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A POSSIBLE REMEDY FOR THE OSCILLATIONS OCCURRING IN THIN-WIRE MOM ANALYSIS OF CYLINDRICAL ANTENNAS

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Abstract—Approximate, non-singular kernels are often used in moment-method formulations coping with thin-wire structures. Their use has important consequences, one of which is the appearance of oscillations in the computed currents when the number of sub-domain basis functions is sufficiently large. These oscillations are not due to round-off errors. In this paper, a smoothing procedure is used in conjunction with Galerkin's formulation with piecewise sinusoidal functions, which yields non-oscillating current distributions. Special attention is paid to the solutions over a wide range of discretization levels (number of basis/testing functions), in order to examine and illustrate the similarities and differences between results obtained with and without the proposed remedy. Finally, a comparison with results derived with the exact kernel is provided.

1. INTRODUCTION

Thin-wire antennas have been at the forefront of electromagnetic (EM) research for many decades, mainly because of their significant theoretical and practical interest, but also due to the difficulties often arising when attempting to validate and crosscheck numerical solutions, some of which will be discussed hereinafter.

The most frequently applied method to wire structures is the so-called method of moments (MoM). In fact, MoM is rather a collective abbreviation for the ensemble of the well-known moment methods, which have been developed in a multitude of variants for EM modeling in the frequency domain [1]. These methods seek for numerical solutions to the integral or integro-differential equations arising from the enforcement of the relevant boundary condition along the wire.

Although there seem to be several ways to formulate the problem, most fall into two categories, depending on whether the exact (full) kernel (EK) or the approximate (reduced) one (AK) is used.

Although the utilization of the AK may often yield acceptable results from a practical point of view, it is associated with important difficulties, the understanding of which explains many interesting phenomena regarding the behavior of the solutions in general. In particular, for Hallén's and Pocklington's equations, adoption of the AK renders the problem non-solvable from a mathematical point of view [2–4] (a relevant study, along with numerical results, can be found in [5]), at least for wires of finite length and for an excitation field being a delta-gap source, a plane wave, or a frill generator. The consequences of this non-solvability are predominantly reflected in a non-converging behavior of the numerical solutions as the number of basis functions is increased (this is a natural consequence of non-solvability), as well as in an oscillating behavior of the calculated currents near wire ends and on both sides of excitation gaps (for gap-driven radiators) when the number of basis functions is large. It is worth stressing that these phenomena are not related to finite wordlength of computers, which imposes further serious complications, some of which may be quite difficult to distinguish and isolate from those discussed above.

The literature on the feeds and/or kernels of wire antennas is very extensive and dates as far back as 1938 [6]; other early references are [7] and certain papers listed in [8]. In particular, there are very many works dealing with the application of moment methods to thin-wire integral equations; an extensive list of pre-1980 such references can be found in [9]. Besides [2–5], upon which the present paper is primarily based, works that mention oscillations are [10–14]. Many other works focus on various aspects of the difficulties associated with the application of moment methods to thin-wire structures, including convergence issues; for example, see [15–21], the references cited therein, as well as the ones cited in [2–5, 10–14]. However, the studies of [2–5] clearly attribute the aforementioned oscillations to non-solvability, which is a deficiency of the integral equation itself, and not to round-off errors or to a deficiency of the numerical method employed. Perhaps the first work to discuss non-solvability is [22]; a more detailed study of non-solvability (with no discussion of moment methods or oscillations) can be found in [23]. A recent study of the integral equations for the currents on cylindrical dipoles of infinite length can be found in [24].

With a single exception, the aforementioned works that mention oscillations [10–14] do not propose a way to obtain useful results from the oscillating solutions, as done in the present paper. The exception is [11], in which it is proposed to smoothly extrapolate the oscillating current values back to the feed region. When the number of basis functions is large, however, the values obtained in this manner seem to be independent of the length of the antenna. The reason for this is that the oscillating values are themselves independent of the length of the antenna, as they are closely approximated by the oscillating values for the case of the antenna of infinite length; this was shown (at least in the delta-gap generator case, for several specific numerical methods, and for a sufficiently large number of basis functions) in [2]. We finally note that the present paper, which deals with computing meaningful results from oscillating values obtained with the AK, differs from [15], which addresses deficiencies of the AK via circumferential integration of the self-term.

A possible way to overcome the aforementioned problem of the oscillating currents can be inferred from the application of the method of auxiliary sources (MAS) to wire antennas [25, 26], which, in general, involves calculations of current distributions from the resulting tangential magnetic field, instead of computations as superpositions of the basis functions, as in moment methods. Hereinafter, it is shown that it is possible to obtain smooth current distributions from divergent basis functions amplitudes, a result that is consistent with the remarks in [27]. For comparison purposes, results obtained with the EK are also presented, which, as expected, are not characterized by any oscillations at all.

2. PROBLEM STATEMENT AND FORMULATION

The geometry under consideration consists in a cylindrical dipole of length L and radius a, lying along the z-axis and centered at the origin of a cylindrical coordinates system. Only sufficiently thin wires are considered, for which k_0a, where $k_0 = 2\pi/\lambda$, is small. Moreover, a time dependence $\exp(j\omega t)$ is assumed and suppressed throughout the analysis.

The MoM formulation of the problem at hand is well known, so only a brief description is provided here, along with some critical remarks and implementation details. The formulation is precisely what one obtains from Pocklington's equation in conjunction with Galerkin's method. Both the basis (expansion) and the testing (weighting) functions are selected to be piecewise sinusoidal functions of length

$2\delta = L/(N+1)$, which are given by

$$f_n(z) = \begin{cases} \sin\left[k_0\left(\delta - |z - z_n|\right)\right], & |z - z_n| \leq \delta \\ 0, & |z - z_n| > \delta \end{cases}, \quad n = 0, \pm 1, \ldots, \pm N, \tag{1}$$

where $z_n = n\delta$ are the centers of the basis/testing functions. For the derivation of the interaction (impedance) matrix, a procedure analogous to the one described in [25, 26] is exploited, without incorporating the terminal basis/testing functions introduced therein, which are not required by Pocklington's equation. When the AK is assumed, the EM field radiated by the dipole is virtually attributed to a filamentary current flowing along the axis of the dipole, which is approximated as a weighted superposition of the basis functions. This assumption is responsible for the coincidence of the matrix equations obtained from the MoM and the MAS in this specific case, under the condition, of course, that the latter is combined with a reaction-matching scheme, as in [25, 26]. On the other hand, in the EK case, there is a tubular current distribution of radius a instead of a filamentary current. In that case, the EM field generated by each basis function can be expressed as continuous superposition (in the sense of a circumferential integral) of the EM fields radiated by cylindrically distributed filamentary currents that form the tubular basis functions.

Either when the EK or the AK is assumed, the algebraic system of equations for the unknown expansion weights w_n is expressed as

$$\sum_{n=-N}^{N} Z_{n,m} w_n = -V_m, \quad m = 0, \pm 1, \ldots, \pm N. \tag{2}$$

Details on the computation of the interaction matrix terms $Z_{n,m}$ in the EK case are given in the Appendix. The voltages V_m denote the reactions integrals of the excitation field. Expressions for these are available in [25] for the delta-gap generator and the receiving/scattering case of plane-wave incidence. Albeit the former is perhaps the most frequently used source model, it is associated with an infinite gap capacitance and an excitation field that tends to infinity, which is responsible for the diverging nature of the imaginary part of the input current (susceptance) accompanying EK solutions. Since it is of interest to examine the behavior of the resulting current distributions over a wide range of variation for N, it is rather preferable to consider other source models, such as a frill generator or a gap generator of finite width Δ, in order to focus on the oscillations under study, without considering any possible influence of the diverging susceptance (arising for $\Delta \to 0$) on the solution behavior. According to [4], for frill-driven dipoles, oscillations are expected to originate only from near the wire ends, but not near the frill generator. Thus, although not necessarily more realistic or adequate compared to other source models, a gap generator of finite width is preferred hereinafter, which is located at an arbitrary point z_g on the wire, with the associated electric field given by

$$E_z^g(z) = -\frac{V_g}{\Delta},\ |z - z_g| < \frac{\Delta}{2}. \tag{3}$$

The voltages V_m of (2) are given by

$$V_m = \int_{z_m-\delta}^{z_m+\delta} E_z^g(z) f_m(z) dz. \tag{4}$$

From (1), (3) and (4), and after relatively simple algebraic manipulations, the voltages V_m are explicitly expressed as

$$V_m = -\frac{V_g}{k_0\Delta}\left[I_m^-\left(z_m^{--}, z_m^{-+}\right) + I_m^+\left(z_m^{+-}, z_m^{++}\right)\right], \tag{5}$$

where $z_m^{--} = \max(z_m - \delta, z_g - \Delta/2)$, $z_m^{-+} = \min(z_m, z_g + \Delta/2)$, $z_m^{+-} = \max(z_m, z_g - \Delta/2)$, $z_m^{++} = \min(z_m + \delta, z_g + \Delta/2)$ and $I_m^\pm$ are given by

$$I_m^\pm\left(z^-, z^+\right) = \begin{cases} \cos\left[k_0\left(\delta \mp z^\pm \pm z_m\right)\right] - \cos\left[k_0\left(\delta \mp z^\mp \pm z_m\right)\right], & z^+ \geq z^- \\ 0, & z^+ < z^- \end{cases}. \tag{6}$$

From the theoretical considerations and numerical results provided in [2–5], it seems that the oscillations typically accompanying AK solutions begin to occur near the wire ends and progressively cover the whole length, as N increases beyond $L/(2a)$. Moreover, for gap-driven antennas (but not for frill-driven ones), additional oscillations in the imaginary part of the current distribution also occur near the driving point.

In any case, after solving (2), the current distribution along the dipole can be obtained either as a superposition of the basis functions themselves or from the tangential magnetic field at $\rho = a$, in agreement with the boundary condition of the magnetic field for perfect conductors. In the former case, the total current is derived by

$$I(z) \approx \sum_{n=-N}^{N} w_n f_n(z). \tag{7}$$

When the AK is used, (7) corresponds to a line current, which is precisely what one obtains from Galerkin's method (with piecewise sinusoids) applied to Pocklington's equation with the AK. According to the preceding, the total current can be alternatively obtained by calculating the magnetic field at $\rho = a$. The aforementioned magnetic field is associated with the surface current density $K(z)$ and the current is $I(z) = 2\pi\alpha K(z)$ or

$$I(z) \approx 2\pi a \sum_{n=-N}^{N} w_n \left[\hat{\phi} \cdot \vec{H}_n(a, z)\right], \tag{8}$$

where $\vec{H}_n$ denotes the magnetic field generated by the sinusoidal current of the basis function $f_n(z)$. When the AK is used, the magnetic field can be readily obtained analytically [25]. On the other hand, when the EK is used, the magnetic field can be expressed as a circumferential integral analogous to (A2), which is non-singular and, thereupon, easily computable.

The proposed remedy consists exactly in the use of (8) instead of (7); this seems to overcome the oscillations discussed above and results in smooth current distributions from oscillating expansion weights w_n, in a manner analogous to the ones exhibited in [25–27].

3. NUMERICAL RESULTS

In what follows, when the AK is used, the current distributions resulting from (8) are referred to as "smoothed", whereas the conventional ones resulting from (7) are referred to as "non-smoothed".

The oscillations accompanying AK solutions typically begin to appear when N increases and reaches $L/(2a)$ [2–5, 25, 26]. Specifically, oscillating quantities are the expansion weights w_n, and these, as a direct outcome, yield oscillating current distributions through (7). For N near $L/(2a)$, oscillations occur only near the wire ends and on both sides of the excitation gap (for gap-driven radiators). The former occur in both the real and imaginary parts of the current distribution, whereas the latter appear only in the imaginary part. As N is further increased, the oscillations increase in magnitude, having a growing spatial frequency in proportion with N. For relatively large N exceeding $L/(2a)$, the oscillations are so great in magnitude and so rapidly varying that the current distributions resulting from (7) exhibit a glaring non-physical behavior, which is still not due to round-off errors, provided that careful numerical calculation (with double-precision arithmetic) is used. To assess the effect of round-off errors, several numerical checks can be performed, including the estimation of the condition number for increasing values of N, the crosschecking of the solutions resulting from independent matrix solvers and the use of different quadrature schemes. Of course, as N continues to grow much beyond $L/(2a)$, the solutions are severely influenced by round-off errors. In this event, it would be rather meaningless to further examine the resulting solutions, as they strongly depend upon the specific hardware and software used. By contrast with the resulting currents in the AK case, the use of the EK always yields smooth and physically acceptable current distributions, since no oscillations occur in the expansion weights w_n as N is increased beyond $L/(2a)$.

All the above considerations are illustrated in Figs. 1–3, for a dipole with $L/\lambda = 0.005$ and $a/\lambda = 0.005$, which is symmetrically driven by a gap generator having $\Delta = a$. The presented results correspond to $N + 1 = 50$, $N + 1 = 100$ and $N + 1 = 150$. Due to symmetry, the presented current distributions are depicted for $z \geq 0$ only. Moreover, for clarity, current distributions are only shown close to the gap ($z = 0$) and the end ($z = L/2$). Apparently, the results are consistent with the above remarks. Even in Fig. 3 where N is large compared to the important parameter $L/(2a) = 50$, the smoothed AK current distributions are not distorted at all by the oscillations that render the non-smoothed AK current distributions non-physical. In Fig. 3(a), certain values of the non-smoothed AK curve near $z = 0$ are out of scale. From the presented results, it is also obvious that the solutions obtained from the EK formulation do not oscillate at all, as expected from [2, 5]. Extensive numerical tests have verified that this behavior is representative of what should be anticipated in general, provided, of course, that N is not exuberantly high, so that the influence of round-off errors is unnoticeable. Similar results were also obtained for $\Delta \to 0$, which corresponds to a delta-gap source. Therefore, any difficulties emerging from the use of the AK should not be blamed exclusively on the delta-gap source.

As for the discrepancies of the results obtained from the different formulations, it is worth noting that the smoothed AK distributions are quite close to the corresponding EK distributions, since their relative differences are smaller than 5% in Figs. 1–3, and typically smaller than 10% for sufficiently thin dipoles with $L/\lambda = 0.5$ and $a/\lambda < 0.01$. Therefore, the smoothing procedure consisting in the utilization of the AK together with (8) is useful, even when N is large enough to yield intensely fluctuating expansion weights w_n, but, of course, before the appearance of severe ill-conditioning effects, which can be detected with the aid of various numerical checks as those discussed above.

One additional question worth addressing regards the stability of the solutions resulting from the use of the AK in conjunction with (8).

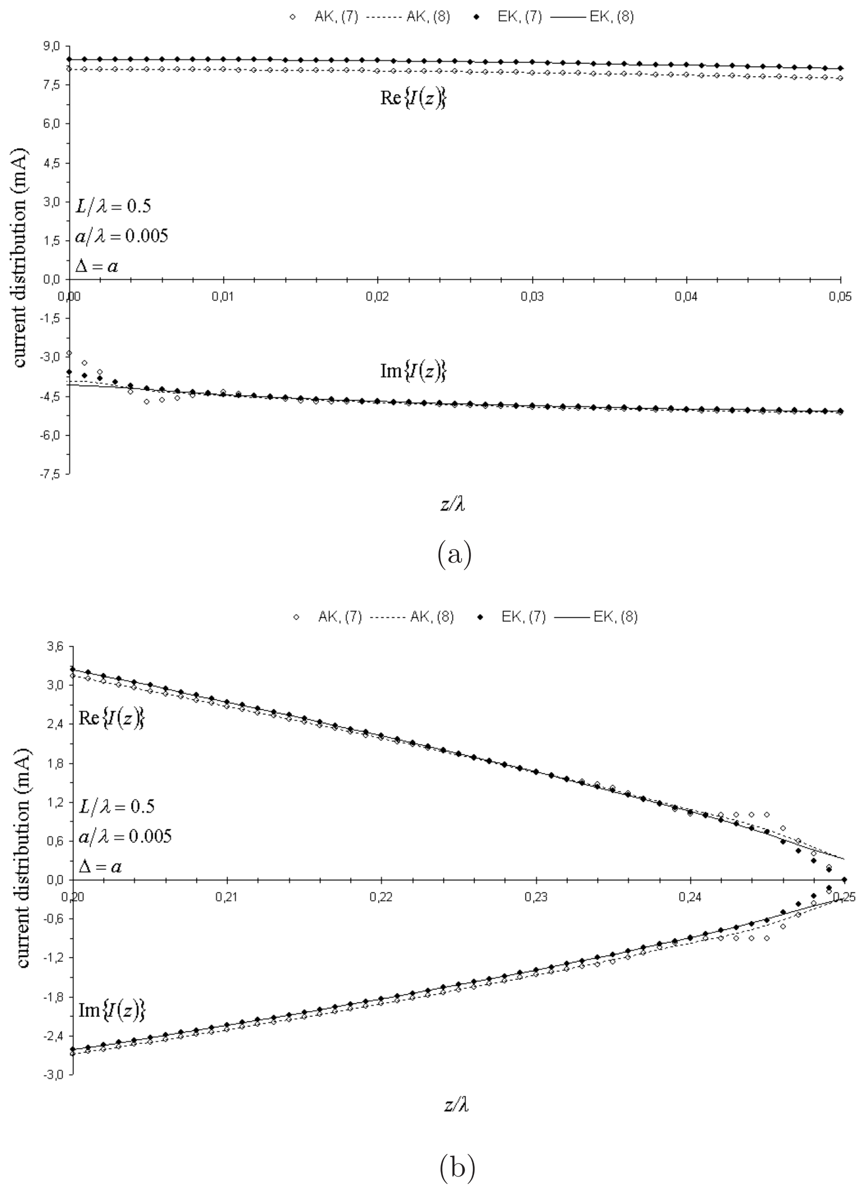
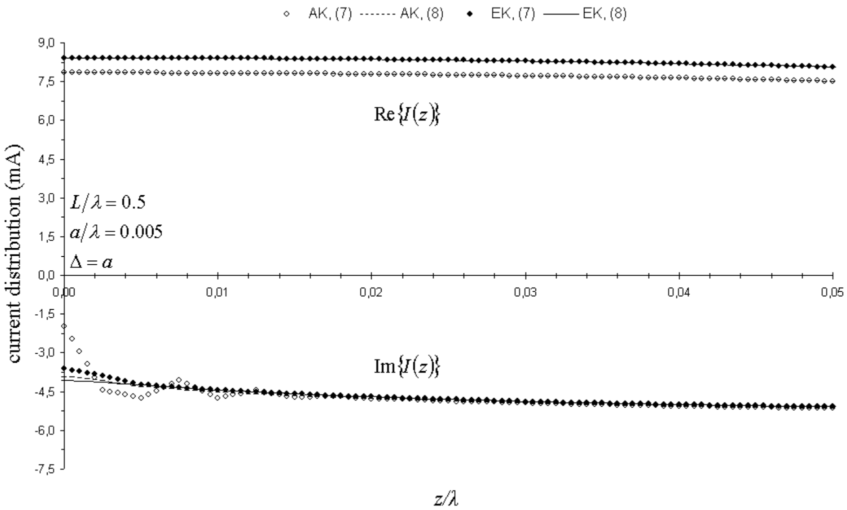


Figure 1. Computed current distribution on a dipole with $L/\lambda = 0.5$ and $a/\lambda = 0.005$, which is symmetrically excited by a finite-gap generator with $\Delta = a$, for $N + 1 = 50$, (a) Current distribution near the ends, (b) Current distribution near the gap.



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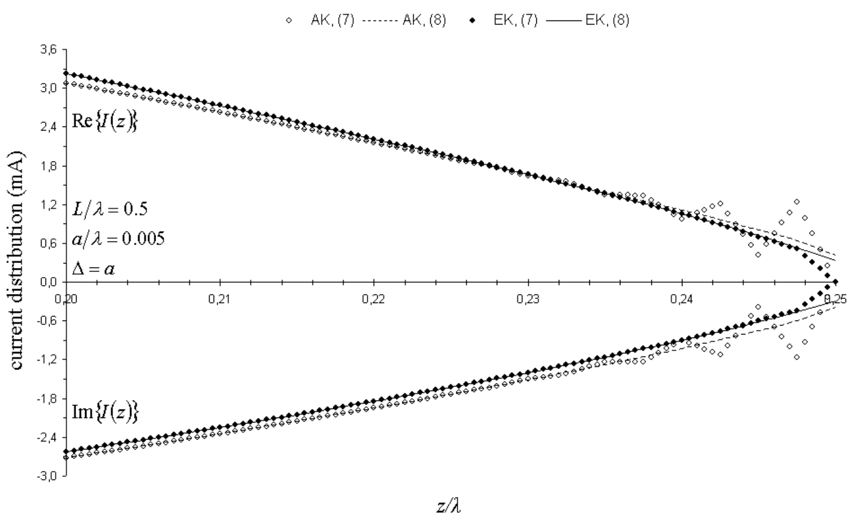
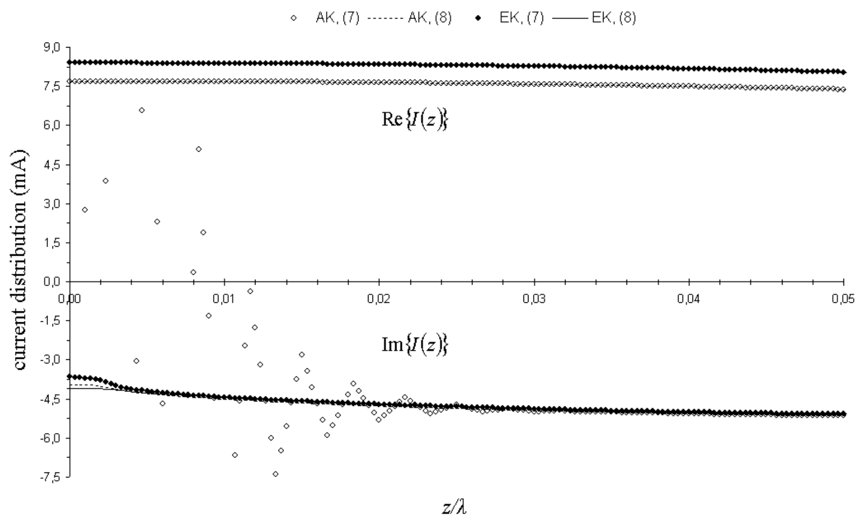


(b)

Figure 2. Computed current distribution on a dipole with $L/\lambda = 0.5$ and $a/\lambda = 0.005$, which is symmetrically excited by a finite-gap generator with $\Delta = a$, for $N + 1 = 100$, (a) Current distribution near the ends, (b) Current distribution near the gap.



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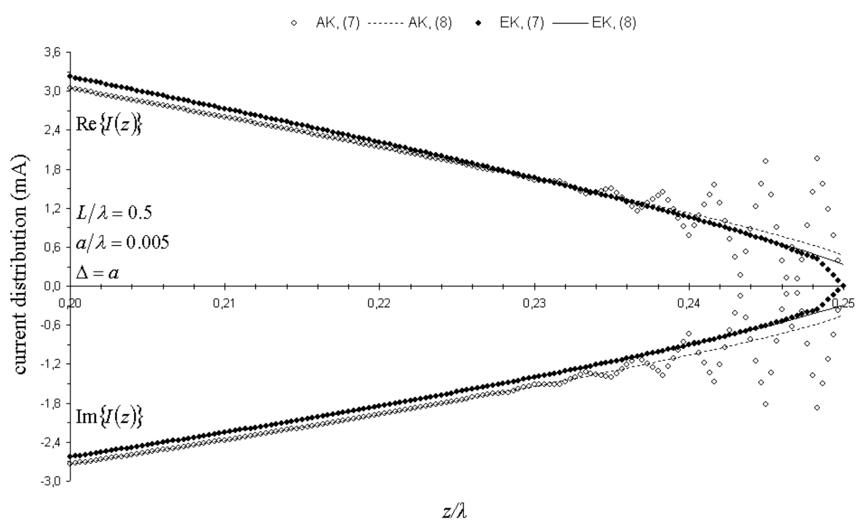

(b)

Figure 3. Computed current distribution on a dipole with $L/\lambda = 0.5$ and $a/\lambda = 0.005$, which is symmetrically excited by a finite-gap generator with $\Delta = a$, for $N + 1 = 150$. In Fig. 3(a), certain values of the non-smoothed AK curve are out of scale, (a) Current distribution near the ends, (b) Current distribution near the gap.

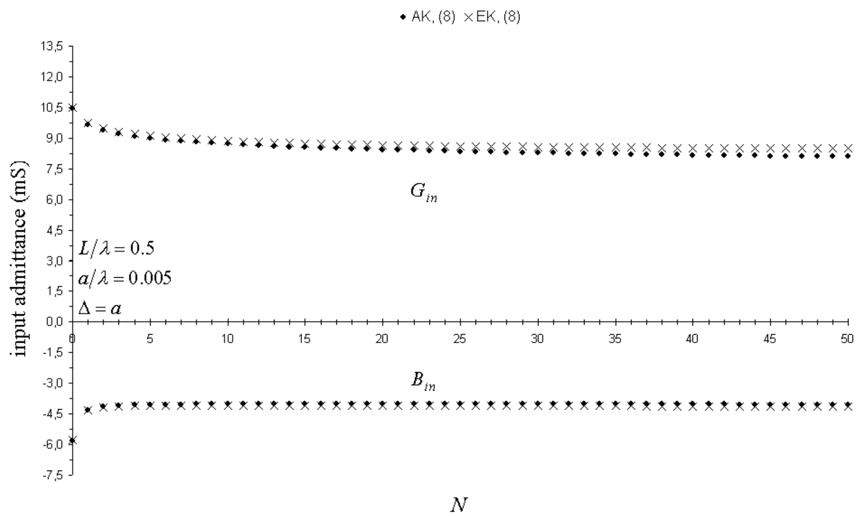

Figure 4. Computed input admittance towards N for a center-driven dipole with $L/\lambda = 0.5$, $a/\lambda = 0.005$ and $\Delta = a$.

Although non-smoothed AK solutions are not expected to converge, it is of interest to examine smoothed AK solutions and, particularly, the closeness of such solutions to the ones derived from the EK formulation. For this purpose, input admittance data are provided in Fig. 4 for the cylindrical dipole of Figs. 1–3, as a function of N. From the presented results, it is obvious that the AK data are quite accurate. Further increase in N reveals that the AK results remain quite close to the EK ones, even for $N > L/(2a)$, as it can be also inferred from Figs. 2 and 3.

4. CONCLUSION

Although oscillations in the expansion coefficients and the associated line currents are unavoidable in AK formulations of thin-wire antennas, meaningful and useful current distributions can be derived by calculating a smoothed current through the magnetic field at a distance a away from the line current. This is true, at least, for the numerical method considered here (Galerkin's formulation with piecewise sinusoidal basis/testing functions). Extensive checks, a small part of which was presented above, revealed that the remedy at hand is effective over a wide range of discretization levels (number of basis/testing functions). In practice, the remedy is restricted only by round-off errors, which render the solutions useless. Moreover, albeit not exhibited here, a similar behavior was ascertained for different excitation types, a fact that further reinforces the use of the proposed remedy for overcoming the complications caused by the oscillations accompanying AK formulations.

APPENDIX A.

Adopting the notation used in [25], the entries of the interaction matrix are expressed as

$$Z_{n,m} = -\int_{z_m-\delta}^{z_m+\delta} \left[\hat{z}\cdot\vec{E}_n(a,z)\right] f_m(z)dz, \tag{A1}$$

where $\vec{E}_n$ stands for the electric field generated by the sinusoidal current of the basis function $f_n(z)$. Due to symmetry, the reaction integrals depend on $|m-n|$ only and the interaction matrix is of symmetric Toeplitz type, which can be constructed from its first or last row/column as $Z_{n,m} = Z_{\pm N\mp|m-n|,\pm N}$. When the EK is used, the basis functions are of tubular nature and, therefore, the field in (A1) can be obtained from the following circumferential integral

$$\hat{z}\cdot\vec{E}_n(a,z) = -\frac{j\zeta_0}{8\pi^2}\int\limits_{2\pi}\left[\frac{e^{-jk_0R_n^+}}{R_n^+}+\frac{e^{-jk_0R_n^-}}{R_n^-}-2\cos\left(k_0\delta\right)\frac{e^{-jk_0R_n}}{R_n}\right]d\phi', \tag{A2}$$

where $\zeta_0 = 120\pi\Omega$ and

$$R_n^{\pm} = \sqrt{2a^2\left(1-\cos\phi'\right)+\left(z-z_n\mp\delta\right)^2}, \tag{A3}$$

$$R_n = \sqrt{2a^2\left(1-\cos\phi'\right)+\left(z-z_n\right)^2}. \tag{A4}$$

The index 2π in any integral herein denotes that the integration can be performed over any range of the form $\xi < \phi' < \xi+2\pi$ with ξ arbitrarily chosen.

The singularities occurring for $\phi'\to 0$ when $z-z_n\to 0$ (notice that $z_n\pm\delta = z_{n\pm1}$ and $R_n^{\pm} = R_{n\pm1}$) are integrable and can be computed by first isolating the singularity arising for $\phi'\to 0$, as follows

$$\int\limits_{2\pi}\frac{e^{-jk_0R_n}}{R_n}d\phi' = \int\limits_{2\pi}\frac{1}{R_n}d\phi' + \int\limits_{2\pi}\frac{e^{-jk_0R_n}-1}{R_n}d\phi'. \tag{A5}$$

The second integral in (A5) has a smooth integrand and, therefore, can be computed with the aid of conventional quadrature algorithms for any z. With regard to the first integral in (A5), this can be expressed in terms of the complete elliptic integral of the first kind. Although there exist extended (open) integration rules for treating improper integrals [28], these sometimes suffer from slow convergence rates. Instead of implementing such techniques, one can further split the integral into distinct parts, one of which is concentrated on a very small region embracing the singularity, defined by $-\gamma < \phi' < \gamma$ with γ small, so that the approximation $R_n \approx \sqrt{(a\phi')^2 + (z - z_n)^2}$ is quite adequate [29, 30]. Then, the singular reaction integrals of (A1) arising for $|m - n| \leq 2$ can be computed with relative ease by splitting them into properly selected ranges. By setting $n = 0$ and exploiting the symmetry about $z = 0$, the integration interval $0 < z < \delta$ is the only one associated with the singularities discussed above. Specifically, the singularities can be treated as follows

$$\left.\begin{array}{l}\displaystyle\int_0^\delta \int_{2\pi} \frac{1}{R_0} f_l(z) d\phi' dz = \int_0^{\gamma a} \int_{-\gamma}^{\gamma} \frac{1}{R_0} f_l(z) d\phi' dz \\ \displaystyle+ \int_0^{\gamma a} \int_{\gamma}^{2\pi-\gamma} \frac{1}{R_0} f_l(z) d\phi' dz + \int_{\gamma a}^{\delta} \int_{2\pi} \frac{1}{R_0} f_l(z) d\phi' dz \end{array}\right., \quad l = 0, 1. \tag{A6}$$

Apparently, the first integral in the right-hand side of (A6) is singular, whereas the other two are not. For sufficiently small values of γ, the condition $\gamma a << \delta$ holds, even when N is several times larger than the important parameter $L/(2a)$. Therefore, for $l = 0$, the current of the corresponding testing function within the interval $0 < z < \gamma a$ can be approximated by $f_0(z) \approx \sin(k_0\delta)\left(1 - k_0^2 z^2/2\right) - \cos(k_0\delta)(k_0 z)$. Similarly, for $l = 1$, the associated current can be approximated by $f_1(z) \approx k_0 z$. Hence, the first (singular) integral in the right-hand side of (A6) can be analytically derived in terms of the integrals T_c and T_s given below

$$\begin{aligned} T_c &= \int_0^{\gamma a} \int_{-\gamma}^{\gamma} \frac{1}{R_0} \left(1 - \frac{k_0^2 z^2}{2}\right) d\phi' dz \\ &= -\frac{\gamma}{6} \left\{ \left[\sqrt{2} + \ln(\sqrt{2} + 1)\right] (k_0 a\gamma)^2 - 24 \ln(\sqrt{2} + 1) \right\}, \end{aligned} \tag{A7}$$

$$T_s = \int_0^{\gamma a} \int_{-\gamma}^{\gamma} \frac{1}{R_0} (k_0 z)\, d\phi' dz = \gamma[\sqrt{2} - 1 + \ln(\sqrt{2} + 1)] k_0 a\gamma. \tag{A8}$$

Numerous tests were performed in order to assess the accuracy of the approximations discussed above. Highly accurate results were obtained for sufficiently small γ and different values for a, due to the properly formed integration ranges embracing the singularities. The numerical results illustrated in this paper were computed with $\gamma = \pi/180$. Finally, it is noted that all the non-singular integrals were computed using two-dimensional Gauss-Legendre rules [28].

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