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EQUIVALENT CIRCUIT MODEL FOR ANALYSIS OF INHOMOGENEOUS GRATINGS

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Abstract—A general circuit model is proposed for frequency domain analysis of inhomogeneous two-dimensional periodic gratings. Each component of electromagnetic fields is expressed by several spatial harmonic plane waves. Then, two differential equations and two boundary conditions are obtained for the electric and magnetic vectors. Finally a circuit model is introduced for the obtained equations. The circuit model consists of loaded and excited nonuniform coupled transmission lines (CTL).

1. INTRODUCTION

Laterally periodic planar layers (gratings) are used in many areas such as electromagnetics [1–3], integrated optics [4], electron beams [5], holography and so on. On the other hand, inhomogeneous planar layers are widely used in electromagnetics as optimum shields and filters and so on [6, 7]. Therefore, many efforts have been done to analyze gratings [1–3, 8–10], inhomogeneous planar layers [11–13] or inhomogeneous gratings [14, 15]. The subject of this paper is to introduce a circuit model for analysis of inhomogeneous two-dimensional periodic gratings. First, each component of TM or TE polarized electromagnetic fields is expressed by several spatial harmonic plane waves. Then, two differential equations and two boundary conditions are obtained for the electric and magnetic vectors. Finally, a circuit model is introduced for the obtained equations. The circuit model consists of a loaded and excited nonuniform coupled transmission lines (CTL). The reflection and transmission coefficients can be obtained through analysis of the CTL model.

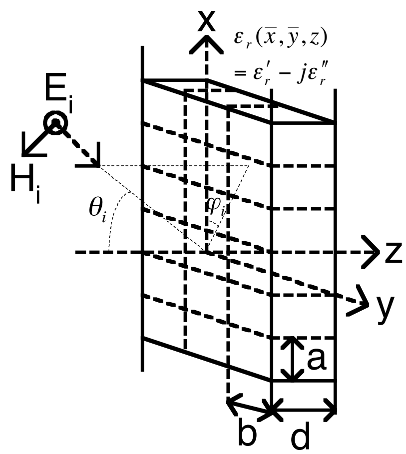

Figure 1. A typical inhomogeneous grating illuminated by a plane wave.

2. THE GRATING STRUCTURE

Figure 1 shows a typical inhomogeneous two-dimensional periodic grating with the relative electric permittivity $\varepsilon_r = \varepsilon_r' - j\varepsilon_r'' = \varepsilon_r' - j\frac{\sigma}{\omega\varepsilon_0}$ and periods of a and b, whose thickness is d. It is assumed that the incident plane wave propagates obliquely towards positive x, y and z direction with an angle of incidence φ_i and θ_i, electric filed strength of E_i and the angular frequency of ω. Regard to the periodicity of the geometry shown in Fig. 1, the electric and magnetic fields are pseudo-periodic functions in x and y with a period of a and b. So, one can use the following Fourier series expansion for an arbitrary three-dimensional function $F(x, y, z)$, which is periodic with respect to x and y with a period of a and b, respectively.

$$F(x,y,z) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}(F)_{m,n|_{z=z}}\exp(-j(U_m x + V_n y)) \tag{1}$$

$$(F)_{m,n|_{z=z}} = \frac{1}{ab}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2} F(x,y,z)\exp(j(U_m x + V_n y))dxdy \tag{2}$$

in which

$$U_m = \frac{2\pi m}{a} \tag{3}$$

$$V_n = \frac{2\pi n}{b} \tag{4}$$

In fact, $(F)_{m,n} = (F)_{m,n}(z)$ denotes the m,n-th Fourier coefficients of $F(x,y,z)$. Of course, in a numerical computation, it is better to truncate the Fourier series expansion of electromagnetic field components by setting

$$(F)_{m,n} = 0, \qquad \text{if } |m| > M \text{ or } |n| > N, \tag{5}$$

where M and N are two positive integers. We use such a truncation $(-M \le m \le M,\ -N \le n \le N)$ in the following sections.

3. THE INCIDENT WAVE

The incident wave is an electromagnetic plane wave consisting of two different polarizations, TE and TM. Thus we can write like as following

$$\begin{aligned} \vec{E}_i &= E_i\left(\alpha_{TE}\hat{a}_{TE} + \alpha_{TM}\hat{a}_{TM}\right)\exp(-j(k_{x0}x + k_{y0}y + k_{z0}z)) \\ &= \left(E_{ix}\hat{a}_x + E_{iy}\hat{a}_y + E_{iz}\hat{a}_z\right)\exp(-j(k_{x0}x + k_{y0}y + k_{z0}z)) \end{aligned} \tag{6}$$

in which

$$k_{x0} = k_0 \sin\theta_i \cos\varphi_i \tag{7}$$

$$k_{y0} = k_0 \sin\theta_i \sin\varphi_i \tag{8}$$

$$k_{z0} = k_0 \cos\theta_i \tag{9}$$

where $k_0 = \omega\sqrt{\mu_0\varepsilon_0}$ is the wave number in the free space. Also, α_{TE} and α_{TM} are the weighting coefficients of TE and TM polarizations, respectively, when $0 \le \alpha_{TE},\ \alpha_{TM} \le 1$ and $\alpha_{TE}^2 + \alpha_{TM}^2 = 1$. Furthermore, $\hat{\alpha}_{TE}$ and $\hat{\alpha}_{TM}$ are the unit vectors related to TE and TM polarizations, respectively, given by

$$\hat{a}_{TE} = -\sin\varphi_i\hat{a}_x + \cos\varphi_i\hat{a}_y \tag{10}$$

$$\hat{a}_{TM} = \cos\theta_i\cos\varphi_i\hat{a}_x + \cos\theta_i\sin\varphi_i\hat{a}_y - \sin\theta_i\hat{a}_z \tag{11}$$

From the Faraday and Amper Laws, the other components of the incident wave can be written versus only x and y components of the electric field, as follows

$$E_{iz} = -\left(\frac{k_{x0}}{k_{z0}}E_{ix} + \frac{k_{y0}}{k_{z0}}E_{iy}\right) = -\alpha_{TM}\sin\theta_i E_i \tag{12}$$

$$H_{ix} = \frac{-1}{\omega\mu_0}\left[\frac{k_{x0}k_{y0}}{k_{z0}}E_{ix} + \left(\frac{k_{y0}^2}{k_{z0}} + k_{z0}\right)E_{iy}\right] \tag{13}$$

$$H_{iy} = \frac{1}{\omega\mu_0}\left[\left(\frac{k_{x0}^2}{k_{z0}} + k_{z0}\right)E_{ix} + \frac{k_{x0}k_{y0}}{k_{z0}}E_{iy}\right] \tag{14}$$

$$H_{iz} = \frac{1}{\omega\mu_0}(-k_{y0}E_{ix} + k_{x0}E_{iy}) \tag{15}$$

These relations can be written in matrix form given by

$$\boldsymbol{e}_{iz} = A_{izx}\boldsymbol{e}_{ix} + A_{izy}\boldsymbol{e}_{iy} \tag{16}$$

$$\boldsymbol{h}_{ix} = Y_{ixx}\boldsymbol{e}_{ix} + Y_{ixy}\boldsymbol{e}_{iy} \tag{17}$$

$$\boldsymbol{h}_{iy} = Y_{iyx}\boldsymbol{e}_{ix} + Y_{iyy}\boldsymbol{e}_{iy} \tag{18}$$

$$\boldsymbol{h}_{iz} = Y_{izx}\boldsymbol{e}_{ix} + Y_{izy}\boldsymbol{e}_{iy} \tag{19}$$

in which the $[(2M+1)\times(2N+1)]\times 1$ column matrices

$$\boldsymbol{f}_{iw} = [0 \quad \cdots \quad F_{iw} \quad \cdots \quad 0]^T \tag{20}$$

($\boldsymbol{f} = \boldsymbol{e}, \boldsymbol{h},\ F = E, H$ and $w = x, y, z$) represent the Fourier coefficient (the same as the amplitude) of the incident electric and magnetic fields components. Also, Y_{iww} and A_{iww} are coefficients obtained from (12)–(15).

4. THE REFLECTED AND TRANSMITTED WAVES

The electric and magnetic fields reflected or transmitted from the gratings, can be represented as $\vec{F}_r = F_{rx}\hat{a}_x + F_{ry}\hat{a}_y + F_{rz}\hat{a}_z$ and $\vec{F}_t = F_{tx}\hat{a}_x + F_{ty}\hat{a}_y + F_{tz}\hat{a}_z$, respectively, where F denotes E or H ($F = E, H$). Each component of these fields are expressed by infinite spatial harmonic plane waves (modes), given by

$$F_{rw}(x,y,z) = \left[\sum_{m=-\infty}^{m=\infty}\sum_{n=-\infty}^{n=\infty}(F_{rw})_{m,n}\exp(-j(U_m x + V_n y) + \gamma_{m,n}z)\right] \times \exp(-j(k_{x0}x + k_{y0}y)) \tag{21}$$

$$F_{tw}(x,y,z) = \left[\sum_{m=-\infty}^{m=\infty}\sum_{n=-\infty}^{n=\infty}(F_{tw})_{m,n}\exp(-j(U_m x + V_n y) - \gamma_{m,n}(z-d))\right] \times \exp(-j(k_{x0}x + k_{y0}y)) \tag{22}$$

where w represents x, y or z ($w = x, y, z$) and also

$$\gamma_{m,n} = \begin{cases} \sqrt{(k_x)_m^2 + (k_y)_n^2 - k_0^2} = \alpha_{m,n}; \\ \qquad\qquad \text{when } k_0 < \sqrt{(k_x)_m^2 + (k_y)_n^2} \\ j\sqrt{k_0^2 - ((k_x)_m^2 + (k_y)_n^2)} = j(k_z)_{m,n}; \\ \qquad\qquad \text{when } k_0 > \sqrt{(k_x)_m^2 + (k_y)_n^2} \end{cases} \tag{23}$$

in which

$$(k_x)_m = k_{x0} + U_m \tag{24}$$
$$(k_y)_n = k_{y0} + V_n \tag{25}$$

are the transverse wave numbers. From the Faraday and Amper Laws, the Fourier coefficients of the other components of the reflected and transmitted waves can be written versus the Fourier coefficients of only x and y components of electric field, as follows

$$(E_{sz})_{m,n} = \frac{\pm j}{\gamma_{m,n}} \left[(U_m + k_{x0})(E_{sx})_{m,n} + (V_n + k_{y0})(E_{sy})_{m,n}\right] \tag{26}$$

$$(H_{sx})_{m,n} = \frac{\pm j}{\omega\mu_0} \left[\frac{(U_m + k_{x0})(V_n + k_{y0})}{\gamma_{m,n}}(E_{sx})_{m,n} + \left(\frac{(V_n + k_{y0})^2}{\gamma_{m,n}} - \gamma_{m,n}\right)(E_{sy})_{m,n}\right] \tag{27}$$

$$(H_{sy})_{m,n} = \frac{\mp j}{\omega\mu_0} \left[\left(\frac{(U_m + k_{x0})^2}{\gamma_{m,n}} - \gamma_{m,n}\right)(E_{sx})_{m,n} + \frac{(U_m + k_{x0})(V_n + k_{y0})}{\gamma_{m,n}}(E_{sy})_{m,n}\right] \tag{28}$$

$$(H_{sz})_{m,n} = \frac{1}{\omega\mu_0} \left[-(V_n + k_{y0})(E_{sx})_{m,n} + (U_m + k_{x0})(E_{sy})_{m,n}\right] \tag{29}$$

in which s represents r or t ($s = r, t$) and the upper and lower signs stand for $s = r$ and $s = t$, respectively. These relations can be written in matrix form given by

$$\boldsymbol{e}_{sz} = \boldsymbol{A}_{szx}\boldsymbol{e}_{sx} + \boldsymbol{A}_{szy}\boldsymbol{e}_{sy} \tag{30}$$
$$\boldsymbol{h}_{sx} = \boldsymbol{Y}_{sxx}\boldsymbol{e}_{sx} + \boldsymbol{Y}_{sxy}\boldsymbol{e}_{sy} \tag{31}$$
$$\boldsymbol{h}_{sy} = \boldsymbol{Y}_{syx}\boldsymbol{e}_{sx} + \boldsymbol{Y}_{syy}\boldsymbol{e}_{sy} \tag{32}$$
$$\boldsymbol{h}_{sz} = \boldsymbol{Y}_{szx}\boldsymbol{e}_{sx} + \boldsymbol{Y}_{szy}\boldsymbol{e}_{sy} \tag{33}$$

in which $s = r,\ t$ and

$$\begin{aligned}\boldsymbol{f}_{rw} &= [(F_{rw})_{-M,-N}\ (F_{rw})_{-M,-N+1}\ \cdots\ (F_{rw})_{-M,N}\ \cdots\ (F_{rw})_{0,0} \\ &\quad \cdots\ (F_{rw})_{M,-N}\ (F_{rw})_{M,-N+1}\ \cdots\ (F_{rw})_{M,N}]^T \qquad (34)\\ \boldsymbol{f}_{tw} &= [(F_{tw})_{-M,-N}\ (F_{tw})_{-M,-N+1}\ \cdots\ (F_{tw})_{-M,N}\ \cdots\ (F_{tw})_{0,0} \\ &\quad \cdots\ (F_{tw})_{M,-N}\ (F_{tw})_{M,-N+1}\ \cdots\ (F_{tw})_{M,N}]^T \qquad (35)\end{aligned}$$

($\boldsymbol{f} = \boldsymbol{e}, \boldsymbol{h}$, $F = E, H$ and $w = x, y, z$) represent the Fourier coefficients of the reflected and transmitted electric and magnetic fields components, respectively. Also, $\boldsymbol{Y}_{sww}$ and $\boldsymbol{A}_{sww}$ are diagonal matrices obtained from (26)–(29).

5. THE INSIDE WAVES

From the Faraday and Amper Laws, the following equations are obtained for the electric and magnetic fields inside the gratings.

$$\partial_z E_x = -j\omega\mu_0 H_y + \partial_x E_z \qquad (36)$$

$$\partial_z E_y = j\omega\mu_0 H_x + \partial_y E_z \qquad (37)$$

$$\partial_z H_x = j\omega\varepsilon_0\varepsilon_r(x, y, z)E_y + \partial_x H_z \qquad (38)$$

$$\partial_z H_y = -j\omega\varepsilon_0\varepsilon_r(x, y, z)E_x + \partial_y H_z \qquad (39)$$

$$E_z = \frac{1}{j\omega\varepsilon_0}\varepsilon_r^{-1}(x, y, z)(\partial_x H_y - \partial_y H_x) \qquad (40)$$

$$H_z = \frac{-1}{j\omega\mu_0}(\partial_x E_y - \partial_y E_x) \qquad (41)$$

The electric and magnetic fields inside the inhomogeneous grating can be written as $\vec{F} = F_x\hat{a}_x + F_y\hat{a}_y + F_z\hat{a}_z$, (where $F = E, H$ and $w = x, y, z$) in which

$$\begin{aligned}F_w(x, y, z) &\approx \left[\sum_{m=-M}^{m=M}\sum_{n=-N}^{n=N}(F_w)_{m,n}|_{z=z}\exp(-j(U_m x + V_n y))\right] \\ &\quad \times \exp(-j(k_{x0}x + k_{y0}y)) \qquad (42)\end{aligned}$$

Using the Fourier series expansion of the field components and that of the permittivity functions in (36)–(41), the following differential equations are obtained for the Fourier coefficients of the fields.

$$\frac{d\boldsymbol{e}(z)}{dz} = -\boldsymbol{Z}(z)\boldsymbol{h}(z) \qquad (43)$$

$$\frac{d\boldsymbol{h}(z)}{dz} = -\boldsymbol{Y}(z)\boldsymbol{e}(z) \qquad (44)$$

where $\boldsymbol{e}(z) = [\boldsymbol{e}_x(z) \quad \boldsymbol{e}_y(z)]^T$ and $\boldsymbol{h}(z) = [\boldsymbol{h}_y(z) \quad -\boldsymbol{h}_x(z)]^T$ are the electric and magnetic fields vectors, respectively, in which

$$\begin{aligned} \boldsymbol{f}_w(z) \;=\; & [(F_w)_{-M,-N} \; (F_w)_{-M,-N+1} \;\cdots\; (F_w)_{-M,N} \;\cdots\; (F_w)_{0,0} \\ & \cdots\; (F_w)_{M,-N} \; (F_w)_{M,-N+1} \;\cdots\; (F_w)_{M,N}]^T_{z=z} \end{aligned} \tag{45}$$

($\boldsymbol{f} = \boldsymbol{e}, \boldsymbol{h}$, $F = E, H$ and $w = x, y, z$) represents the Fourier coefficients of the electric and magnetic field components. Also, the matrices $\boldsymbol{Z}(z)$ and $\boldsymbol{Y}(z)$ in (43)–(44) are defined as follows

$$\boldsymbol{Z}(z) = \begin{bmatrix} \boldsymbol{W}_1(z) & \boldsymbol{W}_2(z) \\ \boldsymbol{W}_3(z) & \boldsymbol{W}_4(z) \end{bmatrix} \tag{46}$$

$$\boldsymbol{Y}(z) = \begin{bmatrix} \boldsymbol{W}_5(z) & \boldsymbol{W}_6 \\ \boldsymbol{W}_7 & \boldsymbol{W}_8(z) \end{bmatrix} \tag{47}$$

where their sub-matrices have been defined as the following

$$\boldsymbol{W}_1(z) = j\omega\mu_0 \boldsymbol{I}_d - \frac{j}{\omega\varepsilon_0} \boldsymbol{K}_x \boldsymbol{Q}(z) \boldsymbol{K}_x \tag{48}$$

$$\boldsymbol{W}_2(z) = \frac{-j}{\omega\varepsilon_0} \boldsymbol{K}_x \boldsymbol{Q}(z) \boldsymbol{K}_y \tag{49}$$

$$\boldsymbol{W}_3(z) = \frac{-j}{\omega\varepsilon_0} \boldsymbol{K}_y \boldsymbol{Q}(z) \boldsymbol{K}_x \tag{50}$$

$$\boldsymbol{W}_4(z) = j\omega\mu_0 \boldsymbol{I}_d - \frac{j}{\omega\varepsilon_0} \boldsymbol{K}_y \boldsymbol{Q}(z) \boldsymbol{K}_y \tag{51}$$

$$\boldsymbol{W}_5(z) = j\omega\varepsilon_0 \boldsymbol{P}(z) - \frac{j}{\omega\mu_0} \boldsymbol{K}_y \boldsymbol{K}_y \tag{52}$$

$$\boldsymbol{W}_6 = \frac{j}{\omega\mu_0} \boldsymbol{K}_y \boldsymbol{K}_x \tag{53}$$

$$\boldsymbol{W}_7 = \frac{j}{\omega\mu_0} \boldsymbol{K}_x \boldsymbol{K}_y \tag{54}$$

$$\boldsymbol{W}_8(z) = j\omega\varepsilon_0 \boldsymbol{P}(z) - \frac{j}{\omega\mu_0} \boldsymbol{K}_x \boldsymbol{K}_x \tag{55}$$

In (48)–(55), $\boldsymbol{I}_d$ is an identity matrix and also

$$\boldsymbol{P}(z) = \begin{bmatrix} (\varepsilon_r)_{0,0} & \cdots & (\varepsilon_r)_{-2M,-2N} \\ \vdots & \ddots & \vdots \\ (\varepsilon_r)_{2M,2N} & \cdots & (\varepsilon_r)_{0,0} \end{bmatrix}_{z=z} \tag{56}$$

and

$$\boldsymbol{Q}(z) = \begin{bmatrix} (\varepsilon_r^{-1})_{0,0} & \cdots & (\varepsilon_r^{-1})_{-2M,-2N} \\ \vdots & \ddots & \vdots \\ (\varepsilon_r^{-1})_{2M,2N} & \cdots & (\varepsilon_r^{-1})_{0,0} \end{bmatrix}_{z=z} \tag{57}$$

are the convolution matrices associated with $\varepsilon_r(z)$ and $\varepsilon_r^{-1}(z)$, respectively. Also,

$$\begin{aligned} \boldsymbol{K}_x &= \mathrm{diag}([\{(k_x)_{-M}\ \cdots\ (k_x)_{-M}\}_{2N+1}\cdots\cdots \\ &\qquad \{(k_x)_0\ \cdots\ (k_x)_0\}_{2N+1}\ \cdots\cdots\ \{(k_x)_M\ \cdots\ (k_x)_M\}_{2N+1}]) \quad (58) \\ \boldsymbol{K}_y &= \mathrm{diag}([\{(k_y)_{-N}\ \cdots\ (k_y)_0\ \cdots\ (k_y)_N\}\cdots\cdots \\ &\qquad \{(k_y)_{-N}\ \cdots\ (k_y)_0\ \cdots\ (k_y)_N\}]_{2M+1}) \quad (59) \end{aligned}$$

are diagonal matrices containing the transverse wave numbers. The indices in (58)–(59) represent the number of repetition of the internal terms.

6. THE BOUNDARY CONDITIONS

From the continuity of transverse components of electric and magnetic fields, there will be four boundary conditions as follows on the surface $z = 0$

$$\boldsymbol{e}_x(0) = \boldsymbol{e}_{rx} + \boldsymbol{e}_{ix} \tag{60}$$
$$\boldsymbol{e}_y(0) = \boldsymbol{e}_{ry} + \boldsymbol{e}_{iy} \tag{61}$$
$$\boldsymbol{h}_x(0) = \boldsymbol{h}_{rx} + \boldsymbol{h}_{ix} \tag{62}$$
$$\boldsymbol{h}_y(0) = \boldsymbol{h}_{ry} + \boldsymbol{h}_{iy} \tag{63}$$

and also other four boundary conditions as follows on the surface $z = d$.

$$\boldsymbol{e}_x(d) = \boldsymbol{e}_{tx} \tag{64}$$
$$\boldsymbol{e}_y(d) = \boldsymbol{e}_{ty} \tag{65}$$
$$\boldsymbol{h}_x(d) = \boldsymbol{h}_{tx} \tag{66}$$
$$\boldsymbol{h}_y(d) = \boldsymbol{h}_{ty} \tag{67}$$

Using (60)–(67) and (31)–(32) (for $s = r$ and t), we can obtain the following main boundary conditions.

$$\boldsymbol{h}_x(d) = \boldsymbol{Y}_{txx}\boldsymbol{e}_x(d) + \boldsymbol{Y}_{txy}\boldsymbol{e}_y(d) \tag{68}$$
$$\boldsymbol{h}_y(d) = \boldsymbol{Y}_{tyx}\boldsymbol{e}_x(d) + \boldsymbol{Y}_{tyy}\boldsymbol{e}_y(d) \tag{69}$$
$$\begin{aligned} \boldsymbol{h}_x(0) &= \boldsymbol{Y}_{rxx}\boldsymbol{e}_x(0) + \boldsymbol{Y}_{rxy}\boldsymbol{e}_y(0) \\ &\quad +(Y_{ixx}\boldsymbol{I}_d - \boldsymbol{Y}_{rxx})\boldsymbol{e}_{ix} + (Y_{ixy}\boldsymbol{I}_d - \boldsymbol{Y}_{rxy})\boldsymbol{e}_{iy} \end{aligned} \tag{70}$$
$$\begin{aligned} \boldsymbol{h}_y(0) &= \boldsymbol{Y}_{ryx}\boldsymbol{e}_x(0) + \boldsymbol{Y}_{ryy}\boldsymbol{e}_y(0) \\ &\quad +(Y_{iyx}\boldsymbol{I}_d - \boldsymbol{Y}_{ryx})\boldsymbol{e}_{ix} + (Y_{iyy}\boldsymbol{I}_d - \boldsymbol{Y}_{ryy})\boldsymbol{e}_{iy} \end{aligned} \tag{71}$$

7. EQUIVALENT CIRCUIT MODEL

We see that two matrix relations (43)–(44) are similar to the differential equations of the lossy or lossless Coupled Tansmission Lines (CTL), whose number of lines is equal to the number of spatial harmonics (modes), if we consider $\boldsymbol{V}(z) = \boldsymbol{e}(z) = [\boldsymbol{e}_x(z) \quad \boldsymbol{e}_y(z)]^T$ and $\boldsymbol{I}(z) = \boldsymbol{h}(z) = [\boldsymbol{h}_y(z) \quad -\boldsymbol{h}_x(z)]^T$ as the voltage and current vectors, respectively. With this assumption, the per-unit-length matrices of the CTL model will be as the following

$$\boldsymbol{R}(z) = \mathrm{Re}\,(\boldsymbol{Z}(z)) \tag{72}$$

$$\boldsymbol{G}(z) = \mathrm{Re}\,(\boldsymbol{Y}(z)) \tag{73}$$

$$\boldsymbol{L}(z) = \frac{1}{j\omega}\mathrm{Im}\,(\boldsymbol{Z}(z)) \tag{74}$$

$$\boldsymbol{C}(z) = \frac{1}{j\omega}\mathrm{Im}\,(\boldsymbol{Y}(z)) \tag{75}$$

These matrices are dependent to z if the gratings are inhomogeneous. It is seen from (46)–(59) that the matrices $\boldsymbol{R}$ and $\boldsymbol{G}$ will be zero (lossless CTL), only if the electric permittivity of the grating is being a real value. Furthermore, the boundary conditions (68)–(71) can be written similar to two following relations

$$\boldsymbol{I}(d) = \boldsymbol{Y}_L\boldsymbol{V}(d) \tag{76}$$

$$\boldsymbol{I}(0) = \boldsymbol{I}_S - \boldsymbol{Y}_S\boldsymbol{V}(0) \tag{77}$$

where

$$\boldsymbol{I}_S = \begin{bmatrix} Y_{iyx}\boldsymbol{I}_d - \boldsymbol{Y}_{ryx} & Y_{iyy}\boldsymbol{I}_d - \boldsymbol{Y}_{ryy} \\ -Y_{ixx}\boldsymbol{I}_d + \boldsymbol{Y}_{rxx} & -Y_{ixy}\boldsymbol{I}_d + \boldsymbol{Y}_{rxy} \end{bmatrix} [\boldsymbol{e}_{ix} \quad \boldsymbol{e}_{iy}]^T \tag{78}$$

$$\boldsymbol{Y}_S = \boldsymbol{Z}_S^{-1} = \begin{bmatrix} -\boldsymbol{Y}_{ryx} & -\boldsymbol{Y}_{ryy} \\ \boldsymbol{Y}_{rxx} & \boldsymbol{Y}_{rxy} \end{bmatrix} \tag{79}$$

$$\boldsymbol{Y}_L = \boldsymbol{Z}_L^{-1} = \begin{bmatrix} \boldsymbol{Y}_{tyx} & \boldsymbol{Y}_{tyy} \\ -\boldsymbol{Y}_{txx} & -\boldsymbol{Y}_{txy} \end{bmatrix} \tag{80}$$

are the current source vector, source admittance matrix and load admittance matrix, respectively. Figure 2 shows the resulted equivalent circuit model considering (76)–(77) as the boundary conditions. Of course, one can write the relation (77) as the following form

$$\boldsymbol{V}(0) = \boldsymbol{V}_S - \boldsymbol{Z}_S\boldsymbol{I}(0) \tag{81}$$

where

$$\boldsymbol{V}_S = \boldsymbol{Y}_S^{-1}\boldsymbol{I}_S = \boldsymbol{Z}_S\boldsymbol{I}_S \tag{82}$$

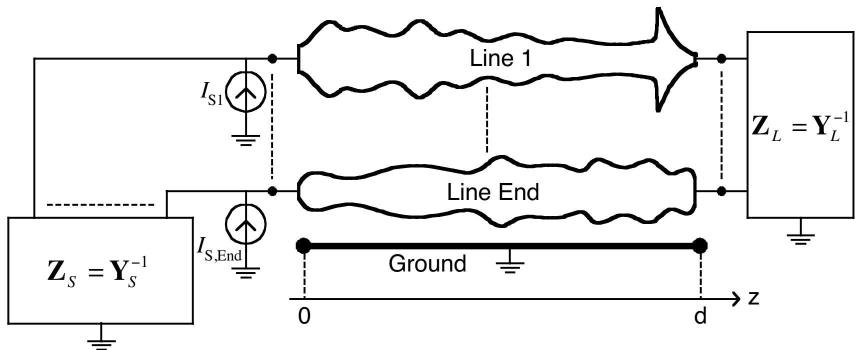

Figure 2. Equivalent circuit model for inhomogeneous gratings using current sources.

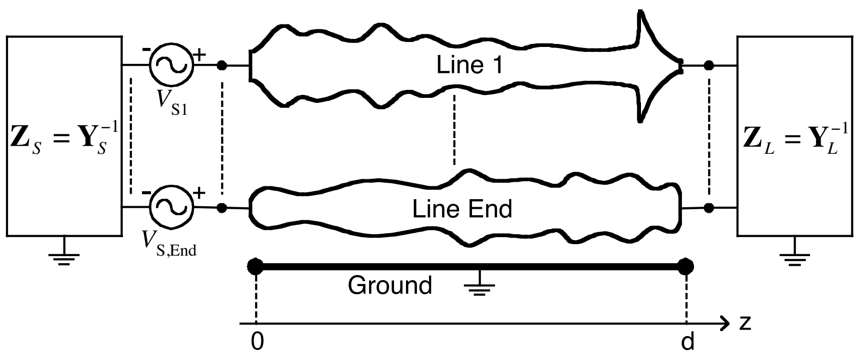

Figure 3. Equivalent circuit model for inhomogeneous gratings using voltage sources.

Figure 3 shows the resulted equivalent circuit model considering (76) and (81) as the boundary conditions. It is noted that for the short-end gratings (gratings which are coated by a perfect electric conductor and are usually utilized as the walls of anechoic chambers), the load impedance will be $\boldsymbol{Z}_L = \mathbf{0}$. The analysis of uniform CTLs is simple [16], in contrary with nonuniform ones. Of course, there are some methods to analyze nonuniform CTLs, which the most straightforward one is subdividing them into many short sections [16, 17]. Also, all softwares designed for high frequency and microwave circuits can be utilized for the analysis of nonuniform gratings, if we use the introduced CTL model.

8. SOME SPECIAL CASES

The equivalent circuit models obtained in the previous section are simplified in a special case. In this section some of the special cases are mentioned considering (46)–(59).

8.1. Homogeneous Gratings

For homogeneous gratings, in which the function ε_r is independent of z, the coupled lines will be uniform and simple to analyze.

8.2. Inhomogeneous Planar Layers

For inhomogeneous planar layers, in which the function ε_r is independent of x and y, only a single nonuniform transmission line will be existed.

8.3. Conductive Gratings

For the gratings consisting of periodic conductors, in which $\sigma \gg \omega\varepsilon_0\varepsilon_r'$, the coupled lines will be uniform and lossy with the following matrices.

$$\boldsymbol{Z}(z) = j\omega\mu_0 \begin{bmatrix} \boldsymbol{I}_d & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_d \end{bmatrix} \tag{83}$$

$$\boldsymbol{Y}(z) = \begin{bmatrix} (\sigma)_{0,0} & \cdots & (\sigma)_{-2M,-2N} \\ \vdots & \ddots & \vdots \\ (\sigma)_{2M,2N} & \cdots & (\sigma)_{0,0} \end{bmatrix} \tag{84}$$

If the thickness of the conductive gratings tends to zero thin Freuency Selective Surfaces (FSS) will be appeared. The equivalent circuit of a thin FSS has been proposed in [18], which is a special case of that of the gratings, proposed in this paper considering (83)–(84) and that $\sigma \to \infty,\ d \to 0$.

9. TRANSMISSION AND REFLECTION COEFFICIENTS

After determining the electric and magnetic fields on the surfaces of grating (or equivalently the voltages and currents at the end of lines of the CTL models), the co- and cross-polarized reflection and the transmission coefficients for propagating (not evanescent) TE and TM modes will be determined as follows

$$(\Gamma_{TX,TY})_{m,n} = \frac{\left\{ \begin{array}{l} [((E_x)_{m,n}\hat{a}_x+(E_y)_{m,n}\hat{a}_y+(E_z)_{m,n}\hat{a}_z)_{|z=0} \\ \quad -(E_{ix}\hat{a}_x + E_{iy}\hat{a}_y + E_{iz}\hat{a}_z)\varepsilon_{m,n}] \cdot \hat{a}_{TX} \end{array} \right\}}{\alpha_{TY} E_i} \tag{85}$$

$$(\Gamma_{TX,TY})_{m,n} = \frac{((E_x)_{m,n}\hat{a}_x+(E_y)_{m,n}\hat{a}_y+(E_z)_{m,n}\hat{a}_z)_{|z=d} \cdot \hat{a}_{TX}}{\alpha_{TY} E_i} \tag{86}$$

in which X, Y represent E or M and $\varepsilon_{m,n}$ is equal to zero except for $m = n = 0$, which is equal to one. Also,

$$(\hat{a}_{TE})_{m,n} = \frac{1}{\sqrt{(k_x)_m^2 + (k_y)_n^2}}(-(k_y)_n\hat{a}_x + (k_x)_m\hat{a}_y) \tag{87}$$

$$(\hat{a}_{TM})_{m,n} = \frac{\mp 1}{k_0\sqrt{(k_x)_m^2 + (k_y)_n^2}}\Big[(k_x)_m(k_z)_{m,n}\hat{a}_x + (k_y)_n(k_z)_{m,m}\hat{a}_y - ((k_x)_m^2 + (k_y)_n^2)\hat{a}_z\Big] \tag{88}$$

are the unit vectors for the electric field of the m, n-th modes of TE and TM polarizations, respectively. The upper and lower signs in (88) stand for calculating the reflection and the transmission coefficients, respectively.

10. CONCLUSIONS

A general circuit model was introduced for frequency domain analysis of inhomogeneous two-dimensional periodic gratings. The equivalent circuit model can be used to give a physical understanding of the operating of gratings that is not possible from numerical wave solvers. Also, all softwares designed for high frequency and microwave circuits can be utilized for the analysis of inhomogeneous gratings, if we use the introduced circuit model. The equivalent circuit model consists of a loaded and excited nonuniform coupled transmission lines (CTL). The excitation of lines is made using current or voltage sources in the circuit model. The co- and cross-polarized reflection and transmission coefficients can be obtained through analysis of the circuit model.

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