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# CHARGE MOMENT TENSOR AND MAGNETIC MOMENT OF ROTATIONAL CHARGED BODIES

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**Abstract**—Based on the strict and delicate analogue relation between the magnetic moment of rotational charged bodies and the rotation inertia of rigid bodies, a new concept of charge moment tensor $I$ which is different from the existent electric multiple moment is introduced in this paper. And by means of eigenvalue theory of tensor $I$, the concept of principal axes and principal-axis scalar charge moment are constructed, and further the scalar charge moment of a charged body and the magnetic moment of a rotational charged body around an arbitrary direction are attained. The relationship between the scalar charge moment distributive law of quadric camber and the positive or negative definiteness of tensor $I$ are discussed. Meanwhile Some principles or theorems are extended, generalized, illustrated, and enumerated.

## 1. INTRODUCTION

There is a strict and delicate analogue relation between the magnetic moment of a rotational charged body and the rotation inertia of a rigid body, [1, 2] i.e.,

$$d\vec{P}_m = \left(\frac{1}{2}\vec{\omega}\right) r_\perp^2 dq \quad \text{Vs} \quad dJ = r_\perp^2 dm \tag{1a}$$

Here $dq$ and $dm$ are respectively the mass element and charge element, $r_\perp$ is the rotation radius of $dq$ or $dm$, $\vec{\omega}$ is the angular velocity of charged bodies.

The other basic analogue relations are

$Q$ (the total charge value) $\longleftrightarrow$ $m$ (the total mass),

$dq$ (the element electric charge) $\longleftrightarrow$ $dm$ (the element mass),

$Ce$ (the electric charge center) $\longleftrightarrow C_m$ (the mass center),
$\sigma_e$ (electrical charge areal density) $\longleftrightarrow \sigma_m$ (mass areal density)
$\rho_e$ (electrical charge body density) $\longleftrightarrow \rho_m$ (mass body density)
$\lambda_e$ (electrical line density) $\longleftrightarrow \lambda_m$ (mass line density)

and the most important analogue relation is:

$$\frac{2}{|\vec{\omega}|}|\vec{P}_m| \longleftrightarrow J \text{ (rotation inertia)} \tag{1b}$$

Based on above analogue relation, references [2] deduce some principles about calculating the magnetic moment of rotational charged bodies such as the (generalized) parallel-axis theorem; the generalized orthogonal-axis theorem, the origin moment theorem and so on. Taking consideration of the same analogue relation, we introduce the concept of charge moment tensor and scalar charge moment in this paper, and perform a orthogonal transformation to diagonalize the charge moment tensor matrix and further construct the concept of principal axes, principal-axis scalar charge moment and charge moment quadric camber. Thus the calculation of charge moment for any charged body with respect to any direction and any point is practicable. And we give a packet of methods to calculate the magnetic moment of a rotational charged body of an arbitrary shape, charge distribution and a given angular velocity of $\vec{\omega}$.

## 2. THE CHARGE MOMENT TENSOR, SCALAR CHARGE MOMENT AND THE MAGNETIC MOMENT OF ROTATIONAL CHARGED BODIES

In an arbitrary body-coordinate System $o$–$xyz$ which is rigidly linked with a charged body of definite volume (area, line or discrete) charge distribution, we introduce the so-called **charge moment tensor** with respect to the given origin $O$:

$$I = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \tag{2a}$$

Under the case of disperse charge distribution, the tensor element

$$I_{\alpha\beta} = \sum_i Q_i \left[ r_i^2 \delta_{\alpha\beta} - x_{i\alpha} x_{i\beta} \right] \quad (\alpha \text{ and } \beta = 1, 2, 3) \tag{2b}$$

here $\vec{r}_i$ is the position vector of point charge $Q_i$, $\vec{r}_i = (x_{i1}, x_{i2}, x_{i3}) = (x_i, y_i, z_i)$, and for the case of continual charge distribution such as a charged body with a volume charge density of $\rho(x_1, x_2, x_3)$:

$$I_{\alpha\beta} = \int \rho_e (x_1, x_2, x_3) \left[ r^2 \delta_{\alpha\beta} - x_\alpha x_\beta \right] dv \qquad (2c)$$

here $dv = dx_1 dx_2 dx_3$, $r^2 = x_1^2 + x_2^2 + x_3^2$.

The concrete expression of $\vec{I}(O)$ is taken the disperse distribution case as an instance,

$$\vec{I}(O) = \begin{pmatrix} \sum\limits_i Q_i \left(y_i^2 + z_i^2\right) & -\sum\limits_i Q_i x_i y_i & -\sum\limits_i Q_i x_i z_i \\ -\sum\limits_i Q_i x_i y_i & \sum\limits_i Q_i \left(z_i^2 + x_i^2\right) & -\sum\limits_i Q_i y_i z_i \\ -\sum\limits_i Q_i x_i z_i & -\sum\limits_i Q_i y_i z_i & \sum\limits_i Q_i \left(x_i^2 + y_i^2\right) \end{pmatrix}$$

According to the definition of charge moment tensor $\vec{I}$ and the arbitrariness of the sign of charge and its distribution, $\vec{I}$ is neither a positive nor a negative definite matrix, but it is a 3-dimension and 2-rank symmetric tensor. Especially, it is different from the existent concepts such as the **electric dipole moment** or the **electric quadruple moment** [3] , i.e., $D_{ij} = \int_v (3x_i' x_j' - r'^2)\rho(\vec{r}\,')dV'$, $(i, j = 1, 2, 3)$.

On the other hand, the vanishing of positive definiteness is $\vec{I}$'s most important characteristic different from that of the inertia tensor of rigid bodies.

Thus the so-called **scalar charge moment** with respect to the same point $O$ and arbitrary direction (provided its direction cosine is $\vec{l} = (\cos\theta_1, \cos\theta_2, \cos\theta_3)$) is

$$\begin{aligned} \vec{l} \cdot I \cdot \vec{l} &= (\cos\theta_1, \cos\theta_2, \cos\theta_3) \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \begin{pmatrix} \cos\theta_1 \\ \cos\theta_2 \\ \cos\theta_3 \end{pmatrix} \\ &= I_{11}\cos^2\theta_1 + I_{22}\cos^2\theta_2 + I_{33}\cos^2\theta_3 + 2I_{12}\cos\theta_1\cos\theta_2 \\ &\quad + 2I_{23}\cos\theta_2\cos\theta_3 + 2I_{31}\cos\theta_3\cos\theta_1 \qquad (3) \end{aligned}$$

Then according to the analogue relation it can be immediately deduced that a rotational charged body with an angular velocity of $\vec{\omega}$ with respect to the same axis $(O, \vec{l}\,)$, must has a **magnetic moment** (given $\vec{\omega} = \omega \vec{l}\,$):

$$\vec{P}_m \left(O, \vec{l}\right) = \frac{1}{2} I_l \vec{\omega} = \frac{1}{2} \left( \frac{\vec{\omega}}{\omega} \cdot I \cdot \frac{\vec{\omega}}{\omega} \right) \vec{\omega} \qquad (4)$$

In fact, to get Equation (3) and (4) is just the aim and importance to introduce the concept of charge moment tensor.

## 3. THE PRINCIPAL AXES OF CHARGE MOMENT AND THE SCALAR CHARGE MOMENT QUADRIC

Using the theory of inertia tensor of rigid bodies for reference [4, 5], We then introduce the concept of principal axes of the charge moment. Taking consideration of the fact that the charge moment tensor $I(O)$ is a real and symmetric matrix, we always can find a orthogonal transformation matrix $R$ to change it into an diagonal matrix [6]. This is just the step of diagonalization. And that orthogonal transformation matrix $R$ is just similar to the mode matrix in the theory of multi-dimensional small Vibration [4]. In order to find this orthogonal matrix $R$, we should solve a eigenvalue equation of $I(O)$'s to look for its eigenvalues $\lambda_i$ and the corresponding eigenvectors $e^{(i)}$ $(i = 1, 2, 3)$.

The **eigenvalue equation** is

$$\sum_{\beta=1}^{3} I_{\alpha\beta} e_\beta = \lambda \cdot e_\alpha \ (\alpha = 1, 2, 3) \text{ or } \sum_{\beta=1}^{3} (I_{\alpha\beta} - \lambda\delta_{\alpha\beta})\, e_\beta = 0 \ \ (\alpha = 1, 2, 3) \tag{5}$$

Here $\begin{pmatrix} e_1^i \\ e_2^i \\ e_3^i \end{pmatrix}$ $(i = 1, 2, 3)$ is the corresponding **normalized eigenvector** of the **eigenvalue** $\lambda_i$ $(i = 1, 2, 3)$.

Collect and arrange them in order, thus form the expected orthogonal transformation matrix $R$ which can make the charge moment tensor diagonal

$$R = \begin{pmatrix} e_1^1 & e_1^2 & e_1^3 \\ e_2^1 & e_2^2 & e_2^3 \\ e_3^1 & e_3^2 & e_3^3 \end{pmatrix} \tag{6}$$

In the new Cartesian coordinate body system $O$-$XYZ$ constructed with above three eigenvectors $e^{(i)}$ $(i = 1, 2, 3)$ as axes ,the charge moment tensor can be expressed as a diagonal form:

$$\vec{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \tag{7}$$

We call $e^{(i)} = \begin{pmatrix} e_1^i \\ e_2^i \\ e_3^i \end{pmatrix}_{\lambda_l}$ $(i = 1, 2, 3)$ the three **principal axes** and $I_i = \lambda_i$ $(i = 1, 2, 3)$ the three **principal-axis scalar charge moments** of the charged body. Thus its scalar charge moment $I_l$ with respect to an arbitrary direction $\vec{l} = (\cos\theta_1, \cos\theta_2, \cos\theta_3)$ and the given point $O$ is

$$I_l = \vec{l} \cdot \vec{I} \cdot \vec{l} = I_1 \cos^2\theta_1 + I_2 \cos^2\theta_2 + I_3 \cos^2\theta_3 \tag{8a}$$

Substituting the above expression into Equation (4), we have

$$\vec{P}_m \left(O, \vec{l}\right) = \frac{1}{2}\left(I_1 \cos^2\theta_1 + I_2 \cos^2\theta_2 + I_3 \cos^2\theta_3\right)\vec{\omega} \tag{8b}$$

For the general case depicted by Equations (3) and (8), $I_1, I_2, I_3$ may not always be of the same sign, that is, they might be of sign different from each other's, because tensor matrix $I$ is neither positive definite nor negative definite due from the arbitrariness of the sign and the distribution of charge $Q_i$ (or charge density $\rho_e$ and so on). Thus the spatial distribution of the scalar charge moment $I_l(O)$ with respect to given point $O$ and given direction $\vec{l}$ might not be definitely a **ellipsoid** but a **quadric** in a general case, such as the ellipsoid camber, the **hyperboloid** (With one or two sheets), the **cylinders** (such as the elliptic cylinder, the hyperbolic cylinder), the **cones**, and so on [8, 9]. They all can be the candidates kind of quadric.

In order to concretely display the quadric distribution law of the scalar charge moment $I_l$ thereafter, we focus our interest and attention on the case of charge distribution with pure positive $Q_i$ or $\rho_e(\vec{r}) > 0$, and the more general Equation (3). Then the charge moment tensor $I$ is a positive definite and 2-rank symmetric tensor; and $I_1, I_2, I_3$ in Equation (8) are all positive.

Provided $\vec{\boldsymbol{R}}_l = (x, y, z)$ and $|\vec{R}| = \frac{1}{\sqrt{I_l}}$, then

$$\vec{l} = (\cos\theta_1, \cos\theta_2, \cos\theta_3) = \left(x\sqrt{I_1}, y\sqrt{I_2}, z\sqrt{I_3}\right).$$

Multiplying the two sides of Equation (3) with factor $R_l^2$, we have:

$$I_{11}x^2 + I_{22}y^2 + I_{33}z^2 + 2I_{12}xy + 2I_{23}yz + 2I_{31}zx = 1 \tag{9}$$

Generally $I_l \neq 0$, so Equation (9) stands for an **ellipsoid camber**.

In the new body coordinate system $O$-$XYZ$ constructed with three orthogonal principal axes $e^{(i)}$ $(i = 1, 2, 3)$, Equation (9) is then reduced to be a positive ellipsoid camber:

$$I_1x^2 + I_2y^2 + I_3z^2 = 1 \tag{10}$$

The lengths of three long-semi axes are respectively $\frac{1}{\sqrt{I_i}}, i = 1, 2, 3$.

The intersection point of the ellipsoid and rotation axis passing by origin $O$ and along with direction $\vec{I}$ is point $P$. And then the distance

$$\overline{PO} = |\vec{R}_l| = \frac{1}{\sqrt{I_l}}, \quad \text{or} \quad I_l = \frac{1}{|\vec{R}_l|^2} = \frac{1}{\overline{OP}^2},$$

When $\vec{\omega} = \omega\vec{l}$, the magnetic moment $\vec{P}_m(O, \vec{l}\,)$ of a rotational charged body with respect to the same axis $(O, \vec{l}\,)$ is

$$\vec{P}_m\left(O, \vec{l}\right) = \frac{1}{2} I_l \vec{\omega} = \frac{1}{2\overline{OP}^2}\vec{\omega} \tag{11}$$

And this is just the physical meaning and importance of scalar-charge-moment quadric. Especially, if we select the charge center $C_e$ as the origin $O$, then this quadric is called **center-charge-moment quadric**.

## 4. THE DETERMINATION OF PARAMETERS OF A CHARGE MOMENT QUADRIC

Subsequently the concrete application steps are given to display how to determine the parameters of a charge moment ellipsoid of a charged body

The parameters of a charge moment quadric include principal axes and principal axis scalar charge moment $I_1$ (or $I_x$), $I_2$ (or $I_y$), $I_3$ (or $I_z$). Based upon the discussion in chapter 3, the key problem is to calculate the charge moment tensor of $I$ a given charged body with respect to given point $O$, and to solve the eigenvalue equation $I \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, to get the three eigenvalues $\lambda_1 (= I_x)$, $\lambda_2 (= I_y)$, $\lambda_3 (= I_z)$ and the corresponding three direction vectors of the principal axes $\begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix}$, $i = 1, 2, 3$. And they stand for three direction vectors of the principal axes. Then the scalar charge moment of this charged body with respect to any direction and the corresponding magnetic moment when the charged body rotate with an angular velocity of $\vec{\omega}$ can be calculated by use of Equations (8a) and (8b), and their spatial distribution is a quadric.

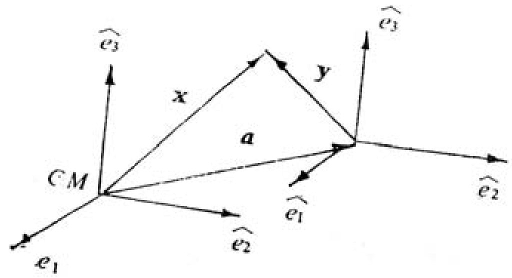



**Figure 1.** The theorem for parallel axes.

## 5. SEVERAL RULES ABOUT THE MAGNETIC MOMENT OF ROTATIONAL CHARGED BODIES

There are some nice and important properties for the charge moment tensor such as the generalized theorem for parallel axes.

Given two coordinate systems $X_1X_2X_3$ and $Y_1Y_2Y_3$ rigidly linked to the charged body and their axes are correspondingly parallel to each other. The origin $O$ of coordinate system $Y_1Y_2Y_3$ depart from the origin $C_e$ (also the charge center) of coordinate system $X_1X_2X_3$ with a displacement vector of $\vec{a}$, just shown as Fig. 1. Then the charge moment tensor element in coordinate system $O$-$Y_1Y_2Y_3$ can be expressed as:

(Suppose $Q$ is the total charge value of the charged body.)

$$I_{\alpha\beta}(O)=\sum_i Q_i\left[\delta_{\alpha\beta}(y_{i1}^2+y_{i2}^2+y_{i3}^2)-y_{i\alpha}y_{i\beta}\right] \text{ (For disperse distribution)}$$

Or

$$I_{\alpha\beta}(O)=\int\rho_e(\vec{y})\left[\delta_{\alpha\beta}(y_1^2+y_2^2+y_3^2)-y_\alpha y_\beta\right]dv \text{ (For continual distribution)}$$

and in the coordinate system $C_e$-$X_1X_2X_3$,

$$I_{\alpha\beta}(C_e)=\sum_i Q_i\left[\delta_{\alpha\beta}(x_{i1}^2+x_{i2}^2+x_{i3}^2)-x_{i\alpha}x_{i\beta}\right] \text{ (For disperse distribution)}$$

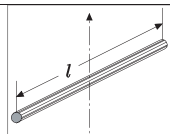
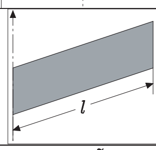
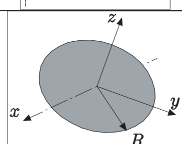
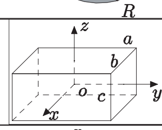
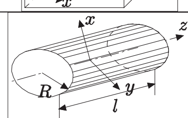
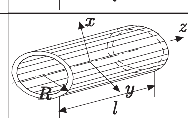
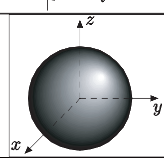
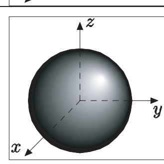
Or

$$I_{\alpha\beta}(C_e)=\int\rho_e(\vec{x})\left[\delta_{\alpha\beta}(x_1^2+x_2^2+x_3^2)-x_\alpha x_\beta\right]dv \text{ (For continual distribution)}$$

Use is made of that $\vec{y}=\vec{x}-\vec{a}$ and the definition of charge center $\vec{x}(C_e)=\frac{1}{Q}\int\rho_e(\vec{x})\vec{x}dv=0$, we have

$$I_{\alpha\beta}(O)=I_{\alpha\beta}(C_e)+Q\left(a^2\delta_{\alpha\beta}-a_\alpha a_\beta\right) \tag{12}$$

**Table 1. The table of calculation rules and examples of magnetic moment for rotational charged bodies** (Provided all matter is evenly charged with total charge value of $Q$ and with a constant volume (surface, linear) charge density).

| Examples | Form of charged bodies | Rules and theorems | Formula of magnetic moment |
|---|---|---|---|
| No. 1. The thin and straight rod (linear) |  | The analogy relation and definition | $\vec{P}_m = \frac{1}{24} Q l^2 \vec{\omega}$ |
| No.2. The rectangular plate (surface distri.) |  | The parallel-axis theorem | $\vec{P}_{mx} = \frac{1}{6} Q l^2 \vec{\omega}$ |
| No. 3. The circular and thin plate (surface distri.) |  | The orthogonal-axes theorem | $\vec{P}_m = \frac{1}{8} Q R^2 \vec{\omega}_x$ ($\vec{P}_{my}$ the similar)<br>$\vec{P}_{mz} = \frac{1}{4} Q R^2 \vec{\omega}_z$ |
| No. 4. The cuboid (volume distri.) |  | The extended orthogonal-axes theorem | $\vec{P}_{mx} = \frac{1}{24} Q (a^2 + c^2) \vec{\omega}_x$<br>($\vec{P}_{my}, \vec{P}_{mz}$ the similar) |
| No. 5. The solid Cylinder (volume distri. and axial sym.) |  | The extended orthogonal axes theorem | $\vec{P}_{mx} = (\frac{1}{8} Q R^2 + \frac{1}{24} Q L^2) \vec{\omega}_x$<br>$\vec{P}_{my}$ the similar |
| No. 6. The thin and hollow cylinder (surface distri. and axial sym.) |  | The extended orthogonal-axes theorem | $\vec{P}_{mx} = (\frac{1}{4} Q R^2 + \frac{1}{24} Q L^2) \vec{\omega}_x$ |
| No.7. The solid globe (volume distri. and spherical sym.) |  | The origin-moment theorem | $\vec{P}_m = \frac{1}{5} Q R^2 \vec{\omega}$ |
| No. 8. The thin Spherical crust (surface distri. and spherical sym.) |  | The origin-moment theorem | $\vec{P}_m = \frac{1}{3} Q R^2 \vec{\omega}$ |

This is the generalized theorem for parallel axes. It constructs a useful and interesting relation for the charge moments between two coordinate systems whose axes are correspondingly parallel. Just because of it, in the special case, the scalar charge moment $I_l(O)$ and $I_l(C_e)$ with respect to the same direction but with a distance of $d$ between the two axes has following so-called parallel-axis theorem:

$$I_l(O) = I_l(C_e) + Qd^2 \tag{13}$$

according to the analogue relation (1a) or (1b),

$$\frac{2}{\omega_0}\left|\vec{P}_m\left(O,\vec{l}\right)\right| = \frac{2}{\omega_e}\left|\vec{P}_m\left(C_e,\vec{l}\right)\right| + Qd^2$$

Where $\omega_o$ and $\omega_e$ are respectively the angular speed of the charged body with respect to corresponding axes. For conciseness and simplification, let $\vec{\omega}_o = \vec{\omega}_e = \vec{\omega}$, then

$$\vec{P}_m\left(O,\vec{l}\right) = \vec{P}_m\left(C_e,\vec{l}\right) + \frac{1}{2}Qd^2\vec{\omega} \tag{14}$$

Which together with other theorems, such as the generalized orthogonal-axes theorem, the origin moment theorem and so on, are already been published in one of my papers [2]. Some principles or theorems are extended, generalized, illustrated, and enumerated in Table 1.

## 6. CONCLUSIONS

The analogue relation between the magnetic moment of rotational charged bodies and the rotation inertia of rigid bodies is not only strict and appropriate but also delicate and perfect, and It is just the opportunity and spring of inspiration to introduce the 3-dimension and 2-rank symmetric charge moment tensor.

The principal axes and the corresponding principal-axis scalar charge moment $I_l$ deduced from the eigenvalue Equation (5) of charge moment tensor $I$ supply us a perfect and systematic theoretic method (8a) and (8b) to calculate the magnetic moment of rotational charged bodies around an arbitrary axes. On the other hand, the quadric spatial distribution law makes the scalar charge moment and the magnetic moment of a rotational charged body more concrete and figurative. Finally, we should lay emphasis on that, the arbitrariness of the distribution and the signs of charge $Q_i$ or $\rho_e(\vec{x})$ make I maybe not a positive or negative definite matrix, and this causes the above quadric maybe not definitely a ellipsoid. The other possible charge distribution and quadric kinds haven't been discussed in this paper and remains to be solved after wards. And this is just the characteristic of the theory of charge moment tensor that is different from that of the inertia tensor. As to the relationship between the parameters of the quadric and the kinds of those quadric, we refer the readers to reference [8].

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