Progress In Electromagnetics Research, PIER 64, 219–238, 2006

ANALYSIS OF LINEAR TAPERED WAVEGUIDE BY TWO APPROACHES

S. Dwari, A. Chakraborty, and S. Sanyal

Department of Electronics and Electrical Communication Engineering
Indian Institute of Technology
Kharagpur-721302, India

Abstract—This paper presents the analysis of linear tapered waveguide. Voltage-standing-wave-ratio (VSWR) is obtained from transmission matrix of the taper waveguide. Taper section is divided into number of section having uniform length. Transmission matrix of taper waveguide is found by multiplication of transmission matrix of each section. Transmission matrix of each section is obtained as the product of three matrices. One is of the initial length of transmission line, second one is due to discontinuity and third one is of the final length of transmission line. Transmission matrix of discontinuity is obtained by two methods. One is by equivalent circuit of step discontinuity and another is by moment method. The results are seen to be in good agreement with [1, 2] and [3].

1. INTRODUCTION

In a tapered section the dimension of waveguide varies smoothly and for this reason there are possibilities of providing a transition from one impedance level to another. Ref. [4] and then [5] analyzed taper line by dividing it into a number of sections of equal length into the direction of propagation. Ref. [1] found the transmission matrix of each section as the product of five matrices and multiplied the matrices of all sections.

This paper proposed that the transmission matrix of each section is obtained as the product of three matrices. They are contributed by initial length of transmission line, discontinuity and the final length of transmission line. Transmission matrix of discontinuity is obtained by two procedures. They are by equivalent circuit of step discontinuity and by moment method. The final transmission matrix of tapered waveguide is obtained by multiplication of matrices of all sections. Voltage-standing-wave-ratio (VSWR) is obtained from final transmission matrix.

2. TRANSMISSION MATRIX OF TAPER SECTION

The linear tapered waveguide is shown in Figure 1 and it can be considered as if it is formed with huge number of step discontinuities. Let the number of sections be N and the length of taper be L. Let $2a_0, 2b_0$ are initial broad and narrow dimensions respectively and $2a_1, 2b_1$ are respective final broad and narrow dimensions.

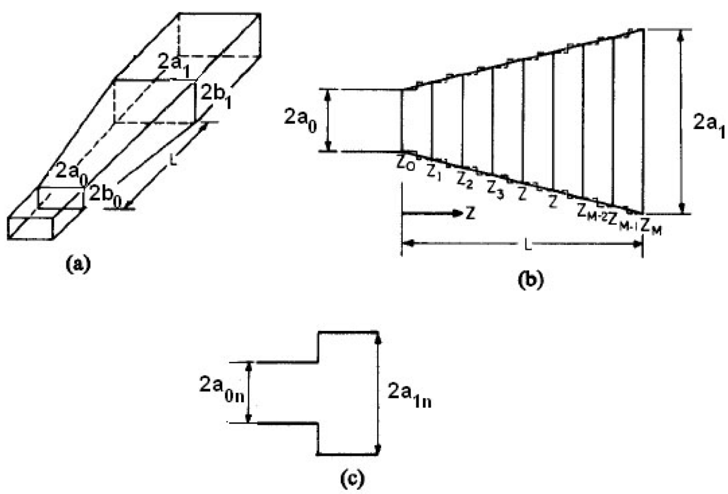

Figure 1. (a) Tapered waveguide of length L connecting uniform waveguides of dimensions $2a_0, 2b_0$ and $2a_1, 2b_1$. (b) Taper waveguide of Figure 1(a) divided into sections. (c) Expanded view of nth section.

The nth section is blown up and shown in Figure 1(c). Each section consists of an initial length of waveguide, a step discontinuity and a final length of waveguide. Each of the constituents can be expressed by a transmission matrix of order 2.

For nth section initial and final broad dimensions are

$$2a_{0n} = 2a_0 + \frac{2(a_1 - a_0)}{N}(n-1) \tag{1}$$

$$2a_{1n} = 2a_0 + \frac{2(a_1 - a_0)}{N}n \tag{2}$$

Narrow wall dimensions are

$$2b_{0n} = 2b_0 + \frac{2(b_1 - b_0)}{N}(n-1) \tag{3}$$

$$2b_{1n} = 2b_0 + \frac{2(b_1 - b_0)}{N}n \tag{4}$$

The matrix of each section is the product of three matrices. Let matrix $[T_1]$ is for initial length, $[T_3]$ is of final length and $[T_2]$ is for discontinuity. So transmission matrix of each section is given by

$$[T] = [T_1][T_2][T_3] \tag{5}$$

The transmission matrices of all the sections are multiplied to obtain the final transmission matrix of the taper.

Suppose $d = \frac{L}{N}$ is the length of each section, k_{0n} is the propagation constant of initial length and k_{1n} is the propagation constant of final length of nth section. Then $[T_1]$ can be written as

$$[T_1] = \begin{bmatrix} e^{jk_{0n}\frac{d}{2}} & 0 \\ 0 & e^{-jk_{0n}\frac{d}{2}} \end{bmatrix} \tag{6}$$

$[T_3]$ can be written as

$$[T_3] = \begin{bmatrix} e^{jk_{1n}\frac{d}{2}} & 0 \\ 0 & e^{-jk_{1n}\frac{d}{2}} \end{bmatrix} \tag{7}$$

Let Γ_{1n} be the reflection co-efficient in the forward direction at discontinuity of nth section and Γ_{2n} be the reflection co-efficient in the backward direction at discontinuity of the nth section. Then $[T_2]$ can be written as

$$[T_2] = \frac{1}{1+\Gamma_{1n}} \begin{bmatrix} 1 & -\Gamma_{2n} \\ \Gamma_{1n} & 1+\Gamma_{1n}+\Gamma_{2n} \end{bmatrix} \tag{8}$$

To obtain $[T_2]$, reflection co-efficients should be determined. Reflection coefficient can be determined by two ways. One is using equivalent circuit representing step discontinuities shown in Figure 2 as described in [6]. Second is by using method of moment.

3. DETERMINATION OF CIRCUIT PARAMETERS OF EQUIVALENT CIRCUIT FOR STEP DISCONTINUITIES

Discontinuity of each section of linear taper waveguide is step discontinuity. This step discontinuity can be represented by equivalent

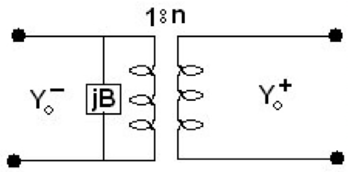

Figure 2. Equivalent circuit of discontinuity.

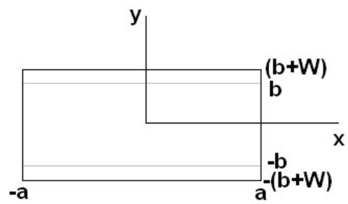

Figure 3. Cross section of waveguide junction for step discontinuity at narrow dimension.

circuit as shown in Figure 2. Following [6] the circuit parameters of the equivalent circuit can be obtained. From circuit parameters reflection coefficients can be calculated. Evaluation of circuit parameters of equivalent circuit for step discontinuities (1) in narrow dimension only and (2) in both narrow & broad dimension are as follows:

3.1. Evaluation of Circuit Parameters of Equivalent Circuit for Step Discontinuities in Narrow Dimension

Consider the waveguide junction as shown in Figure 3. Let junction is at $z = 0$. Waveguide of dimension $2a \times 2b$ extends up to $z = -\propto$ and waveguide of dimension $2a \times 2(b + W)$ extends upto $z = +\propto$. The dimensions are such that only dominant mode can propagate in each section. Let there be an incident wave from $-z$ direction.

In the region $z < 0$ transverse fields are

$$\begin{aligned} E_t^- &= \left(e^{-j\beta z} + \Gamma e^{j\beta z}\right) \frac{V_0}{1+\Gamma} e_0 + \sum_i V_i e^{\alpha_i z} e_i \\ H_t^- &= Y_0^- \left(e^{-j\beta z} - \Gamma e^{j\beta z}\right) \frac{V_0}{1+\Gamma} h_0 - \sum_i Y_i V_i e^{\alpha_i z} h_i \end{aligned} \tag{9}$$

where e_i, h_i are the mode vectors, α_i are the cutoff mode-attenuation constants, Y_i are the characteristics admittances, and Γ is the reflection coefficient for the dominant mode. The subscript 0 signifies the dominant mode. It is assumed that at $z = \propto$ matched load is there. So in the region $z > 0$

$$\begin{aligned} E_t^+ &= \hat{V}_0 e^{-j\hat{\beta}z}\hat{e}_0 + \sum_i \hat{V}_i e^{-\hat{\alpha}_i z}\hat{e}_i \\ H_t^+ &= Y_0^+ \hat{V}_0 e^{-j\hat{\beta}z}\hat{h}_0 + \sum_i \hat{Y}_i \hat{V}_i e^{-\hat{\alpha}_i z}\hat{h}_i \end{aligned} \tag{10}$$

where the carets are used to differentiate the above parameters from there $z < 0$ counterparts. At junction region

$$\iint_{z=0} E^+ \times H^+ \cdot ds = \iint_{z=0} E^- \times H^- \cdot ds \tag{11}$$

So

$$Y_0^+ \hat{V}_0^2 + \sum_i \hat{Y}_i \hat{V}_i^2 = \frac{1-\Gamma}{1+\Gamma} Y_0^- V_0^2 - \sum_i Y_i V_i^2 \tag{12}$$

Relative admittance observed from $z < 0$ is

$$\frac{1-\Gamma}{1+\Gamma} = \frac{Y}{Y_0^-} = \frac{G}{Y_0^-} + j\frac{B}{Y_0^-} \tag{13}$$

Y_0 is real and Y_i are imaginary. For real V_i and $\hat{V}_i$

$$\frac{jB}{Y_0^-} = \frac{\sum_i Y_i V_i^2 + \sum_i \hat{Y}_i \hat{V}_i^2}{Y_0^- V_0^2} \tag{14}$$

$$\frac{G}{Y_0^-} = \frac{Y_0^+ \hat{V}_0^2}{Y_0^- V_0^2} \tag{15}$$

For equivalent circuit of Figure 2, with matched condition at $z = \propto$, it is evident that

$$\frac{G}{Y_0^-} = n^2 \frac{Y_0^+}{Y_0^-} \tag{16}$$

From Eq. (15) and Eq. (16)

$$n^2 = \frac{\hat{V}_0^2}{V_0^2} \tag{17}$$

For the junction shown in Figure 3, the dominant mode vectors [6] are

$$\vec{e}_0 = \hat{u}_y \sqrt{\frac{1}{2ab}} \sin\frac{\pi}{2a}(x+a) \tag{18}$$

$$\vec{\hat{e}}_0 = \hat{u}_y \sqrt{\frac{1}{2a(b+W)}} \sin\frac{\pi}{2a}(x+a) \tag{19}$$

Assumed tangential electric field in the aperture

$$\vec{E}_t^a = \hat{u}_y \sin\frac{\pi}{2a}(x+a) \tag{20}$$

Therefore following [6]:

$$V_0 = \int_{x=-a}^{a}\int_{y=-b}^{b} \frac{1}{\sqrt{2ab}}\sin^2\frac{\pi(x+a)}{2a}dxdy = \sqrt{2ab} \tag{21}$$

$$\hat{V}_0 = \int_{x=-a}^{a}\int_{y=-b}^{b} \frac{1}{\sqrt{2a(b+W)}}\sin^2\frac{\pi(x+a)}{2a}dxdy = \sqrt{\frac{2a}{b+W}b} \tag{22}$$

Therefore

$$\frac{\hat{V}_0}{V_0} = \sqrt{\frac{b}{b+W}} \tag{23}$$

$$n^2 = \frac{b}{b+W} \tag{24}$$

This is the transformation ratio of the transformer of equivalent circuit. Now the first summation in the numerator of Equation (14) is zero and second summation is related to aperture susceptance B_a by

$$\sum_i \hat{Y}_i\hat{V}_i^2 = j|V|^2 B_a = j4b^2 B_a \tag{25}$$

Now

$$V_0 = \int_{x=-a}^{a}\int_{y=-b}^{b} \sin\frac{\pi(x+a)}{2a}\frac{1}{\sqrt{2ab}}\sin\frac{\pi(x+a)}{2a}dxdy = \sqrt{2ab} \tag{26}$$

So

$$B = \frac{2B_a b}{a} \tag{27}$$

3.1.1. Determination of Aperture Susceptance B_a

Let $E_x = 0$ and $E_y = f(x,y)$ be known over the cross section at $z = 0$. For TE to x mode at $z > 0$ scalar potential function is:

$$\psi = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} A_{mn}\sin\frac{m\pi}{2a}(x+a)\cos\frac{n\pi}{2(b+W)}(y+b+W)e^{-\gamma_{mn}z} \tag{28}$$

where A_{mn} are mode amplitudes and γ_{mn} are the mode propagation constants. In particular E_y at $z = 0$ is given by

$$E_y|_{z=0} = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\gamma_{mn}A_{mn}\sin\frac{m\pi}{2a}(x+a)\cos\frac{n\pi}{2(b+W)}(y+b+W) \quad (29)$$

This is the form of double Fourier series: a sine series in x and a cosine series in y. It is thus evident that $\gamma_{mn}A_{mn}$ are fourier coefficients of E_y, or

$$\begin{aligned}\gamma_{mn}A_{mn} \;&=\; E_{mn} = \frac{\epsilon_n}{2a(b+W)}\int_{-a}^{a}\int_{-(b+W)}^{(b+W)} E_y|_{z=0}\sin\frac{m\pi}{2a}(x+a)\\ &\qquad\times\cos\frac{n\pi}{2(b+W)}(y+b+W)dxdy \qquad (30)\end{aligned}$$

where $\epsilon_n = 1$ for $n = 0$ and $\epsilon_n = 2$ for $n > 0$ (Numann's number). The A_{mn} and hence the field are now evaluated. The z directed complex power at $z = 0$ is

$$\begin{aligned}P \;&=\; \iint_{z=0}\left(\vec{E}\times\vec{H}^*\right)\cdot\hat{u}_z ds\\ &=\; -\int_{-a}^{a}\int_{-(b+W)}^{(b+W)}\left[E_yH_x^*\right]_{z=0}dxdy\\ &=\; \int_{-a}^{a}\int_{-(b+W)}^{(b+W)}\left[\sum_{m,n}E_{mn}\sin\frac{m\pi(x+a)}{2a}\cos\frac{n\pi(y+b+W)}{2(b+W)}\right]\\ &\qquad\left[\sum_{p,q}\frac{k^2-\left(\dfrac{p\pi}{2a}\right)^2}{j\omega\mu\gamma_{pq}^*}E_{pq}^*\sin\frac{p\pi(x+a)}{2a}\cos\frac{q\pi(y+b+W)}{2(b+W)}\right]dxdy\\ &\qquad\qquad (31)\end{aligned}$$

This reduces to

$$P = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}(Y_0)_{mn}^*|E_{mn}|^2\frac{2a(b+W)}{\epsilon_{mn}} \quad (32)$$

Assume

$$E_y|_{z=0} = \begin{cases} \sin\dfrac{\pi(x+a)}{2a} & -b < y < b \\ 0 & |y| > b \end{cases} \tag{33}$$

Only non zero amplitudes are:

$$E_{10} = \gamma_{10} A_{10} = \frac{b}{b+W} \tag{34}$$

$$E_{1n} = \gamma_{1n} A_{1n} = \frac{2}{n\pi}\left[\sin\frac{n\pi}{b+W}\left(b+\frac{W}{2}\right) - \sin\left(\frac{n\pi W}{2(b+W)}\right)\right] \tag{35}$$

Therefore

$$P = \frac{2ab^2}{b+W}\left[(Y_0)^*_{10} + 8\sum_{n=1}^{\infty}(Y_0)^*_{1n}\left(\frac{\sin\left(\frac{n\pi b}{b+W}\right)\cos\left(\frac{n\pi}{2}\right)}{\frac{n\pi b}{b+W}}\right)^2\right] \tag{36}$$

Now

$$(Y_0)_{10} = \frac{k^2 - \left(\frac{\pi}{2a}\right)^2}{\omega\mu\beta} = \frac{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}{\eta} \tag{37}$$

$$(Y_0)_{1n} = \frac{k^2 - \left(\frac{\pi}{2a}\right)^2}{-j\omega\mu\alpha} = \frac{j4(b+W)(Y_0)_{10}}{\lambda_g\sqrt{n^2-\left(\frac{4(b+W)}{\lambda_g}\right)^2}} \tag{38}$$

Therefore

$$\frac{P^*}{|V|^2} = (Y_0)_{10}\left[\frac{a}{2(b+W)} + j\frac{16a}{\lambda_g}\sum_{n=1}^{\infty}\frac{\sin^2\left(\frac{n\pi b}{2(b+W)}\right)\cos^2\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi b}{b+W}\right)^2\sqrt{n^2-\left(\frac{4(b+W)}{\lambda_g}\right)^2}}\right] \tag{39}$$

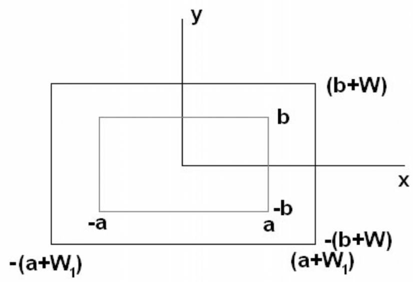

Figure 4. Cross section of waveguide junction for step discontinuity at both broad and narrow dimension.

where $V = 2b$. The imaginary part of this is the aperture susceptance B_a.

$$B_a = \frac{16a}{\lambda_g Z_0} \sum_{n=1}^{\infty} \frac{\sin^2\left(\dfrac{n\pi b}{2(b+W)}\right)\cos^2\left(\dfrac{n\pi}{2}\right)}{\left(\dfrac{n\pi b}{b+W}\right)^2 \sqrt{n^2 - \left(\dfrac{4(b+W)}{\lambda_g}\right)^2}} \tag{40}$$

3.2. Evaluation of Circuit Parameters of Equivalent Circuit for Step Discontinuities at Both Broad and Narrow Dimension

Consider the waveguide junction as shown in Figure 4. Let junction is at $z = 0$. Waveguide of dimension $2a \times 2b$ extends upto $z = -\propto$ and waveguide of dimension $2(a+W_1) \times 2(b+W)$ extends upto $z = +\propto$. The dimensions are such that only dominant mode can propagate in each section. Let there be an incident wave from $-z$ direction. Following similar procedure as described earlier it can be shown:

$$n^2 = \frac{16(a+W_1)^3 ab}{\pi^2(b+W)W_1^2(2a+W_1)^2} \cos^2 \frac{\pi a}{2(a+W_1)} \tag{41}$$

$$B = \frac{2B_a b}{a} \tag{42}$$

$$B_a = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \frac{4a^2\lambda^2\left[k^2 - \left(\dfrac{m\pi}{2(a+W_1)}\right)^2\right](b+W)}{n^2\pi^6 b^2 \eta (a+W_1)} \left[\frac{1}{1 - \left(\dfrac{ma}{a+W_1}\right)^2}\right]^2$$

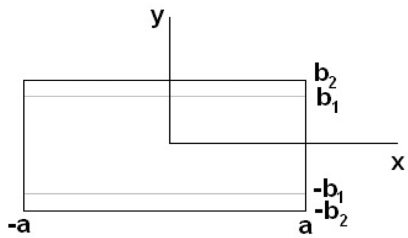

Figure 5. Cross section of waveguide junction for step discontinuity at narrow dimension.

$$\frac{\sin^2\left(\frac{m\pi}{2}\right)\cos^2\left(\frac{n\pi}{2}\right)\cos^2\left(\frac{m\pi a}{2(a+W_1)}\right)\sin^2\left(\frac{n\pi b}{2(b+W)}\right)}{\sqrt{\left(\frac{m\lambda}{2(a+W_1)}\right)^2+\left(\frac{n\lambda}{2(b+W)}\right)^2-1}} \tag{43}$$

4. REFLECTION COEFFICIENT FOR STEP DISCONTINUITIES BY METHOD OF MOMENT

Method of moment can be used for the analysis of waveguide discontinuities. Evaluation of reflection coefficient using moment method for step discontinuities (1) in narrow dimension only and (2) in both narrow & broad dimension are as follows:

4.1. Reflection Coefficient for Step Discontinuities in Narrow Dimension by Method of Moment

Consider the waveguide junction as shown in Figure 5. Let junction is at $z = 0$. Waveguide of dimension $2a \times 2b_1$ extends up to $z = -\propto$ and waveguide of dimension $2a \times 2b_2$ extends upto $z = +\propto$. It is assumed that $b_2 > b_1$. The dimensions are such that only dominant mode can propagate in each section. Let there be an incident wave from $-z$ direction. In the present analysis, the following assumptions are made:

- the x component of the electric field at the plane of the aperture is ignored;
- only the x component of the magnetic field at the aperture plane is considered;
- the excitation in the feed waveguide is TE_{10} dominant wave with the incident electric field y-directed and uniform in the same direction, the assumption of uniform y-directed electric

field having no component in the x-direction follows as a natural consequence;

- the electric field E_y in the aperture is assumed to vary only in the x-direction and is constant in the y-direction.

At the region of window the tangential component of the magnetic field on both sides should be identical. In this analysis two sources producing these fields — the source in the waveguide exciting the TE_{10} mode and the magnetic current source at the discontinuity are considered. Using the principle of superposition, the x components of the magnetic field at the plane of discontinuity are derived.

When aperture is shortened (no magnetic current source) the x component of magnetic fields are denoted by $H_x^{I/1}$ and $H_x^{II/1}$ on guides with narrow dimensions $2b_1$ & $2b_2$ respectively.

When the generator is shortened the x component of magnetic fields are denoted by $H_x^{I/2}$ and $H_x^{II/2}$ on guides with narrow dimensions $2b_1$ & $2b_2$ respectively.

At the plane of discontinuity using principle of superposition it is possible to write

$$H_x^{I/1} + H_x^{I/2} = H_x^{II/1} + H_x^{II/2} \tag{44}$$

Electric field at plane of discontinuity is considered as

$$\vec{E}_{dis} = \hat{u}_y \sum_{p=1}^{M} E'_p e'_p, \qquad p = 1, 2, \ldots\ldots, M. \tag{45}$$

Where basis function $e'_p, (p = 1, 2, \ldots\ldots, M)$ are defined by

$$e'_p = \begin{cases} \sin \dfrac{p\pi(x+a)}{2a}, & \begin{matrix} -a < x < a \\ -b_1 < y < b_1 \end{matrix} \\ 0, & \text{otherwise} \end{cases} \tag{46}$$

As in [6] internally scattered electric field is given by

$$\vec{E}^{scat} = \sum_m \sum_n V_{mn}^e \vec{e}_{mn}^{\,e} + V_{mn}^m \vec{e}_{mn}^{\,m} \tag{47}$$

Magnetic fields are related to the electric fields as follows:

$$\left.\begin{aligned} H_x^e &= Y_{mn1}^e E_y^e \\ H_x^m &= Y_{mn1}^m E_y^m \end{aligned}\right\} \text{for wave propagating in } -z \text{ direction.} \tag{48}$$

$$\left.\begin{aligned} H_x^e &= -Y_{mn2}^e E_y^e \\ H_x^m &= -Y_{mn2}^m E_y^m \end{aligned}\right\} \text{for wave propagating in } +z \text{ direction.} \tag{49}$$

Where Y^e_{mn1} & Y^m_{mn1} are the characteristic admittances of TE_{mn} & TM_{mn} modes for the waveguide of $-z$ direction Y^e_{mn2} & Y^m_{mn2} are the characteristic admittances of TE_{mn} & TM_{mn} modes for the waveguide of $+z$ direction.

For waveguide of $-z$ direction modal vectors are given by

$$\begin{aligned} \vec{e}^{\,e}_{mn} &= \frac{1}{\pi}\sqrt{\frac{ab_1\epsilon_m\epsilon_n}{(mb_1)^2+(na)^2}}\left[\hat{u}_x\left(\frac{n\pi}{2b_1}\right)\cos\left\{\frac{m\pi}{2a}(x+a)\right\}\right. \\ &\quad\times\sin\left\{\frac{n\pi}{2b_1}(y+b_1)\right\}-\hat{u}_y\left(\frac{m\pi}{2a}\right)\sin\left\{\frac{m\pi}{2a}(x+a)\right\} \\ &\quad\left.\times\cos\left\{\frac{n\pi}{2b_1}(y+b_1)\right\}\right] \end{aligned} \tag{50}$$

$$\begin{aligned} \vec{e}^{\,m}_{mn} &= -\frac{2}{\pi}\sqrt{\frac{ab_1}{(mb_1)^2+(na)^2}}\left[\hat{u}_x\left(\frac{m\pi}{2a}\right)\cos\left\{\frac{m\pi}{2a}(x+a)\right\}\right. \\ &\quad\times\sin\left\{\frac{n\pi}{2b_1}(y+b_1)\right\}+\hat{u}_y\left(\frac{n\pi}{2b_1}\right)\sin\left\{\frac{m\pi}{2a}(x+a)\right\} \\ &\quad\left.\times\cos\left\{\frac{n\pi}{2b_1}(y+b_1)\right\}\right] \end{aligned} \tag{51}$$

Similarly for waveguide of $+z$ direction modal vectors are given by

$$\begin{aligned} \vec{e}^{\,e}_{mn} &= \frac{1}{\pi}\sqrt{\frac{ab_2\epsilon_m\epsilon_n}{(mb_2)^2+(na)^2}}\left[\hat{u}_x\left(\frac{n\pi}{2b_2}\right)\cos\left\{\frac{m\pi}{2a}(x+a)\right\}\right. \\ &\quad\times\sin\left\{\frac{n\pi}{2b_2}(y+b_2)\right\}-\hat{u}_y\left(\frac{m\pi}{2a}\right)\sin\left\{\frac{m\pi}{2a}(x+a)\right\} \\ &\quad\left.\times\cos\left\{\frac{n\pi}{2b_2}(y+b_2)\right\}\right] \end{aligned} \tag{52}$$

$$\begin{aligned} \vec{e}^{\,m}_{mn} &= -\frac{2}{\pi}\sqrt{\frac{ab_2}{(mb_2)^2+(na)^2}}\left[\hat{u}_x\left(\frac{m\pi}{2a}\right)\cos\left\{\frac{m\pi}{2a}(x+a)\right\}\right. \\ &\quad\times\sin\left\{\frac{n\pi}{2b_2}(y+b_2)\right\}+\hat{u}_y\left(\frac{n\pi}{2b_2}\right)\sin\left\{\frac{m\pi}{2a}(x+a)\right\} \\ &\quad\left.\times\cos\left\{\frac{n\pi}{2b_2}(y+b_2)\right\}\right] \end{aligned} \tag{53}$$

where ϵ_m, ϵ_n are the Neumann's number satisfying

$$\epsilon_i = \begin{cases} 1 & \text{for } i=0 \\ 2 & \text{otherwise} \end{cases} \tag{54}$$

Following [6] modal voltages of waveguide of $-z$ direction are given by

$$\begin{aligned} V_{mn}^{e} &= \iint\limits_{aperture} \vec{E}_{dis} \cdot \vec{e}_{mn}^{e} ds \\ &= \frac{mb_1 \sin(n\pi)}{n\pi} \sqrt{\frac{ab_1 \epsilon_m \epsilon_n}{(mb_1)^2 + (na)^2}} \sum_{p=1}^{M} E'_p \left[\frac{\sin(p+m)\pi}{\pi(p+m)} - \frac{\sin(p-m)\pi}{\pi(p-m)} \right] \end{aligned} \tag{55}$$

$$\begin{aligned} V_{mn}^{m} &= \iint\limits_{aperture} \vec{E}_{dis} \cdot \vec{e}_{mn}^{m} ds \\ &= \frac{2a \sin(n\pi)}{\pi} \sqrt{\frac{ab_1}{(mb_1)^2 + (na)^2}} \sum_{p=1}^{M} E'_p \left[\frac{\sin(p+m)\pi}{\pi(p+m)} - \frac{\sin(p-m)\pi}{\pi(p-m)} \right] \end{aligned} \tag{56}$$

Similarly modal voltages of waveguide of $+z$ direction are given by

$$\begin{aligned} V_{mn}^{e} &= \frac{mb_2}{n\pi} \sqrt{\frac{ab_2 \epsilon_m \epsilon_n}{(mb_2)^2 + (na)^2}} \sum_{p=1}^{M} E'_p \left[\frac{\sin(p+m)\pi}{\pi(p+m)} - \frac{\sin(p-m)\pi}{\pi(p-m)} \right] \\ &\quad \times \left[\sin \frac{n\pi(b_2+b_1)}{2b_2} - \sin \frac{n\pi(b_2-b_1)}{2b_2} \right] \end{aligned} \tag{57}$$

$$\begin{aligned} V_{mn}^{m} &= \frac{2a}{\pi} \sqrt{\frac{ab_2}{(mb_2)^2 + (na)^2}} \sum_{p=1}^{M} E'_p \left[\frac{\sin(p+m)\pi}{\pi(p+m)} - \frac{\sin(p-m)\pi}{\pi(p-m)} \right] \\ &\quad \times \left[\sin \frac{n\pi(b_2+b_1)}{2b_2} - \sin \frac{n\pi(b_2-b_1)}{2b_2} \right] \end{aligned} \tag{58}$$

Now

$$H_x^{I/1} = 2H_x^{inc} = -2Y_0 \cos \frac{\pi x}{2a} \tag{59}$$

Where H_x^{inc} is the x component of magnetic field due to incident TE_{10} mode and Y_0 is the characteristic admittance of the line due to TE_{10} mode.

Now

$$H_x^{II/1} = 0 \tag{60}$$

$$\begin{aligned} H_x^{I/2} &= -\sum_m \sum_n \sum_{p=1}^{M} E'_p [\sin c(p+m)\pi - \sin c(p-m)\pi] \left[\frac{ab_1}{(mb_1)^2 + (na)^2} \right] \\ &\quad \times (\sin c(n\pi)) \left[\frac{m^2 b_1 \epsilon_m \epsilon_n Y_{mn1}^{e}}{2a} + \frac{2n^2 a}{b_1} Y_{mn1}^{m} \right] \end{aligned}$$

$$\times \sin\left\{\frac{m\pi}{2a}(x+a)\right\}\cos\left\{\frac{n\pi}{2b_1}(y+b_1)\right\} \tag{61}$$

$$\begin{aligned} H_x^{II/2} = & \sum_m \sum_n \sum_{p=1}^{M} E'_p \left[\sin c(p+m)\pi - \sin c(p-m)\pi\right]\left[\frac{ab_2}{(mb_2)^2+(na)^2}\right] \\ & \times \left(\frac{1}{n\pi}\right)\left[\frac{m^2 b_2 \epsilon_m \epsilon_n Y^e_{mn2}}{2a} + \frac{2n^2 a}{b_2} Y^m_{mn2}\right] \\ & \times \left[\sin\frac{n\pi(b_2+b_1)}{2b_2} - \sin\frac{n\pi(b_2-b_1)}{2b_2}\right] \\ & \times \sin\left\{\frac{m\pi}{2a}(x+a)\right\}\cos\left\{\frac{n\pi}{2b_2}(y+b_2)\right\} \end{aligned} \tag{62}$$

Therefore continuity equation will be

$$H_x^{I/1} + H_x^{I/2} = H_x^{II/2} \tag{63}$$

The weighting function W_q is selected to be the same form as the basis function.

$$W_q = \begin{cases} \sin\dfrac{q\pi(x+a)}{2a}, & \begin{matrix} -a < x < a \\ -b_1 < y < b_1 \end{matrix} \\ 0, & \text{otherwise} \end{cases} \tag{64}$$

where $q = 1, 2, \ldots\ldots, M$.

Inner multiplication is defined as

$$< H, W_q >= \iint\limits_{Aperture\ at\ discontinuity} H \cdot W_q \quad dxdy \tag{65}$$

So

$$\left\langle H_x^{I/1}, W_q\right\rangle + \left\langle H_x^{I/2}, W_q\right\rangle = \left\langle H_x^{II/2}, W_q\right\rangle \tag{66}$$

or

$$L_1 + L_2 = L_3 \tag{67}$$

Therefore

$$L_1 = \frac{4Y_0 ab_1}{\pi}\left[\frac{1}{q+1}\cos\frac{\pi}{2}(2q+1) + \frac{1}{q-1}\cos\frac{\pi}{2}(2q-1)\right] \tag{68}$$

$$\begin{aligned} L_2 = & 2\sum_{p=1}^{M} E'_p \left[\sum_m \sum_n \left[\frac{m^2 b_1 \epsilon_m \epsilon_n Y^e_{mn1}}{2a} + \frac{2n^2 a Y^m_{mn1}}{b_1}\right]\right. \\ & \times \left[\frac{a^2 b_1^2}{(mb_1)^2+(na)^2}\right](\sin(n\pi))^2 \left[\sin c(p+m)\pi - \sin c(p-m)\pi\right] \end{aligned}$$

$$\times\left[\sin c(q+m)\pi-\sin c(q-m)\pi\right]\Big] \tag{69}$$

$$\begin{aligned} L_3 \;=\; & -2\sum_{p=1}^{M}E'_p\left[\sum_m\sum_n\left[\frac{m^2b_2\epsilon_m\epsilon_nY^m_{mn2}}{2a}+\frac{2n^2aY^m_{mn2}}{b_2}\right]\right.\\ & \times\left[\frac{a^2b_2^2}{(mb_2)^2+(na)^2}\right]\left(\frac{1}{n\pi}\right)^2\left[\sin\frac{n\pi(b_2+b_1)}{2b_2}-\sin\frac{n\pi(b_2-b_1)}{2b_2}\right]^2\\ & \left.\times\left[\sin c(p+m)\pi-\sin c(p-m)\pi\right]\left[\sin c(q+m)\pi-\sin c(q-m)\pi\right]\right] \end{aligned} \tag{70}$$

To solve the unknown coefficients $E'_p(p=1,2,\ldots\ldots,M)$, Equation (67) can be written in the matrix form for all p and q as below:

$$[A_1]+[A_2]\left[E'_p\right]=[A_3]\left[E'_p\right] \tag{71}$$

or,

$$\left[E'_p\right] = \{[A_3]-[A_2]\}^{-1}[A_1] \tag{72}$$

$$E_i = \text{Incident electric field} = \cos\frac{\pi x}{2a} \tag{73}$$

$$\begin{aligned} E_R &= \text{Reflected electric field}\\ &= -\cos\frac{\pi x}{2a}-\sum_{p=1}^{M}E'_p\left[\sin c(p+1)\pi-\sin c(p-1)\pi\right]\sin\frac{\pi(x+a)}{2a} \end{aligned} \tag{74}$$

Reflection coefficient is given by

$$\Gamma_1=-1-\sum_{p=1}^{M}E'_p\left[\sin c(p+1)\pi-\sin c(p-1)\pi\right] \tag{75}$$

Similarly if there be an incident wave from $+z$ direction then

$$E_i = \text{Incident electric field} = \cos\frac{\pi x}{2a} \tag{76}$$

$$\begin{aligned} E_R &= \text{Reflected electric field}\\ &= -\cos\frac{\pi x}{2a}-\frac{b_1}{b_2}\sum_{p=1}^{M}E'_p\left[\sin c(p+1)\pi-\sin c(p-1)\pi\right]\sin\frac{\pi(x+a)}{2a} \end{aligned} \tag{77}$$

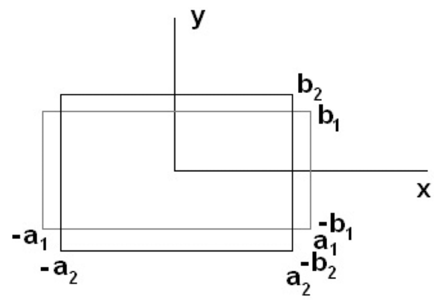

Figure 6. Cross section of waveguide junction for step discontinuity at both broad and narrow dimension.

Reflection coefficient is given by

$$\Gamma_2 = -1 - \frac{b_1}{b_2} \sum_{p=1}^{M} E'_p \left[\sin c(p+1)\pi - \sin c(p-1)\pi\right] \tag{78}$$

4.2. Reflection Coefficient for Step Discontinuities at Both Broad and Narrow Dimension by Method of Moment

Consider the waveguide junction as shown in Figure 6. Let junction is at $z = 0$. Waveguide of dimension $2a_1 \times 2b_1$ extends up to $z = -\propto$ and waveguide of dimension $2a_2 \times 2b_2$ extends upto $z = +\propto$. It is considered that $a_1 > a_2$ and $b_2 > b_1$. The dimensions are such that only dominant mode can propagate in each section. Following similar procedure as described in previous section it can be shown:

If there be an incident wave from $-z$ direction then reflection coefficient:

$$\begin{aligned} \Gamma_1 \quad = \quad & -1 - \frac{a_2}{a_1} \sum_{p=1}^{M} E'_p \left[\cos \frac{\pi(p+1)}{2} \sin c \frac{\pi(pa_1 + a_2)}{2a_1} \right. \\ & \left. - \cos \frac{\pi(p-1)}{2} \sin c \frac{\pi(pa_1 - a_2)}{2a_1}\right] \end{aligned} \tag{79}$$

If there be an incident wave from $+z$ direction then reflection coefficient:

$$\Gamma_2 = -1 - \frac{b_1}{b_2} \sum_{p=1}^{M} E'_p \left[\sin c(p+1)\pi - \sin c(p-1)\pi\right] \tag{80}$$

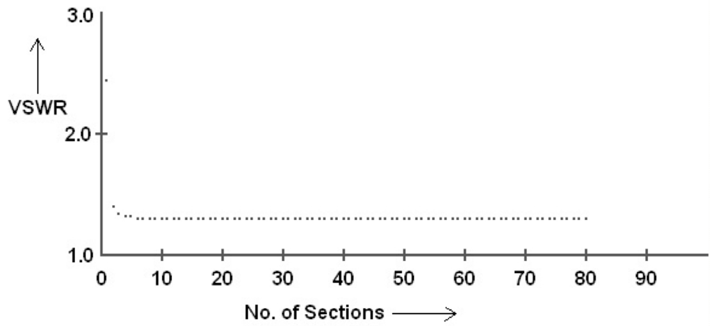

Figure 7. VSWR versus number of sections for tapering in narrow dimensions.

5. RESULTS

From final transmission matrix of taper waveguide reflection coefficient and the VSWR are calculated. For tapering in narrow dimension considering $2a_0 = 5.81\,\text{cm}$, $2b_0 = 1.20\,\text{cm}$, $2a_1 = 5.81\,\text{cm}$, $2b_1 = 2.91\,\text{cm}$, $f = 3.96\,\text{GHz}$, and $L = 4\,\text{cm}$, VSWR is plotted against number of sections N in Figure 7 using equivalent circuit representation of step discontinuities. For large number of sections VSWR is converged. In convergence region the plot of VSWR with taper length is shown in Figure 8 along with previously published results.

Calculating reflection coefficient of step discontinuities by moment method, for tapering in narrow dimension with same dimensional parameter as above, the plot of VSWR against length is shown in Figure 9 along with previously published results. Similarly for tapering in both dimension with $2a_0 = 0.90\,\text{inch}$, $2b_0 = 0.40\,\text{inch}$, $2a_1 = 0.75\,\text{inch}$, $2b_1 = 0.60\,\text{inch}$, $f = 8.7\,\text{GHz}$ and $L = 2.85\,\text{inch}$, VSWR is plotted against number of sections N in Figure 10 using equivalent circuit representation of step discontinuities. In convergence region the plot of VSWR with frequency is shown in Figure 11 along with previously published results.

Calculating reflection coefficient of step discontinuities by moment method, for double taper section with same dimensional parameter as above, the plot of VSWR against frequency is shown in Figure 12 along with previously published results.

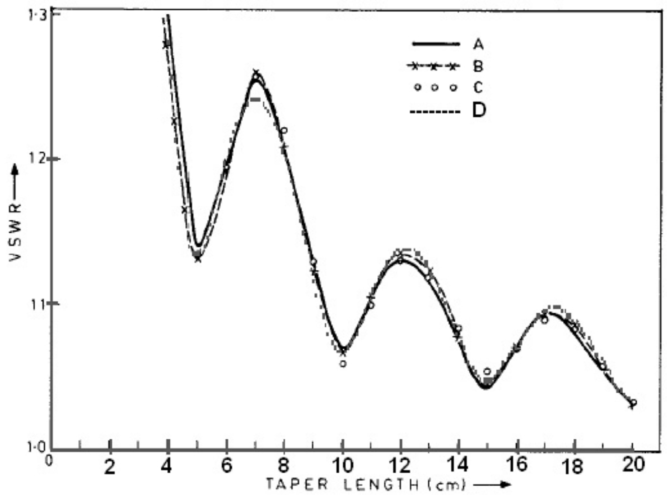

Figure 8. VSWR versus taper length of a taper at a fixed frequency of 3.96 GHz. A- Theoretical curve by Johnson's [3] method. B- Theoretical curve by chakraborty & Sanyal's [1] method. C- Experimental points as reported by Matsumaru [2]. D- By equivalent circuit.

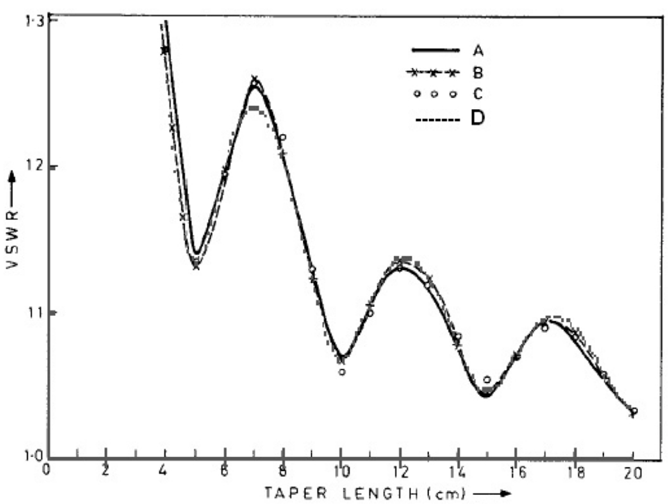

Figure 9. VSWR versus taper length of a taper at a fixed frequency of 3.96 GHz. A- Theoretical curve by Johnson's [3] method. B- Theoretical curve by chakraborty & Sanyal's [1] method. C- Experimental points as reported by Matsumaru [2]. D- By moment circuit.

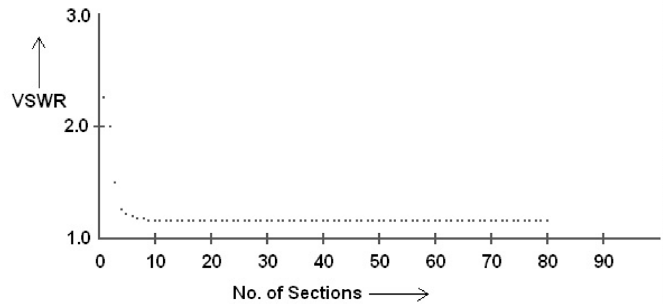

Figure 10. VSWR versus number of sections for double taper section.

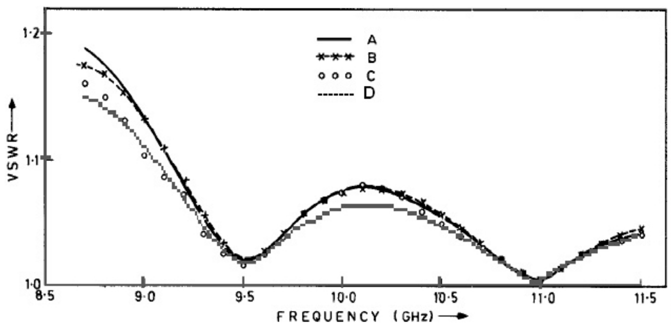

Figure 11. VSWR versus frequency for a linear double taper. A- Theoretical curve by Johnson's [3] method. B- Theoretical curve by chakraborty & Sanyal's [1] method. C- Experimental points as reported by Johnson [3]. D- By equivalent circuit.

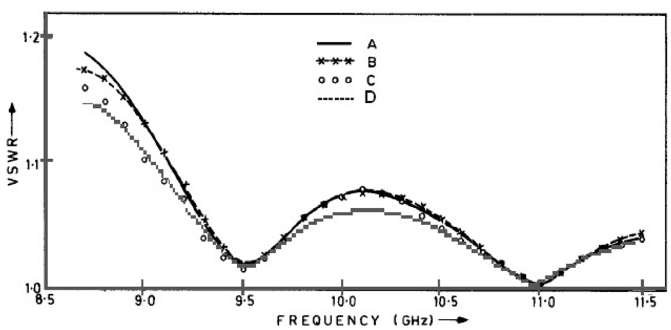

Figure 12. VSWR versus frequency for a linear double taper. A- Theoretical curve by Johnson's [3] method. B- Theoretical curve by chakraborty & Sanyal's [1] method. C- Experimental points as reported by Johnson [3]. D- By moment circuit.

6. CONCLUSION

The results obtained in the presented methods are to be in good agreement with results of [1, 2] and [3]. Here it is assumed that only TE_{10} mode is propagating along the tapered line. Even if other modes are generated they are not supported by the structure.

REFERENCES

1. Chakraborty, A. and G. S. Sanyal, "Transmission matrix of a linear double taper in rectangular waveguides," *IEEE Trans. Microw. Theory Tech.,* Vol. MTT-28, No. 6, 577–579, Jun. 1980.
2. Matsumaru, K., "Reflection coefficient of *E*-plane tapered waveguides," *IRE Trans. Microw. Theory Tech.,* Vol. MTT-6, 143–149, Apr. 1958.
3. Johnson, R. C., "Design of linear double tapers in rectangular waveguides," *IRE Trans. Microw. Theory Tech.,* Vol. MTT-7, 374–378, July 1959.
4. Ghosh, R. N., *Microwave Circuit Theory and Analysis,* Ch. 12, 352–355, McGraw-Hill, New York, 1963.
5. Mallick, A. K. and G. S. Sanyal, "Electromagnatic wave propagation in a rectangular waveguide with sinusoidally varying width," *IEEE Trans. Microw. Theory Tech.,* Vol. MTT-26, 243–249, Apr. 1978.
6. Harrington, R. F., *Time Harmonic Electromagnetic Fields,* McGraw-Hill, New York, 1961.