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SOME ELLIPTIC TRAVELING WAVE SOLUTIONS TO THE NOVIKOV-VESELOV EQUATION

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Abstract—An approach is proposed to obtain some exact explicit solutions in terms of elliptic functions to the Novikov-Veselov equation (NVE$[V(x,y,t)] = 0$). An expansion ansatz $V \to \psi = \sum_{j=0}^{2} a_j f^j$ is used to reduce the NVE to the ordinary differential equation $(f')^2 = R(f)$, where $R(f)$ is a fourth degree polynomial in f. The well-known solutions of $(f')^2 = R(f)$ lead to periodic and solitary wave like solutions V. Subject to certain conditions containing the parameters of the NVE and of the ansatz $V \to \psi$ the periodic solutions V can be used as start solutions to apply the (linear) superposition principle proposed by Khare and Sukhatme.

1. INTRODUCTION

The Novikov-Veselov (NV) equation [1] as a "natural" two-dimensional generalization of the celebrated Korteweg-de Vries (KV) equation [2] has relevance in nonlinear physics (in particular in inverse scattering theory) [3, 4] and mathematics (cf. e.g., [5, 6]).

As regards to physics, Tagami [3] derived solitary solutions of the NV equation by means of the Hirota method. Cheng [4] investigated the NV equation associated with the spectral problem $(\partial_x\partial_y + u)\psi = 0$ in the plane and presented solutions by applying the inverse scattering transform. With regards to mathematics, Taimanov [5] investigated applications of the (modified) NV equation to differential geometry of surfaces. Ferapontov [6] used the (stationary) NV equation to describe a certain class of surfaces in projective differential geometry (the so-called isothermally asymptotic surfaces). Apart from these applications solutions of the NV equation are interesting in and of themselves.

In the following we derive some solutions of the NV equation by combining a symmetry reduction method [7, 8] and the Khare-Sukhatme superposition principle [9–12].

2. ELLIPTIC SOLUTIONS

2.1. General Considerations

Following Novikov and Veselov [1] we consider the system

$$V_t = \partial^3 V + \overline{\partial}^3 V + 3\partial(uV) + 3\overline{\partial}(\overline{u}V), \tag{1}$$

$$\overline{\partial}u = \partial V, \tag{2}$$

where $\partial = \frac{1}{2}(\partial_x - i\partial_y)$, $\overline{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$ are the Cauchy-Riemann operators in $\mathbb{R}^2$. System (1), (2) is equivalent to

$$V_t = \frac{1}{4}(V_{xxx} - 3V_{xyy}) + 3V(u_{1x} + u_{2y}) + 3(u_1V_x + u_2V_y), \tag{3}$$

$$V_x = u_{1x} - u_{2y},\ -V_y = u_{1y} + u_{2x} \tag{4}$$

with $u(x, y, t) = u_1(x, y, t) + iu_2(x, y, t)$, where u is defined up to an arbitrary holomorphic function $\varphi = \varphi_1 + i\varphi_2$ so that $\varphi_{1x} = \varphi_{2y}$, $\varphi_{1y} = -\varphi_{2x}$. (4) imply

$$u_1 = -2\partial_x^{-1}\partial_y\widetilde{D}V + V + \varphi_1,\ \ u_2 = -2\widetilde{D}V + \varphi_2. \tag{5}$$

The operator $\widetilde{D} := (\partial_x^{-1}\partial_y + \partial_y^{-1}\partial_x)^{-1}$ is well-defined [13, (6)], so that u_1, u_2 can be inserted into (3). Traveling wave solutions

$$V(x, y, t) = \psi(z),\ z = x + ky - vt \tag{6}$$

imply $\partial_x^{-1} = k\partial_y^{-1}$ and thus lead to $\varphi \equiv \text{const.} = C_0 + iC_1$. Hence, (3) can be written as

$$-v\psi_z = \frac{1 - 3k^2}{4}\psi_{zzz} + 6\frac{1 - 3k^2}{k^2 + 1}\psi\psi_z + 3\psi_z(C_0 + C_1k). \tag{7}$$

Following an approach outlined previously [7, 8, 14] it seems useful to find elliptic (traveling wave) solutions of the form ($p = 2$ follows from balancing the linear term of highest order with the nonlinear term in (7))

$$\psi(z) = \sum_{j=0}^{p} a_j f(z)^j, \quad p = 2 \tag{8}$$

with [15]

$$\left(\frac{df(z)}{dz}\right)^2 = \alpha f^4 + 4\beta f^3 + 6\gamma f^2 + 4\delta f + \epsilon \equiv R(f). \tag{9}$$

The coefficients a_0, a_1, a_2, α, β, γ, δ, ϵ are assumed to be real but otherwise either arbitrary or interrelated.

Inserting (8) into (7) and using (9) we obtain a system of algebraic equations that can be reduced to yield the nontrivial solutions

$$\alpha = 0, \;\; \beta = -\frac{2a_1}{1+k^2}, \;\; \gamma = -\frac{4a_0}{1+k^2} + \frac{2F}{3(3k^2-1)}, \;\; \delta, \;\; \epsilon \text{ arbitrary},$$
$$\text{subject to} \quad a_2 = 0, \; 3k^2 - 1 \neq 0, \tag{10}$$

$$\alpha = -\frac{2a_2}{1+k^2}, \; \beta = -\frac{a_1}{1+k^2}, \; \gamma = \frac{F}{6(3k^2-1)} - \frac{a_1^2 + 4a_0a_2}{4a_2(1+k^2)},$$
$$\delta = \frac{1}{8a_2^2}\left(\frac{a_1^3 - 12a_0a_1a_2}{1+k^2} + \frac{2a_1a_2F}{3k^2-1}\right), \; \epsilon \text{ arbitrary},$$
$$\text{subject to} \quad a_2 \neq 0, \; 3k^2 - 1 \neq 0 \tag{11}$$

with $F = v + 3C_0 + 3kC_1$.

Thus, the coefficients of the polynomial $R(f)$ are (partly) determined leading to solutions $f(z)$ of (9). As is well known [15, p. 4–16], [16, p. 454] $f(z)$ can be expressed in terms of Weierstrass' elliptic function $\wp(z; g_2, g_3)$ according to

$$f(z) = f_0 + \frac{R'(f_0)}{4\left[\wp(z; g_2, g_3) - \frac{1}{24}R''(f_0)\right]}, \tag{12}$$

where the primes denote differentiation with respect to f and f_0 is a simple root of $R(f)$.

The invariants g_2, g_3 of $\wp(z; g_2, g_3)$ and the discriminant $\Delta = g_2^3 - 27g_3^2$ are related to the coefficients of $R(f)$ [17, p. 44]. They are suitable to classify the behaviour of $f(z)$ and to discriminate between periodic and solitary wave like solutions [8].

Solitary wave like solutions are determined by (cf. (12) and Ref. [18, p. 651–652])

$$f(z) = f_0 + \frac{R'(f_0)}{4\left[e_1 - \dfrac{R''(f_0)}{24} + 3e_1 \operatorname{csch}^2(\sqrt{3e_1}z)\right]}, \quad \triangle = 0, \quad g_3 < 0, \tag{13}$$

where $e_1 = \frac{1}{2}\sqrt[3]{|g_3|}$ in (13).

In general, $f(z)$ (according to (12)) is neither real nor bounded. Conditions for real and bounded solutions $f(z)$ can be obtained by considering the phase diagram of $R(f)$ [19, p. 15]. They are denoted as "phase diagram conditions" (PDC) in the following. An example of a phase diagram analysis is given in [14].

2.2. Periodic Solutions

At first the coefficients according to (10) are considered. For simplicity we assume $\epsilon = 0$, so that $f_0 = 0$ is a simple root of (9). The solution (12) can be evaluated to yield

$$V(x,y,t) = a_0 + a_1 \cdot \frac{3(1+k^2)(1-3k^2)\delta}{(1+k^2)F + 6a_0(1-3k^2) + 3(1+k^2)(1-3k^2)\wp(x+ky-vt; g_2, g_3)} \tag{14}$$

with g_2, g_3 according to (10) and [8].

Evaluating (12) with coefficients according to (11) (with $\epsilon = 0$ for simplicity) in the same manner we obtain periodic solutions depending on a_0, a_1 and a_2.

2.3. Solitary Wave Like Solutions

To find the subset of solitary wave like solutions of the NV equation according to (10), (13) the discriminant Δ must vanish. This is given if $\delta = 0$ or $\delta = -\frac{(6a_0(1-3k^2)+(1+k^2)F)^2}{8a_1(1-3k^2)^2(1+k^2)}$.

For $g_3 < 0$ we obtain solitary wave like solutions and here the PDC is fulfilled automatically for $g_3 < 0$.

If $\delta = 0$, $\epsilon = 0$, $f_0 = \frac{6a_0(1-3k^2)+(1+k^2)F}{2a_1(3k^2-1)}$, we obtain (cf. (8), (13))

$$\begin{aligned} V(x,y,t) \;&=\; a_0 + \frac{6a_0(1-3k^2)+(1+k^2)F}{2(3k^2-1)}\text{sech}^2 \\ &\cdot\left[\sqrt{-\frac{6a_0}{1+k^2}+\frac{F}{3k^2-1}}(x+ky-vt)\right]. \end{aligned} \tag{15}$$

If $\delta = -\frac{(6a_0(1-3k^2)+(1+k^2)F)^2}{8a_1(1-3k^2)^2(1+k^2)}$, $\epsilon = 0$, $f_0 = 0$, (8) reads

$$\begin{aligned} V(x,y,t) \;&=\; a_0 + \frac{6a_0(1-3k^2)+(1+k^2)F}{4(3k^2-1)}\tanh^2 \\ &\cdot\left[\sqrt{\frac{F}{2(1-3k^2)}+\frac{3a_0}{1+k^2}}(x+ky-vt)\right]. \end{aligned} \tag{16}$$

Subject to (10) (15), (16) represent general physical traveling solitary wave solutions of the NV equation for $\epsilon = 0$. While periodic solutions depend on a_0 and a_1, solitary solutions only depend on a_0.

Solitary wave like solutions according to (11) can be obtained by an analogous procedure.

3. SUPERPOSITION SOLUTIONS

Khare and Sukhatme proposed a superposition principle for nonlinear wave and evolution equations (NLWEEs) [9]. They have shown that suitable linear combinations of periodic traveling-wave solutions expressed by Jacobian elliptic functions lead to new solutions of the nonlinear equation in question. Combining the approach above with this superposition principle we have evaluated the following start solutions for superposition [20]

$$f(z) = \begin{cases} -\frac{3\gamma+\sqrt{9\gamma^2-16\beta\delta}}{4\beta}\text{dn}^2\left(\frac{1}{2}\sqrt{3\gamma+\sqrt{9\gamma^2-16\beta\delta}}\,z, \frac{2\sqrt{9\gamma^2-16\beta\delta}}{3\gamma+\sqrt{9\gamma^2-16\beta\delta}}\right), & \beta\delta > 0,\ \gamma > 0, \\ \frac{4\delta}{-3\gamma+\sqrt{9\gamma^2-16\beta\delta}}\text{sn}^2\left(\frac{1}{2}\sqrt{-3\gamma+\sqrt{9\gamma^2-16\beta\delta}}\,z, \frac{3\gamma+\sqrt{9\gamma^2-16\beta\delta}}{3\gamma-\sqrt{9\gamma^2-16\beta\delta}}\right), & \beta\delta > 0,\ \gamma < 0, \\ -\frac{3\gamma+\sqrt{9\gamma^2-16\beta\delta}}{4\beta}\text{cn}^2\left(\frac{(9\gamma^2-16\beta\delta)^{\frac{1}{4}}}{\sqrt{2}}z, \frac{3\gamma+\sqrt{9\gamma^2-16\beta\delta}}{2\sqrt{9\gamma^2-16\beta\delta}}\right), & \beta\delta < 0. \end{cases} \tag{17}$$

In (10) we choose $\epsilon = 0$ for simplicity and thus, we obtain start solutions for superposition according to (17). As an example we consider solutions of the form dn^2 for $p = 3$, further results for cn^2, sn^2 and according to (11) can be obtained in the same manner.

According to (8), (10) the start solution for superposition reads

$$V(x,y,t) = a_0 + a_1\, A\, \mathrm{dn}^2(\mu(x + ky - vt), m), \tag{18}$$

with A, μ, m according to (17), so that the superposition ansatz can be written as

$$\widetilde{V}(x,y,t) = a_0 + a_1\, A \sum_{i=1}^{3} \mathrm{dn}^2 \left[\mu(x + k\, y - v_3\, t) + \frac{2(i-1)K(m)}{3}, m\right]. \tag{19}$$

Inserting $\widetilde{V}(x,y,t)$ (denoting $\mathrm{d}_i = \mathrm{dn}\Big(\mu(x + ky - v_3 t) + \frac{2(i-1)K(m)}{3}, m\Big)$) into (7) ($v \to v_3$) and using well known relations for c_i^2 and s_i^2 [22, p. 16] leads to

$$\begin{aligned}
&6Aa_1\mu m(1-3k^2)\left(\mu^2 - \frac{2Aa_1}{1+k^2}\right)\sum_{i=1}^{3} \mathrm{c}_i \mathrm{d}_i^3 \mathrm{s}_i \\
&-\frac{12A^2a_1^2 m\mu(1-3k^2)}{1+k^2}\sum_{i=1}^{3}\mathrm{d}_i^2 \sum_{j\neq i}^{3} \mathrm{c}_j \mathrm{d}_j \mathrm{s}_j \\
&-2Aa_1\mu m\left(\frac{6a_0(1-3k^2)}{1+k^2} + 3(C_0+C_1k) + (2-m)(1-3k^2)\mu^2 + v_3\right) \\
&\sum_{i=1}^{3}\mathrm{c}_i\mathrm{d}_i\mathrm{s}_i = 0.
\end{aligned} \tag{20}$$

Remarkably, $\mu^2 - \frac{2Aa_1}{1+k^2}$ vanishes automatically [20, (13)]. By use of [23], (20) reads

$$\begin{aligned}
&-2Aa_1\mu m\left(\frac{6a_0(1-3k^2)}{1+k^2} + 3(C_0+C_1k) + (2-m)(1-3k^2)\mu^2 + v_3\right) \\
&\sum_{i=1}^{3}\mathrm{c}_i\mathrm{d}_i\mathrm{s}_i - 2Aa_1\mu m\left(-\frac{12Aa_1(1-3k^2)(m-1+q^2)}{(1+k^2)(1-q^2)}\right)\sum_{i=1}^{3}\mathrm{c}_i\mathrm{d}_i\mathrm{s}_i = 0.
\end{aligned} \tag{21}$$

Thus, the speed v_3 in the superposition solution (19) is given by

$$v_3 = \frac{6a_0(3k^2-1)}{1+k^2} - 3(C_0 + C_1 k) + (2-m)(3k^2-1)\mu^2 + \frac{12Aa_1(3k^2-1)(m-1+q^2)}{(1+k^2)(q^2-1)}. \tag{22}$$

The start solution V and the superposition solution $\widetilde{V}$ are shown in Fig. 1.

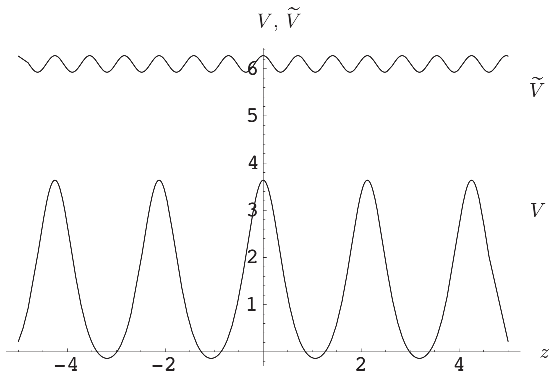

Figure 1. V and $\widetilde{V}$ (cf. (18), (19)) for $c = -1$, $k = 1$, $a_0 = -1$, $a_1 = -1$, $C_0 = 1$, $C_1 = 1$, $\delta = 4$ (therefore: $v_3 = -8.66008$).

4. CONCLUSION

For the NV equation we have shown that a rather broad set of traveling wave solutions according to (6), (8) and subject to the nonlinear ordinary differential equation (9) can be obtained. Periodic and solitary wave solutions can be presented in compact form in terms of Weierstrass' elliptic function and its limiting cases ($\triangle = 0$, $g_3 \leq 0$), respectively. The phase diagram conditions (PDC) yield constraints for real and bounded solutions. Finally, it is shown that application of the Khare-Sukhatme superposition principle yields new periodic (real, bounded) solutions of the NV equation.

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23. cf. Ref. [12], Eq. (11); cf. Ref. [9], Eqs. (7), (8).