\overline{\varepsilon}} - k^2\overline{\overline{I}}\right) = k^4\overline{\overline{I}} + k_0^2k^2\left(\overline{\overline{\varepsilon}} - \left(\overline{\overline{\varepsilon}}\right)_t\overline{\overline{I}}\right) + k_0^4 adj\overline{\overline{\varepsilon}} \tag{19}$$

where subscript $t$ represents the trace of the matrix.

When (19) is substituted into (18), we obtain

$$k^4\left(\hat{k}\cdot\overline{\overline{\varepsilon}}\cdot\hat{k}\right)+k^2k_0^2\hat{k}\cdot\left[adj\overline{\overline{\varepsilon}}-(adj\overline{\overline{\varepsilon}})_t\,\overline{\overline{I}}\right]\cdot\hat{k}+k_0^4\left|\overline{\overline{\varepsilon}}\right|=0 \tag{20}$$

The following relations can be obtained using (1) as

$$\left|\overline{\overline{\varepsilon}}\right| = \varepsilon_3\left(\varepsilon_1^2-\varepsilon_2^2\right) \tag{21}$$

$$adj\overline{\overline{\varepsilon}} = \varepsilon_1\varepsilon_3\overline{\overline{I}}-i\varepsilon_2\varepsilon_3\left(\hat{b}_0\times\overline{\overline{I}}\right)+(\varepsilon_1-\varepsilon_3)\,\hat{b}_0\hat{b}_0 \tag{22}$$

$$(adj\overline{\overline{\varepsilon}})_t = \varepsilon_1^2-\varepsilon_2^2+2\varepsilon_1\varepsilon_3 \tag{23}$$

Substitution of (21)–(23) into (20) gives

$$k^4\left[\varepsilon_1\sin^2\theta+\varepsilon_3\cos^2\theta\right]+k^2k_0^2\left[\left(\varepsilon_2^2-\varepsilon_1^2+\varepsilon_1\varepsilon_3\right)\sin^2\theta-2\varepsilon_1\varepsilon_3\right]$$
$$+k_0^4\left[\varepsilon_3\left(\varepsilon_1^2-\varepsilon_2^2\right)\right]=0 \tag{24}$$

Note that $\theta$ is the angle between the wave vector $\overline{k}$ and the applied magnetic field $\overline{B}_0$. Equation (24) has two roots in $k^2$. The roots for the fourth order equation in (24) are

$$\frac{k_I^2}{k_0^2}=\frac{1}{2\left[\varepsilon_1\sin^2(\theta)+\varepsilon_3\cos^2(\theta)\right]}\left[\left(\varepsilon_1^2-\varepsilon_2^2\right)\sin^2(\theta)+\varepsilon_1\varepsilon_3\left(1+\cos^2(\theta)\right)\right.$$
$$\left.+\left[\left(\varepsilon_1^2-\varepsilon_2^2-\varepsilon_1\varepsilon_3\right)^2\sin^4(\theta)+4\varepsilon_2^2\varepsilon_3^2\cos^2(\theta)\right]^{1/2}\right] \tag{25}$$

$$\frac{k_{II}^2}{k_0^2}=\frac{1}{2\left[\varepsilon_1\sin^2(\theta)+\varepsilon_3\cos^2(\theta)\right]}\left[\left(\varepsilon_1^2-\varepsilon_2^2\right)\sin^2(\theta)+\varepsilon_1\varepsilon_3\left(1+\cos^2(\theta)\right)\right.$$
$$\left.-\left[\left(\varepsilon_1^2-\varepsilon_2^2-\varepsilon_1\varepsilon_3\right)^2\sin^4(\theta)+4\varepsilon_2^2\varepsilon_3^2\cos^2(\theta)\right]^{1/2}\right] \tag{26}$$

Equation (25) and (26) represent the two types of waves — type I wave which is represented by $k_I$, and type II wave which is represented by $k_{II}$. The wave numbers in (25)–(26) check with the results given in [9, 10]. We can now represent the dispersion relation in (16) using (25)–(26) as

$$\left|\overline{\overline{W}}_E\right|=k_0^2\left[\left(\varepsilon_1\sin^2\theta+\varepsilon_3\cos^2\theta\right)\left(k^2-k_I^2\right)\left(k^2-k_{II}^2\right)\right]=0 \tag{27}$$

### 2.2. Dispersion Relation in terms of $k_z$ — Method II

When Equations (3)–(5) are substituted into (16), we get

$$\left|\overline{\overline{W}}_E\right| = \begin{vmatrix} k_0^2\varepsilon_1 - k_y^2 - k_z^2 & k_xk_y - i\varepsilon_2k_0^2 & k_xk_z \\ k_xk_y + i\varepsilon_2k_0^2 & k_0^2\varepsilon_1 - k_x^2 - k_z^2 & k_yk_z \\ k_xk_z & k_yk_z & k_0^2\varepsilon_3 - k_x^2 - k_y^2 \end{vmatrix} = 0$$

Expansion of $\left|\overline{\overline{W}}_E\right|$ leads to the fourth order equation in $k_z$ as

$$\begin{aligned} \left|\overline{\overline{W}}_E\right| &= k_z^4k_0^2\varepsilon_3 + k_z^2\left[k_0^2k_\rho^2(\varepsilon_1+\varepsilon_3) - 2k_0^4\varepsilon_1\varepsilon_3\right] \\ &\quad + \left[k_0^6\varepsilon_3\left(\varepsilon_1^2-\varepsilon_2^2\right) - k_0^4k_\rho^2\left(\varepsilon_1^2-\varepsilon_2^2+\varepsilon_1\varepsilon_3\right) + k_0^2k_\rho^4\varepsilon_1\right] \\ &= 0 \end{aligned} \tag{28}$$

Equation (28) has two roots in $k_z^2$ as

$$\frac{k_{zI}^2}{k_0^2} = \frac{\left[2\varepsilon_1\varepsilon_3 - \frac{k_\rho^2}{k_0^2}(\varepsilon_1+\varepsilon_3)\right] + \left[\frac{k_\rho^4}{k_0^4}(\varepsilon_1-\varepsilon_3)^2 + 4\varepsilon_2^2\varepsilon_3\left(\varepsilon_3 - \frac{k_\rho^2}{k_0^2}\right)\right]^{1/2}}{2\varepsilon_3} \tag{29}$$

$$\frac{k_{zII}^2}{k_0^2} = \frac{\left[2\varepsilon_1\varepsilon_3 - \frac{k_\rho^2}{k_0^2}(\varepsilon_1+\varepsilon_3)\right] - \left[\frac{k_\rho^4}{k_0^4}(\varepsilon_1-\varepsilon_3)^2 + 4\varepsilon_2^2\varepsilon_3\left(\varepsilon_3 - \frac{k_\rho^2}{k_0^2}\right)\right]^{1/2}}{2\varepsilon_3} \tag{30}$$

Similarly, the wave numbers given in (29) and (30) correspond to the type I and type II waves, respectively. We can express the dispersion relation for an electrically gyrotropic medium in terms of $k_z^2$ using (29)–(30) as

$$\left|\overline{\overline{W}}_E\right| = k_0^2\varepsilon_3\left(k_z^2 - k_{zI}^2\right)\left(k_z^2 - k_{zII}^2\right) = 0 \tag{31}$$

The results obtained in this part are new results and can be used for multilayered gyroelectric media when the stratification is in $z$-direction. Representing the wave vectors in terms of $\theta$ as

$$k_\rho = k\sin\theta \tag{32a}$$
$$k_x = k\sin\theta\cos\phi \tag{32b}$$
$$k_y = k\sin\theta\sin\phi \tag{32c}$$
$$k_z = k\cos\theta \tag{32d}$$

and substituting (32a) and (32d) into (28) leads to (24), which validates the results obtained in (29)–(30).

### 2.3. Dispersion Relation in Terms of $k$ Using Matrix Algebra — Method III

In this section, the dispersion relation given in Method I is obtained using an alternate method, which involves field vectors using simple matrix algebra. For this, we look for the solution of the monochromatic plane wave of the form

$$\overline{E} = \overline{E}_0 e^{i\overline{k}\cdot\overline{r}} \tag{33}$$

$$\overline{H} = \overline{H}_0 e^{i\overline{k}\cdot\overline{r}} \tag{34}$$

The wave vector $\overline{k}$ is defined by (4). The Helmholtz equation can be obtained in the source free region using Maxwell's Equations (7)–(8) as

$$\nabla \times \nabla \times \overline{E} = \omega^2 \mu_0 \varepsilon_0 \overline{\overline{\varepsilon}} \cdot \overline{E} \tag{35}$$

Substituting (33) into (35) gives

$$-\overline{k} \times \overline{k} \times \overline{E} = \omega^2 \mu_0 \varepsilon_0 \overline{\overline{\varepsilon}} \cdot \overline{E} \tag{36}$$

$$\overline{E}_0 - \hat{k}\left(\hat{k} \cdot \overline{E}_0\right) = \frac{\omega^2 \mu_0 \varepsilon_0}{k^2} \overline{\overline{\varepsilon}} \cdot \overline{E}_0 \tag{37a}$$

or

$$\overline{E}_0 - \hat{k}\left(\hat{k} \cdot \overline{E}_0\right) = \frac{k_0^2}{k^2} \overline{\overline{\varepsilon}} \cdot \overline{E}_0 \tag{37b}$$

When (37) is decomposed into its $x, y, z$, components, we obtain

$$E_{0x}\left(k_0^2\varepsilon_1 - k^2 + k^2\sin^2\theta\cos^2\phi\right) + E_{0y}\left(k^2\sin^2\theta\sin\phi\cos\phi - i\varepsilon_2 k_0^2\right) + E_{0z}\left(k^2\sin\theta\cos\theta\cos\phi\right) = 0 \tag{38a}$$

$$E_{0x}\left(k^2\sin^2\theta\sin\phi\cos\phi - ik_0^2\varepsilon_2\right) + E_{0y}\left(k_0^2\varepsilon_1 - k^2 + k^2\sin^2\theta\sin^2\phi\right) + E_{0z}\left(k^2\sin\theta\cos\theta\sin\phi\right) = 0 \tag{38b}$$

$$E_{0x}\left(k^2\sin\theta\cos\theta\cos\phi\right) + E_{0y}\left(k^2\sin\theta\cos\theta\sin\phi\right) + E_{0z}\left(k_0^2\varepsilon_3 - k^2 + k^2\cos^2\theta\right) = 0 \tag{38c}$$

The vector equations given in (38) can be written as

$$[E_{0x}\ \ E_{0y}\ \ E_{0z}]\bullet \begin{bmatrix} k_0^2\varepsilon_1 - k^2 + k^2\sin^2\theta\cos^2\phi & k^2\sin^2\theta\sin\phi\cos\phi - ik_0^2\varepsilon_2 & k^2\sin\theta\cos\theta\cos\phi \\ k^2\sin^2\theta\sin\phi\cos\phi - i\varepsilon_2 k_0^2 & k_0^2\varepsilon_1 - k^2 + k^2\sin^2\theta\sin^2\phi & k^2\sin\theta\cos\theta\sin\phi \\ k^2\sin\theta\cos\theta\cos\phi & k^2\sin\theta\cos\theta\sin\phi & k_0^2\varepsilon_3 - k^2 + k^2\cos^2\theta \end{bmatrix} = 0 \quad (39a)$$

Using (32), we can write (39a) as

$$[E_{0x}\ \ E_{0y}\ \ E_{0z}]\bullet \begin{bmatrix} k_0^2\varepsilon_1 - k^2 + k_x^2 & k_xk_y - ik_0^2\varepsilon_2 & k_xk_z \\ k_xk_y - i\varepsilon_2 k_0^2 & k_0^2\varepsilon_1 - k^2 + k_y^2 & k_yk_z \\ k_xk_z & k_yk_z & k_0^2\varepsilon_3 - k^2 + k_z^2 \end{bmatrix} = 0 \quad (39b)$$

When (39) is carefully reviewed, it is seen that this equation is nothing but the transposed version of Equation (15). Hence, (39) can be rewritten as,

$$\left[[E_{0x}\ \ E_{0y}\ \ E_{0z}]\bullet \begin{bmatrix} k_0^2\varepsilon_1 - k^2 + k_x^2 & k_xk_y - ik_0^2\varepsilon_2 & k_xk_z \\ k_xk_y - i\varepsilon_2 k_0^2 & k_0^2\varepsilon_1 - k^2 + k_y^2 & k_yk_z \\ k_xk_z & k_yk_z & k_0^2\varepsilon_3 - k^2 + k_z^2 \end{bmatrix}\right]^T = \overline{\overline{W}}_E \cdot \overline{E}_0 = 0 \quad (40)$$

As a result, the dispersion relation is obtained using simply matrix method as an alternative to the first method.

## 3. WAVE PROPAGATION IN A GYROELECTRIC MEDIUM

In this section, we analyze the wave propagation in a gyroelectric medium. When the direction of wave propagation coincides with the direction of the imposed magnetic field ($\theta = 0°$, or $180°$), we have the phenomenon known as *longitudinal propagation*. When the direction of wave propagation is perpendicular to the direction of the imposed magnetic field ($\theta = 90°$), we have what is known as *transverse propagation*. The cases $\theta = 0°$ and $\theta = 90°$, which correspond to longitudinal and transverse propagations, generate two uncoupled waves which are called *principal waves* [9]. These special cases are considered in the following subsections with details.

### 3.1. Longitudinal Propagation, $\theta = 0°$

When the propagation is parallel to $\overline{B}_0$ ($\theta = 0°$), (25) and (26) reduce to

$$k_I = k_0\sqrt{\varepsilon_1 + \varepsilon_2} \tag{41}$$
$$k_{II} = k_0\sqrt{\varepsilon_1 - \varepsilon_2} \tag{42}$$

Equation (38) can be rewritten accordingly assuming the propagation is also on the on *yz-plane* ($\phi = 90°$) as shown in Fig. 1 as

$$E_{0x}\left(k_0^2\varepsilon_1 - k^2\right) - i\varepsilon_2 E_{0y} = 0 \tag{43a}$$
$$-i\varepsilon_2 k_0^2 E_{0x} + E_{0y}\left(k_0^2\varepsilon_1 - k^2\right) = 0 \tag{43b}$$
$$E_{0z}\left(k_0^2\varepsilon_3\right) = 0 \tag{43c}$$

From (43c), it is clear that $E_{0z} = 0$. Hence for a longitudinal propagation, there is no electric field component in the direction of propagation. Also, it can be shown that the magnetic field $\overline{H}$ is transverse to the direction of propagation. Consequently, the two waves that travel parallel to $\overline{B}_0$ are transverse electromagnetic (TEM) waves. When (41) is substituted into (43), we obtain

$$\frac{E_{0x}}{E_{0y}} = -i \tag{44a}$$

Equation (44a) corresponds to a right-handed circularly polarized (RHCP) wave. If (42) is substituted into (43), we obtain

$$\frac{E_{0x}}{E_{0y}} = i \tag{44b}$$

Equation (44b) corresponds to a left-handed circularly polarized (LHCP) wave. Therefore, the electric field vectors of the two waves traveling parallel to $\overline{B}_0$ can be written as

$$\overline{E}_I = (\hat{x} + i\hat{y})\, E_I e^{ik_I z} \tag{45a}$$
$$\overline{E}_{II} = (\hat{x} - i\hat{y})\, E_{II} e^{ik_{II} z} \tag{45b}$$

$E_I$ and $E_{II}$ are amplitudes for the field vectors $\overline{E}_I$ and $\overline{E}_{II}$, respectively. $\overline{E}_I$ represents a RHCP wave, and $\overline{E}_{II}$ represents a LHCP wave. The sum of these two waves gives the following composite wave

$$\overline{E}_I + \overline{E}_{II} = \hat{x}\left(E_{II}e^{ik_{II}z} + E_I e^{ik_I z}\right) + \hat{y}\left(-iE_{II}e^{ik_{II}z} + iE_I e^{ik_I z}\right) \tag{46}$$

The polarization of the composite wave in (46) can be found using the ratio $\frac{E_x}{E_y}$ as

$$\frac{E_x}{E_y} = -i\frac{1 + (E_{II}/E_I)\, e^{i(k_{II}-k_I)z}}{1 - (E_{II}/E_I)\, e^{i(k_{II}-k_I)z}} \tag{47}$$

When the amplitudes of $E_I$ and $E_{II}$ are equal, (47) simplifies to

$$\frac{E_x}{E_y} = -\cot\left(\frac{k_{II}-k_I}{2}z\right) \text{ when } E_I = E_{II} \tag{48}$$

If the condition $E_I = E_{II}$ is met, then the ratio in (48) becomes real. As a consequence, the composite wave in (46) at any position $z$ is linearly polarized. Furthermore, the composite wave goes under *Faraday rotation* [9, 11]. Because, the orientation angle of its plane of polarization depends on $z$ and rotates as $z$ changes. For a plasma medium, the electrons circulating along the magnetic field lines cause this effect. From (48), the Faraday rotation angle, $\theta_F$ through which the resultant vector $\overline{E}$ rotates as the wave travels a unit distance can be written as

$$\theta_F = \frac{k_{II}-k_I}{2} \tag{49}$$

Since always $k_I > k_{II}$ from (41)–(42), the rotation is clockwise. Using (41), (42) with (6), $\theta_F$ can be expressed as

$$\theta_F = \frac{k_0}{2}\left[\sqrt{1 - \frac{\omega_p^2}{\omega(\omega+\omega_b)}} - \sqrt{1 - \frac{\omega_p^2}{\omega(\omega-\omega_b)}}\right] \tag{50}$$

## 3.2. Transverse Propagation, $\theta = 90°$

When the direction of wave propagation is perpendicular to the direction of the imposed magnetic field ($\theta = 90°$), we have what is known as *transverse propagation.* When the propagation is perpendicular to $\overline{B}_0$, (25) and (26) reduce to

$$k_I = k_0\sqrt{\frac{\varepsilon_1^2 - \varepsilon_2^2}{\varepsilon_1}} \tag{51}$$

and

$$k_{II} = k_0\sqrt{\varepsilon_3} \tag{52}$$

Equation (38) can be written accordingly as

$$E_{0x}\left(k_0^2\varepsilon_1 - k^2\right) + E_{0y}\left(-i\varepsilon_2 k_0^2\right) = 0 \tag{53a}$$

$$E_{0x}\left(-ik_0^2\varepsilon_2\right) + E_{0y}\left(k_0^2\varepsilon_1\right) = 0 \tag{53b}$$

$$E_{0z}\left(k_0^2\varepsilon_3 - k^2\right) = 0 \tag{53c}$$

When (52) is substituted into (53), $E_{0x}$ and $E_{0y}$ are found to be equal to zero. So, the electric field vector has only the $E_{0z}$ component. Since the propagation constant $k_{II}$ given in (52) is independent of $\overline{B}_0$ and equal to the propagation constant of a wave in the isotropic plasma, this TEM wave, known as an *ordinary wave*, is independent of $\overline{B}_0$ in its propagation properties and behaves as it was a TEM wave in isotropic plasma. Thus, we see that one of the two waves traveling in the $y$ direction is a linearly polarized TEM wave whose electric vector is parallel to $\overline{B}_0$ and has the form

$$\overline{E}_{II} = \hat{z}E_{II}e^{ik_{II}y} \tag{54}$$

When (51) is substituted into (53), it is seen that $E_{0z}$ becomes zero and we can write

$$\frac{E_{0x}}{E_{0y}} = i\frac{\varepsilon_1}{\varepsilon_2} \tag{55}$$

The electric field vector of this wave, known as an *extraordinary wave*, can be put into the following form

$$\overline{E}_I = \left(i\hat{x}\frac{\varepsilon_1}{\varepsilon_2} + \hat{y}\right)E_I e^{ik_1 y} \tag{56}$$

The magnetic field vector $\overline{H}_I$ is obtained by substituting (56) into (8) as

$$\overline{H}_I = -i\hat{z}\frac{k_I}{\omega\mu_0}\frac{\varepsilon_1}{\varepsilon_2}E_I e^{ik_I y} \tag{57}$$

From (56) and (57), we see that the extraordinary wave traveling perpendicular to $\overline{B}_0$ is a transverse magnetic (TM) wave with its magnetic vector parallel to $\overline{B}_0$. Hence from Section 3.1 and Section 3.2 it is seen that the cases $\theta = 0°$ and $\theta = 90°$, which correspond to longitudinal and transverse propagations, generate two uncoupled waves, which are called *principal waves*.

## 4. CUT OFF AND RESONANCE CONDITIONS

In this section, we extend the analysis of wave propagation that is discussed in Section 3, and investigate the cut offs and resonances for principle waves. From the results, we will obtain eight regions for the CMA diagram on $X$-$Y^2$-plane. The term *cut off* for any type of wave occurs when $k = 0$ or the phase velocity, $v_p = \omega/k = \infty$, and the term *resonance* is used when $k = \infty$ or the phase velocity, $v_p = \omega/k = 0$ [7]. Cut offs and resonances distinguish values of the plasma parameters in which $k^2/k_0^2$ is positive or negative and hence the region of propagation or non-propagation. The attenuation $\sqrt{-k^2}$ is small just beyond cut-off but large just beyond resonance. The characteristics of cut-offs and resonances are listed below in Table 1 [8].

**Table 1.** Characteristics of cut-offs and resonances.

| Cut-Off | Resonance |
|---|---|
| $v_p = \infty$ | $v_p = 0$ |
| $k = 0$ | $k = \infty$ |

The dispersion relation given in (24) can be expressed as

$$\begin{aligned} \left|\overline{\overline{W}}_E\right| &= k^4\left[\varepsilon_1\sin^2(\theta)+\varepsilon_3\cos^2(\theta)\right] \\ &\quad +k_0^2k^2\left[\left(\varepsilon_2^2-\varepsilon_1^2-\varepsilon_1\varepsilon_3\right)\sin^2(\theta)-2\varepsilon_1\varepsilon_3\right] \\ &\quad +k_0^4\varepsilon_3\left(\varepsilon_1^2-\varepsilon_2^2\right)=0 \end{aligned} \tag{58}$$

or

$$\left|\overline{\overline{W}}_E\right| = Ak^4+Bk^2+C=0 \tag{59}$$

where

$$A = \left[\varepsilon_1\sin^2(\theta)+\varepsilon_3\cos^2(\theta)\right] \tag{60a}$$

$$B = k_0^2\left[\left(\varepsilon_2^2-\varepsilon_1^2-\varepsilon_1\varepsilon_3\right)\sin^2(\theta)-2\varepsilon_1\varepsilon_3\right] \tag{60b}$$

$$C = k_0^4\varepsilon_3\left(\varepsilon_1^2-\varepsilon_2^2\right) \tag{60c}$$

As seen from the above relations, if $C = 0$ and either $A \neq 0$ or $B \neq 0$, at least one root of the equation is zero. This represents then the cut-off condition. As $C$ is independent of $\theta$, the cut-off condition does not depend on the direction of propagation. Similarly, $A = 0$ represents the resonance condition. This condition is defined by

$$\tan^2(\theta) = -\frac{\varepsilon_3}{\varepsilon_1} \tag{61}$$

In contrast to cut-off condition, resonance condition depends on $\theta$. The expressions in (25) and (26) have always real values.

$$\frac{k_{I,II}^2}{k_0^2} = (\beta - i\alpha)^2 \tag{62}$$

When the value of $k_{I,II}^2/k_0^2$ in (62) is positive, it is equal to $\beta^2$; when it is negative it is equal to $-\alpha^2$. Cold plasma is an example of a gyroelectric medium which satisfies the dispersion relation (16) with the permittivity tensor given by (3). In the magneto-ionic theory it is customary to use notations $X$ and $Y$ which are used to describe the elements of permittivity tensor for the cold plasma in (3). They are given by

$$X = \frac{\omega_p^2}{\omega^2} \qquad Y = \frac{\omega_b}{\omega} \tag{63}$$

$\omega_b$ and $\omega_p$ are defined by (6b). So in our analysis, we will use $X$ and $Y$ to describe the dispersion curves in a gyroelectric medium such as cold plasma for three different cases, namely, the isotropic case, the longitudinal propagation and the transverse propagation. The following results reproduce some of the results given in [7, 8] while the method of analysis in our approach differs from what has been reported before.

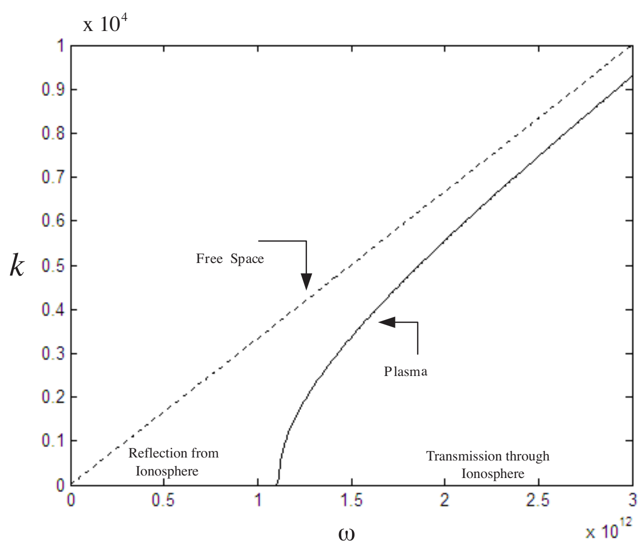


**Figure 2.** $k$-$\omega$ diagram for isotropic plasma when $w_p = 1.11{*}10^{12}$ [rad/sec].

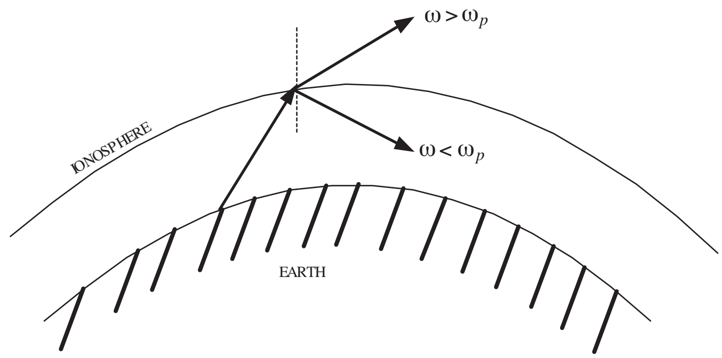


**Figure 3.** Wave propagation in the ionosphere-earth waveguide.

### 4.1. Isotropic Case, No Magnetic Field, $Y = 0$

It is conventional to represent the dispersion of electromagnetic waves by a plot of the propagation constant $k$ against $\omega$ as shown in Figure 2. For a field-free plasma, $\overline{B}_0 = 0$, this gives a hyperbola which *cuts off* below the plasma frequency $\omega_p$. Consequently, radio waves of frequency less than the plasma frequency $\omega_p$ for the ionosphere are reflected back to earth. This can be illustrated in Figure 3. When the external applied magnetic field is zero, i.e., $Y = 0$, (25) and (26) reduce to

$$\frac{k_I^2}{k_0^2} = \frac{k_{II}^2}{k_0^2} = \varepsilon_3 = 1 - X = 1 - \frac{\omega_p^2}{\omega^2} \tag{64}$$

At this stage the plasma is known as *isotropic plasma* because there is no distinction between the type I and type II waves.

### 4.2. The Longitudinal Propagation, $\theta = 0°$

In this section we analyze the cut off and the resonance conditions of the type I and type II waves for the longitudinal propagation. We use $X$-$Y^2$ diagram and $k$-$\omega$ diagram to illustrate the results that we obtain.

#### *4.2.1. The Cut Off and Resonance Conditions for Type I and Type II Waves*

*Case I — Type I Wave*

When the cut off condition is met for the type I wave for the longitudinal propagation, i.e., $\theta = 0°$ and $k_I = 0$ in (41), the cut off frequency can be obtained as

$$\omega_{cI_{long}} = \frac{\omega_b}{2} + \sqrt{\frac{\omega_b^2}{4} + \omega_p^2} \tag{65}$$

For waves to propagate, the necessary condition is $k_I^2 > 0$. This requires that

$$\omega > \omega_{cI_{long}} \quad \text{or} \quad Y < 1 - X \tag{66}$$

When the resonance condition is met for type I wave by setting $k_I = \infty$, we get

$$\omega_{rI_{long}} = \omega_b \quad \text{or} \quad Y = 1 \tag{67}$$

The results given by (65)–(67) can be plotted on the $X$-$Y^2$ plane as shown in Figure 4.

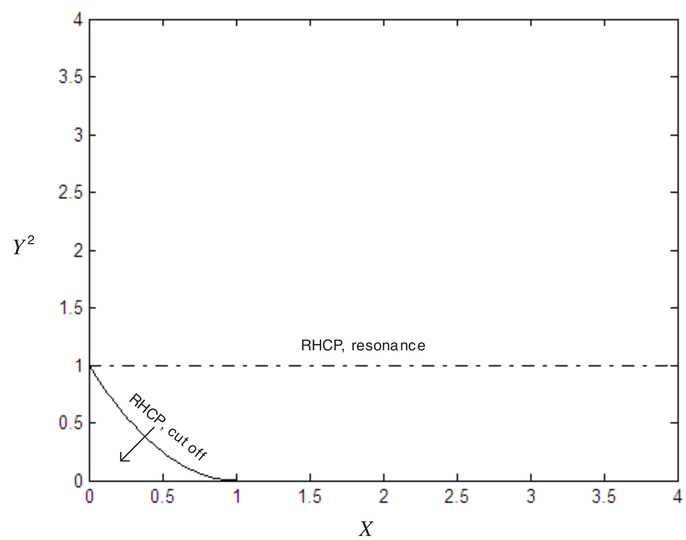


**Figure 4.** $X$-$Y^2$ diagram showing the resonance and cut off conditions for longitudinal propagation $\theta = 0°$, for the type I wave.

*Case II — Type II Wave*

Similarly, if the cut off condition is met for type II wave, we obtain the cut off frequency as

$$\omega_{cII_{long}} = -\frac{\omega_b}{2} + \sqrt{\frac{\omega_b^2}{4} + \omega_p^2} \tag{68}$$

The wave propagation exists for type II wave when $k_{II}^2 > 0$. This requires that

$$\omega > \omega_{cII_{long}} \quad \text{or} \quad Y > X - 1 \tag{69}$$

For longitudinal propagation, no resonance occurs for type II waves. The results can be plotted similarly on the $X$-$Y^2$ plane as shown in Figure 5.

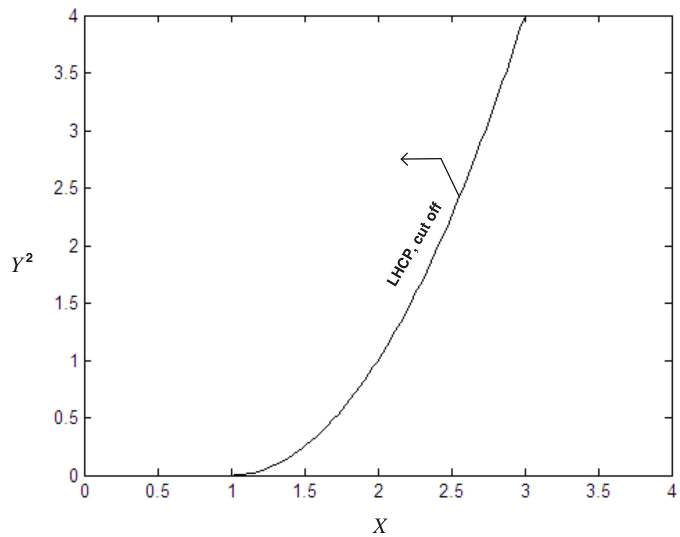


**Figure 5.** $X$-$Y^2$ diagram showing the resonance and cut off conditions for longitudinal propagation $\theta = 0°$, for the type II wave.

The same information given by the $X$-$Y^2$ diagram on Figures 4 and 5 can be plotted using the $k$-$\omega$ diagram illustrated as in Figure 6.

### 4.3. Transverse Propagation, $\theta = 90°$

In this section we analyze the cut off and the resonance conditions of the type I and type II waves for the transverse propagation. Similar to Section 4.2, we use $X$-$Y^2$-diagram and $k$-$\omega$ diagram to illustrate the results that we obtain.

*4.3.1. The Cut Off and Resonance Conditions for Type I and Type II Waves*

*Case I — Type I Wave*

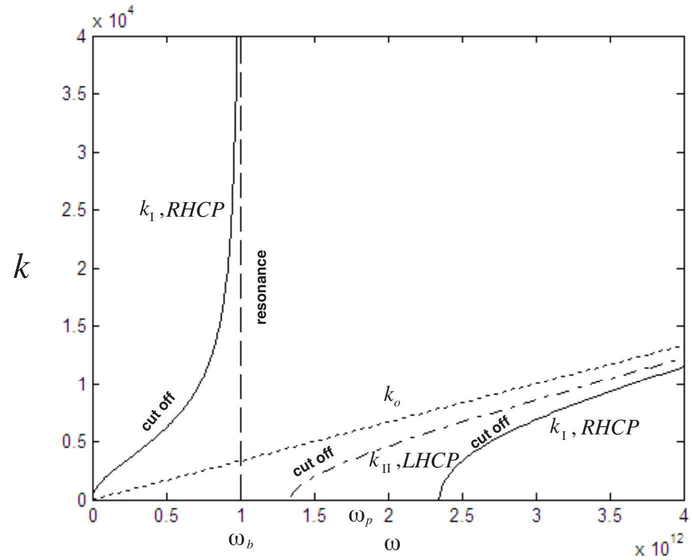


**Figure 6.** $k$-$\omega$ diagram showing the resonance and cut off conditions for longitudinal propagation $\theta = 0°$ of the type I and type II waves when $\omega_p^2/\omega_b^2 = 10^{1/2}$, $w_b = 1{*}10^{12}$ [rad/sec].

When the cut off condition is met for the type I wave, from Eq. (51), we derive the cut off frequency as

$$\omega_{cI_{tran}} = \pm\frac{\omega_b}{2} + \sqrt{\frac{\omega_b^2}{4} + \omega_p^2} \tag{70}$$

For waves to propagate, the necessary condition is $k_I^2 > 0$. This requires that

$$\omega > \omega_{cI_{tran}} \tag{71}$$

or in terms of $X$ and $Y$ notation

$$Y < 1 - X \quad \text{when} \quad \omega_{cI_{tran}} = \frac{\omega_b}{2} + \sqrt{\frac{\omega_b^2}{4} + \omega_p^2} \tag{72}$$

and

$$Y > 1 - X \quad \text{when} \quad \omega_{cI_{tran}} = -\frac{\omega_b}{2} + \sqrt{\frac{\omega_b^2}{4} + \omega_p^2} \tag{73}$$

When the resonance condition is met for the type I wave using (51),

$$\omega_{rI_{tran}} = \sqrt{\omega_b^2 + \omega_p^2} \quad \text{or} \quad Y^2 = 1 - X \tag{74}$$

The results given by (70)–(74) can be plotted on the $X$-$Y^2$ plane as shown in Figure 7.

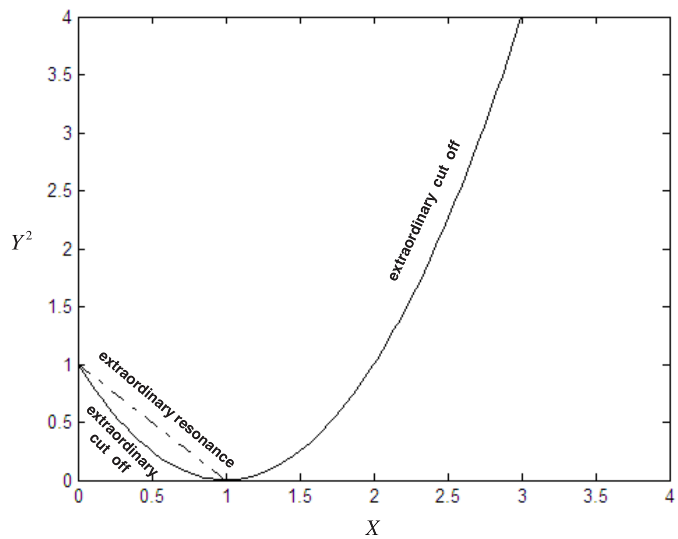


**Figure 7.** $X$-$Y^2$ diagram showing the resonance and cut off conditions for transverse propagation $\theta = 90°$, for the type I wave.

*Case II — Type II Wave*

When the cut off condition is met for type II wave, we obtain the cut off frequency from (52) as

$$\omega_{cII_{tran}} = \omega_p \tag{75}$$

For waves to propagate, the necessary condition is $k_{II}^2 > 0$. This requires that

$$\omega > \omega_{cII_{tran}} \quad \text{or} \quad X < 1 \tag{76}$$

No resonance occurs for type II wave when there is transverse propagation. The results can be plotted similarly on the $X$-$Y^2$-plane as shown in Figure 8.

The same information given by the $X$-$Y^2$ diagram on Figures 7 and 8 can be plotted using the $k$-$\omega$ diagram illustrated as in Figure 9.

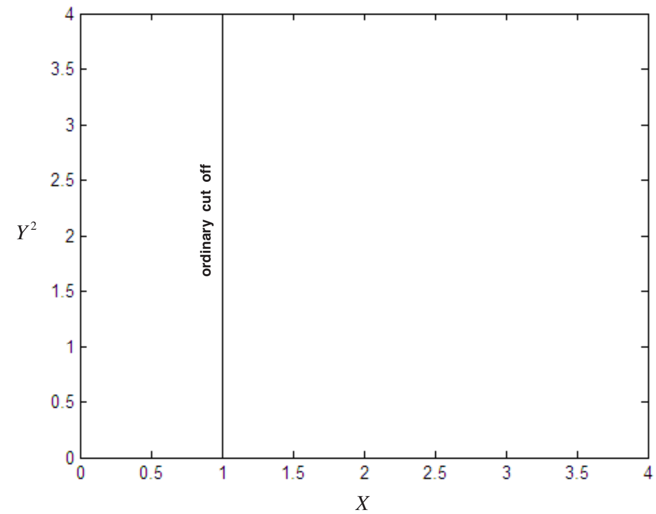


**Figure 8.** $X$-$Y^2$ diagram showing the resonance and cut off conditions for transverse propagation $\theta = 90°$ for the type II wave.

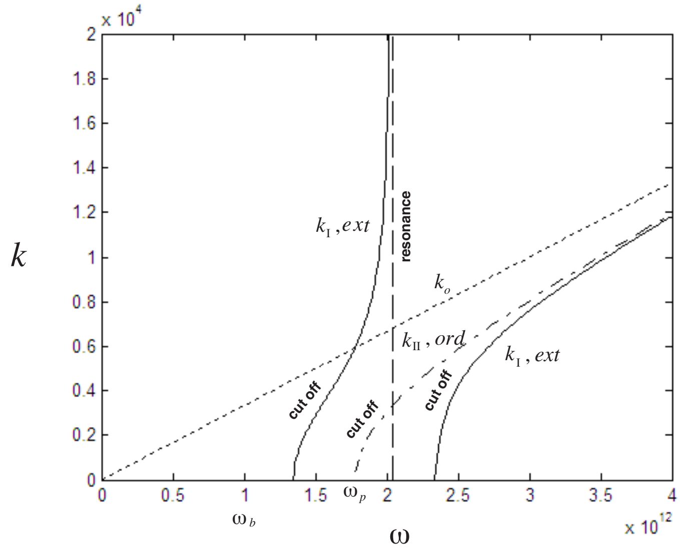


**Figure 9.** $k$-$\omega$ diagram showing the resonance and cut off conditions for transverse propagation $\theta = 90°$ of the type I and type II waves when $\omega_p^2/\omega_b^2 = 10^{1/2}$.

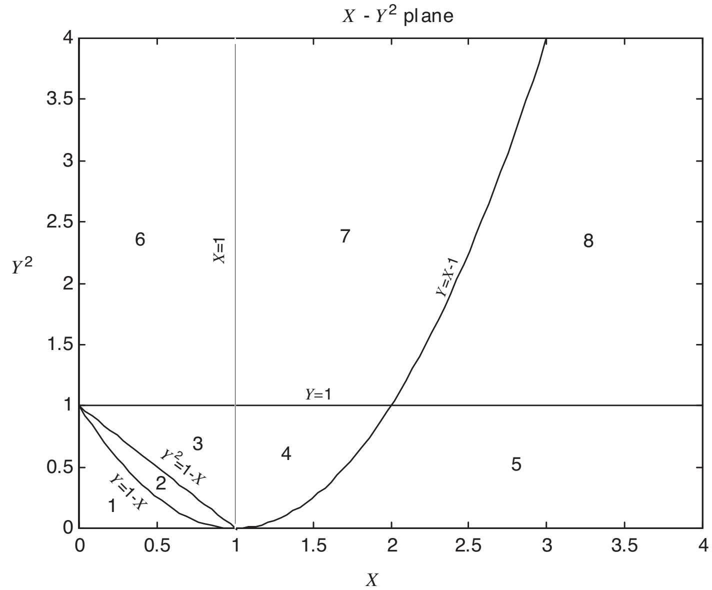


**Figure 10.** The $X$-$Y^2$ plane divided into eight regions.

## 5. CLEMMOW-MUALLY-ALLIS (CMA) DIAGRAM

Overall, we identified the boundaries using principle waves on the $X$-$Y^2$ plane with Equations (65)–(76) which divide this plane into eight regions. As a result, we obtained the frequency bands for each region over which wave can propagate. Hence, we not only constructed a single diagram showing the boundaries for the cut off and resonance conditions for the principle waves but also we identified the frequency bands in each region over which wave can propagate with an arbitrary angle to the magnetic field. This is illustrated in Fig. 10. This figure agrees with the results of H. Weil and D. Walsh [12].

When the wave normal surfaces are plotted for each region shown in Fig. 10, we obtain the CMA diagram. The wave normal surface is the polar plot of the phase velocity where the distance from the center to the point on the curves denotes the magnitude of the phase velocity in that direction. When the wave normal surface is plotted for each region, it illustrates the propagating modes and non-propagating modes for type I and type II waves. This makes CMA diagram a powerful tool to analyze the wave propagation in a gyroelectric medium such as cold plasma. The CMA diagram in Fig. 11 is obtained using our results and agrees with the results given in [7]. The results shown in the

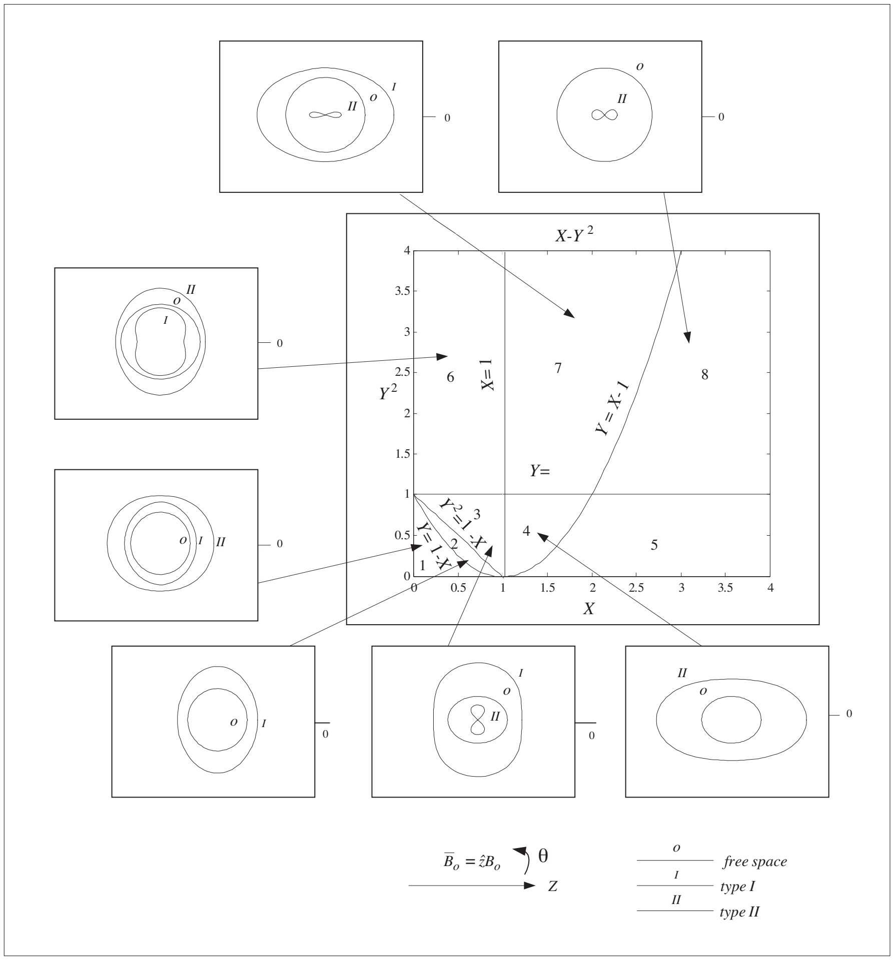


**Figure 11.** CMA diagram for a cold plasma.

CMA diagram are tabulated in Table 2 to show the frequency bands and the corresponding waves that can propagate in each region, which is reported first time on this paper according to our knowledge. By using Table 2 that we introduce, it is possible to choose the frequency of operation in the region of propagation or non-propagation for type I or type II wave without going through any analysis on the CMA diagram. This feature is a need to make the design nonreciprocal devices predictable and practical. The results in Table 2 also can be used in the radiation or scattering problems when the medium under consideration is cold plasma.

**Table 2.** Region of frequency bands for propagation on the CMA diagram.

| | Frequency Band | Wave Propagates |
|---|---|---|
| **Region 1** | $\omega > \omega_1,\ \omega_1 = \frac{\omega_b}{2} + \sqrt{\frac{\omega_b^2}{4} + \omega_p^2}$ | Type I, Type II |
| **Region 2** | $\omega_1 > \omega > \omega_2,\ \omega_2 = \sqrt{\omega_b^2 + \omega_p^2}$ | Type I |
| **Region 3** | $\omega_2 > \omega > \max(\omega_p, \omega_b)$ | Type I, Type II |
| **Region 4** | $\omega_p > \omega > \max(\omega_b, \omega_3),\ \omega_3 = \sqrt{\frac{\omega_b^2}{4} + \omega_p^2} - \frac{\omega_b}{2}$ | Type II |
| **Region 5** | $\omega_3 > \omega > \omega_b$ | No Propagation |
| **Region 6** | $\omega_b > \omega > \omega_p$ | Type I, Type II |
| **Region 7** | $\min(\omega_p, \omega_b) > \omega > \omega_3$ | Type I, Type II |
| **Region 8** | $\min(\omega_3, \omega_b) > \omega > 0$ | Type II |

## 6. CONCLUSION

In this paper, the general dispersion relations for a gyroelectric medium are derived using three different methods. One of them is a new method which can be applied when the stratification of the layers is in the $z$-direction. The dispersion relations are then obtained in two distinct forms. From dispersion relations, we show that there exist two types of waves in a gyroelectric medium. The detailed analysis of wave propagation and the dispersion characteristics of a gyroelectric medium is presented. The polarization of the waves for a gyroelectric medium is analyzed and given for the principle waves. The resonance and cut off conditions are investigated for the principle waves. The wave propagation and the dispersion characteristics for a gyroelectric medium such as cold plasma are inspected by constructing the CMA diagram. The frequency bands for which wave can propagate in each region are given on this diagram. Our results obtained for the dispersion relations of a gyroelectric medium given in Section 2 and the dispersion curves with the CMA diagram given in Section 4 are compared with the existing results whenever possible. The results are tabulated for the first time to show the conditions and types of the waves that can propagate when the specific condition is met in each region on the CMA diagram. The results presented in this paper can be used in the development of the nonreciprocal devices, ionospheric radiation and scattering problems.

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