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TRANSIENT ANALYSIS OF LOSSY NONUNIFORM TRANSMISSION LINES USING A TIME-STEP INTEGRATION METHOD

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Abstract—This paper presents an effective numerical method for the transient analysis of lossy transmission lines. With the discretization of the spatial variation of voltages and currents along the transmission lines while remaining the temporal derivatives unchanged, a semi-discrete model is derived from the telegrapher's equations. The time-step integration method is utilized to derive the recursive scheme of time advancing. A large time step can be used in the computation, meanwhile, its accuracy is guaranteed. Numerical examples are presented to demonstrate the stability and accuracy of the proposed method.

1. INTRODUCTION

As the speed of integrated circuits increases, the transmission-line behavior of interconnects has significantly affected the signal integrity, and their accurate modeling and simulation have become an essential part of the design process. This issue has been discussed in a number of works and various methodologies have been presented for the analysis of transmission lines [1–13]. For uniform transmission lines with or without frequency dependent parameters, both traditional numerical methods (e.g., IFFT [2], AWE [5] and CFH [6]) and recently developed macromodeling methods (MoC [10] and MRA [11]) can provide efficient solutions. However, they become complex to deal with nonuniform transmission lines, which will degrade both simplicity and efficiency of the algorithms.

The finite-difference time-domain (FDTD) method is widely used in solving various kinds of electromagnetic problems. It is also a common way to simulate transient response of lossy multiconductor transmission lines (MTLs) [7, 8]. This method discretizes the telegrapher's equations both in time and space and the resulting difference equations are solved using the leap-frog scheme. Due to its simplicity and flexibility, it doesn't need to decouple transmission lines in the simulation, and it is straightforward to deal with nonuniform cases. However, this algorithm is restricted by the Courant–Friedrich–Levy (CFL) condition to ensure its stability. The Crank-Nicolson (CN) scheme [14] and the alternating-direction implicit (ADI) technique [15] are two typical solutions to eliminate the CFL limit. Although they are unconditionally stable, the dispersion errors of results will become evident with the increase of time step.

In this paper, an efficient and precise time-step integration method is proposed for the transient analysis of transmission lines. In contrast to FDTD method, this approach only discretizes the spatial derivatives in the telegrapher's equations, while the temporal derivatives remain unchanged. In this way, a semi-discrete model is derived. Similar to the FDTD algorithm, the modal decomposition is not needed in this model. Thus, nonuniform transmission lines with arbitrary coupling status can be easily dealt with. The analytical solution of the semi-discrete model can be achieved using the time-step integration algorithm. Unlike other unconditionally stable methods, its accuracy is guaranteed even for a large time step. Therefore, the computational efficiency can be improved significantly. The stability and accuracy of this method are illustrated by two numerical examples.

2. DEVELOPMENT OF THE SEMI-DISCRETE MODEL

Consider N-coupled transmission lines represented by the telegrapher's equations as

$$\frac{\partial}{\partial x}\boldsymbol{V}(x,t) = -\boldsymbol{R}(x)\boldsymbol{I}(x,t) - \boldsymbol{L}(x)\frac{\partial}{\partial t}\boldsymbol{I}(x,t) \tag{1}$$

$$\frac{\partial}{\partial x}\boldsymbol{I}(x,t) = -\boldsymbol{G}(x)\boldsymbol{V}(x,t) - \boldsymbol{C}(x)\frac{\partial}{\partial t}\boldsymbol{V}(x,t) \tag{2}$$

where

$$\boldsymbol{V}(x,t) = [V1(x,t), V2(x,t), \ldots, VN(x,t)]^T$$
$$\boldsymbol{I}(x,t) = [I1(x,t), I2(x,t), \ldots, IN(x,t)]^T$$

and $\boldsymbol{R}(x)$, $\boldsymbol{L}(x)$, $\boldsymbol{C}(x)$, and $\boldsymbol{G}(x)$ are per-unit-length (p.u.l.) parameter matrices of the transmission lines.

Several techniques have been presented for the spatial representation of voltages and currents along the transmission lines, such as finite difference [7, 8], Chebyshev polynomial [1, 16], and wavelets [17], and corresponding semi-discrete models can be developed, respectively. In order to compare with the FDTD method, the finite-difference scheme is used in this paper.

As the FDTD algorithm, we interlace the $M + 1$ voltage points $(\boldsymbol{V}_1, \boldsymbol{V}_2, \cdots, \boldsymbol{V}_{M+1})$ and the M current points $(\boldsymbol{I}_1, \boldsymbol{I}_2, \cdots, \boldsymbol{I}_M)$ along the transmission lines. The current points at two ends are $\boldsymbol{I}_0$ and $\boldsymbol{I}_{M+1}$. Each voltage and adjacent current solution point is separated by $\Delta x/2$ as shown in Figure 1.

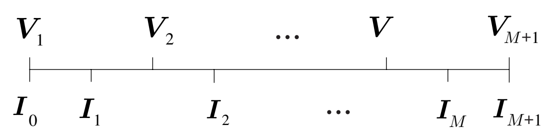

Figure 1. Discretization of voltages and currents along the lines.

Next, the spatial derivatives of voltage and current points are approximated by central differences. At the two end points, forward and backward difference schemes are used. Then, the telegrapher's equations after discretization are given by

$$\frac{d}{dt}\boldsymbol{I}_i(t) = -\frac{1}{\Delta x}\boldsymbol{L}_i^{-1}\left[\boldsymbol{V}_{i+1}(t) - \boldsymbol{V}_i(t)\right] - \boldsymbol{L}_i^{-1}\boldsymbol{R}_i\boldsymbol{I}_i(t) \quad (i = 1, 2, \cdots, M) \tag{3}$$

$$\frac{d}{dt}\boldsymbol{V}_{i+1}(t) = -\frac{1}{\Delta x}\boldsymbol{C}_{i+1}^{-1}\left[\boldsymbol{I}_{i+1}(t - \boldsymbol{I}_i(t)\right] - \boldsymbol{C}_{i+1}^{-1}\boldsymbol{G}_{i+1}\boldsymbol{V}_{i+1}(t) \quad (i = 1, 2, \cdots, M-1) \tag{4}$$

where $\boldsymbol{L}_i = \boldsymbol{L}[(i-0.5)\Delta x]$, $\boldsymbol{R}_i = \boldsymbol{R}[(i-0.5)\Delta x]$, $\boldsymbol{C}_i = \boldsymbol{C}[(i-1)\Delta x]$, and $\boldsymbol{G}_i = \boldsymbol{G}[(i-1)\Delta x]$. The discrete equations at the two end points of the lines are

$$\frac{d}{dt}\boldsymbol{V}_1(t) = -\frac{2}{\Delta x}\boldsymbol{C}_1^{-1}\left[\boldsymbol{I}_1(t) - \boldsymbol{I}_0(t)\right] - \boldsymbol{C}_1^{-1}\boldsymbol{G}_1\boldsymbol{V}_1(t) \tag{5}$$

$$\frac{d}{dt}\boldsymbol{V}_{M+1}(t) = -\frac{2}{\Delta x}\boldsymbol{C}_{M+1}^{-1}[\boldsymbol{I}_{M+1}(t) - \boldsymbol{I}_M(t)] - \boldsymbol{C}_{M+1}^{-1}\boldsymbol{G}_{M+1}\boldsymbol{V}_{M+1}(t) \tag{6}$$

Combining (3)–(6) and writing them in the matrix form yield:

$$\frac{d\boldsymbol{X}}{dt} = \boldsymbol{H}X + \boldsymbol{F} \tag{7}$$

where $X = \begin{pmatrix} \boldsymbol{V}_1 & \boldsymbol{I}_1 & \boldsymbol{V}_2 & \cdots & \boldsymbol{I}_M & \boldsymbol{V}_{M+1} \end{pmatrix}^T$

$$\boldsymbol{H}=\begin{bmatrix} -\boldsymbol{C}_1^{-1}\boldsymbol{G}_1 & -\frac{2}{\Delta x}\boldsymbol{C}_1^{-1} & 0 & & & & 0 \\ \frac{1}{\Delta x}\boldsymbol{L}_1^{-1} & -\boldsymbol{L}_1^{-1}\boldsymbol{R}_1 & -\frac{1}{\Delta x}\boldsymbol{L}_1^{-1} & & & & \\ & \frac{1}{\Delta x}\boldsymbol{C}_2^{-1} & -\boldsymbol{C}_2^{-1}\boldsymbol{G}_2 & -\frac{1}{\Delta x}\boldsymbol{C}_2^{-1} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \frac{1}{\Delta x}\boldsymbol{L}_M^{-1} & -\boldsymbol{L}_M^{-1}\boldsymbol{R}_M & -\frac{1}{\Delta x}\boldsymbol{L}_M^{-1} \\ 0 & & & 0 & \frac{2}{\Delta x}\boldsymbol{C}_{M+1}^{-1} & -\boldsymbol{C}_{M+1}^{-1}\boldsymbol{G}_{M+1} \end{bmatrix}$$

$$\boldsymbol{F} = \begin{pmatrix} \frac{2}{\Delta x}\boldsymbol{C}_1^{-1}\boldsymbol{I}_0 & 0 & 0 & \cdots & 0 & -\frac{2}{\Delta x}\boldsymbol{C}_{M+1}^{-1}\boldsymbol{I}_{M+1} \end{pmatrix}^T$$

The terminal networks of the transmission lines are usually characterized using the state-variable formulation [8], which is compatible to the form of (7). Therefore, the state-variable equations can be easily combined into (7), and the resulting equation is given by

$$\frac{d\boldsymbol{X}_1}{dt} = \boldsymbol{H}_1\boldsymbol{X}_1 + \boldsymbol{F}_1 \tag{8}$$

where $\boldsymbol{X}_1$, $\boldsymbol{H}_1$, and $\boldsymbol{F}_1$ are the modified matrices of $\boldsymbol{X}$, $\boldsymbol{H}$, and $\boldsymbol{F}$, respectively. Provided that the terminal networks are linear, only the variables associated with input signals will exist in $\boldsymbol{F}_1$.

3. TIME-STEP INTEGRATION CALCULATION

The well-known solution of Equation (8) can be written as

$$\boldsymbol{X}_1(t) = \exp\left(\boldsymbol{H}_1 \cdot t\right)\boldsymbol{X}_1(0) + \int_0^t \exp\left[\boldsymbol{H}_1 \cdot (t-\zeta)\right]\boldsymbol{F}_1(\zeta)d\zeta \tag{9}$$

Assuming the time step is τ, for a time interval (t_j, t_{j+1}) where $t_j = j\tau\,(j = 0, 1, 2, \cdots)$, we have

$$\boldsymbol{X}_1(t_{j+1}) = \exp\left(\boldsymbol{H}_1 \cdot \tau\right)\boldsymbol{X}_1\left(t_j\right) + \int_{t_j}^{t_{j+1}} \exp\left[\boldsymbol{H}_1 \cdot (t_{j+1}-\zeta)\right]\boldsymbol{F}_1(\zeta)d\zeta \tag{10}$$

In order to calculate the integration item in an analytical way, we assume that $\boldsymbol{F}_1$ is linear within this time interval as

$$\boldsymbol{F}_1 = \boldsymbol{r}_0 + \boldsymbol{r}_1\left(t - t_j\right) \tag{11}$$

where $\boldsymbol{r}_0$ and $\boldsymbol{r}_1$ are known vectors. It is worthy noting that the inputs of high-speed circuits are generally characterized in the piecewise-linear form (e.g., trapezoidal pulses), this assumption is therefore quite reasonable for the practical application. Substituting (11) into (10), the time-step integration expression can be transformed into the recursive form:

$$\boldsymbol{X}_1^{j+1} = \boldsymbol{T} \cdot \boldsymbol{X}_1^j + \boldsymbol{Q} \tag{12}$$

where

$$\boldsymbol{Q} = (\boldsymbol{T} - \boldsymbol{I})\, \boldsymbol{H}_1^{-1} \left(\boldsymbol{r}_0 + \boldsymbol{H}_1^{-1}\boldsymbol{r}_1\right) - \tau \boldsymbol{H}_1^{-1}\boldsymbol{r}_1 \tag{13}$$

and $\boldsymbol{T} = \exp[\boldsymbol{H}_1 \cdot \tau]$. In (12), superscripts j and $j+1$ of $\boldsymbol{X}_1$ denote its value at t_j and t_{j+1}, respectively. During an arbitrary linear period of input signal waveforms, $\boldsymbol{Q}$ is a constant vector. In particular, for the constant region of input signals, $\boldsymbol{Q}$ can be reduced to a simpler one:

$$\boldsymbol{Q} = (\boldsymbol{T} - \boldsymbol{I})\, \boldsymbol{H}_1^{-1}\boldsymbol{r}_0 \tag{14}$$

Therefore, when the initial value is given, $\boldsymbol{X}_1$ can be calculate in this recursive way.

The matrix exponential $\boldsymbol{T}$ is the key factor in (12). Extensive work has been done to achieve algorithms for computing the matrix exponential [18]. The scaling and squaring method [19] is one of the most effective ways. It has been implemented in MATLAB's expm function and is used to calculate $\boldsymbol{T}$ in this paper.

4. NUMERICAL RESULTS

In order to describe the characteristics of the proposed method, a simple transmission line system is considered firstly, as shown in Figure 2. The p.u.l. parameters of the line are $R = 8.24\,\Omega/\text{m}$, $L = 309\,\text{nH/m}$, $C = 144\,\text{pF/m}$, and $G = 905\,\text{nS/m}$, with the length of 30 cm. The input voltage source is a 1-V pulse with a 0.5-ns rise/fall time and a width of 5 ns.

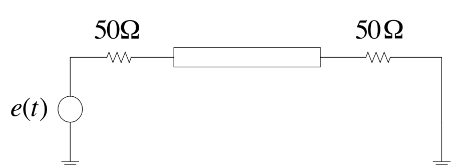

Figure 2. A single transmission line system.

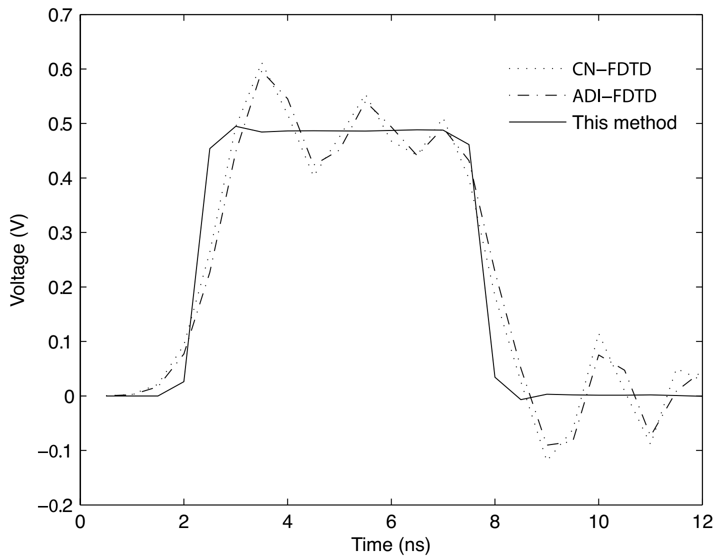

Figure 3. Transient response at the far end of the line ($\tau = 0.5\,\text{ns}$).

It is assumed that the transmission line is divided into 30 segments. As two unconditionally stable algorithms, both CN-FDTD method (the CN parameter is equal to 0.5) and ADI-FDTD method are utilized for comparison. When the time step is small, all these methods can achieve accurate results. In order to improve computational efficiency, the time step should be enlarge. As an extreme scenario, we choose the time step equal to the rise time of the input voltage (0.5 ns). The transient voltages at the far end of the transmission line are depicted in Figure 3. It is shown that although CN- and ADI-FDTD methods are stable, large discrepancies are observed in the results, which are not tolerable for accurate circuit simulation. By contrast, the result of the proposed method is both stable and accurate.

As the second example, we consider a two-coupled nonuniform transmission line system as shown in Figure 4. The input voltage source is a 1-V pulse with a 0.5-ns rise/fall time and a width of 3 ns. The length of the coupled line is 5 cm, and the p.u.l. parameters are represented as follows:

$$\boldsymbol{L} = \begin{bmatrix} L(x) & Lm(x) \\ Lm(x) & L(x) \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} C(x) & Cm(x) \\ Cm(x) & C(x) \end{bmatrix}, \boldsymbol{R} = \begin{bmatrix} R(x) & 0 \\ 0 & R(x) \end{bmatrix},$$

$$\boldsymbol{G} = \begin{bmatrix} G(x) & 0 \\ 0 & G(x) \end{bmatrix}$$

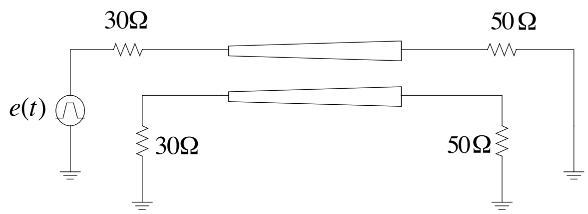

Figure 4. Circuit of nonuniform transmission lines.

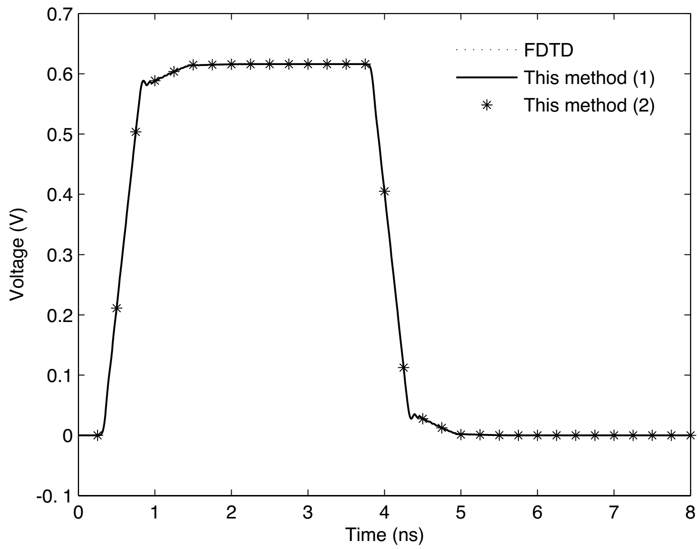

Figure 5. Transient response at far end of the active line.

where

$$L(x) = 387/[1 + K(x)] \text{ nH/m}$$
$$Lm(x) = k(x)L(x)$$
$$C(x) = 104.3/[1 - k(x)] \text{ pF/m}$$
$$Cm(x) = -k(x)C(x)$$
$$R(x) = 30/[1 + k(x)] \ \Omega/\text{m}$$
$$G(x) = 0.001/[1 - k(x)] \text{ S/m}$$
$$k(x) = 0.25[1 + \sin(6.25\pi x + 0.25\pi)]$$

The transient voltages at the far end of the lines are depicted in Figures 5 and 6. It is assumed that the nonuniform lines are

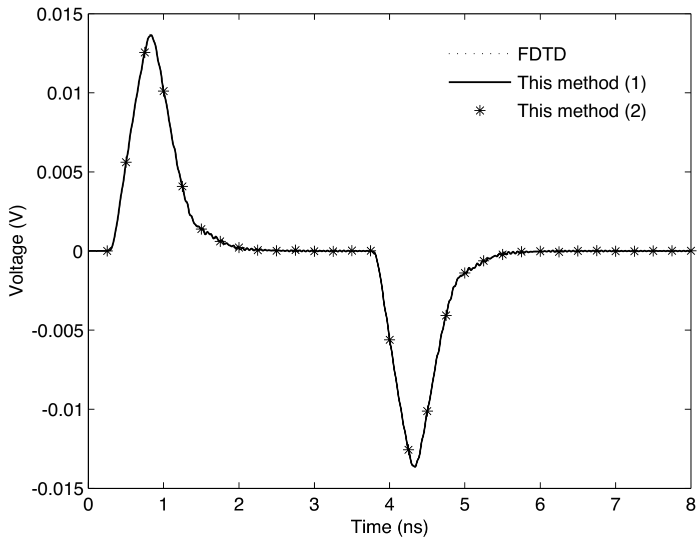

Figure 6. Transient response at far end of the victim line.

divided into 20 segments in the length and the time step is chosen to be 10 ps in both the proposed method and FDTD method. The results are described by the dotted line and solid line, respectively. For the proposed method, the CPU time on Pentium IV PC (3.0 GHz) is 0.075 s. Larger time step can be used to improve the computational efficiency. As described in the last section, the time step will not affect the accuracy of results. Accurate results with $\tau = 250$ ps ('star' symbol) are also depicted in Figures 5 and 6. The CPU time is then reduced to 0.012 s. As an extreme scenario, we can even choose the time step equal to the rise time of the input voltage (500 ps). However, constrained by the CFL condition, the results of the FDTD method become not convergent as the time step increases (e.g., $\tau = 20$ ps in this example).

5. CONCLUSION

An effective numerical method is presented for the analysis of transient response of lossy transmission lines. Based on the semi-discrete model developed in this paper, the time-step integration method is utilized to calculate the time response of transmission lines. Due to its analytical solution in the time domain, large time step can be chosen in the calculation. Numerical examples show that the results of this method are both accurate and stable.

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