

ANALYSIS OF NONUNIFORM TRANSMISSION LINES USING THE EQUIVALENT SOURCES

M. Khalaj-Amirhosseini

College of Electrical Engineering
Iran University of Science and Technology
Tehran, Iran

Abstract—A new method is introduced to analyze arbitrary nonuniform transmission lines. In this method, the equations of nonuniform transmission lines are converted to the equations of uniform transmission lines, which have been excited by distributed equivalent sources. Then, the voltage and current distributions are obtained using an iterative method. The validity of the method is verified using a comprehensive example.

1. INTRODUCTION

Nonuniform Transmission Lines (NTL) are widely used in microwave circuits as resonators [1], impedance matchers [1, 2], delay equalizers [3], filters [4], wave shapers [5], analog signal processors [6] and etc. The differential equations describing these structures have non-constant coefficients because their primary parameters vary along the line. So, except for a few special cases, no analytical solution exists for NTLs. Therefore, many efforts have been done to analyze NTLs. The most used method is subdividing the NTLs into many short sections [7–10]. Analysis of arbitrary NTLs using Taylor's and the Fourier series expansion of the primary parameters has been introduced in [11–14]. In this paper, a new method is introduced to analyze arbitrary NTLs, also. In this method, the equations of nonuniform transmission lines are converted to the equations of uniform transmission lines, which have been excited by distributed equivalent sources. Then, the voltage and current distributions are obtained using an iterative method. This method is applicable to all arbitrary lossy and dispersive NTLs. The validity of the method is verified using a comprehensive example.

2. DIFFERENTIAL EQUATIONS OF NTLS

In this section, the equations related to NTLs in the frequency domain are reviewed. It is assumed that the principal propagation mode of the lines is TEM or quasi-TEM. This assumption is valid when widths in the cross section are small enough compared to the wavelength. Fig. 1 shows a typical NTL whose length is d and has been terminated by arbitrary loads $Z_S(\omega)$ and $Z_L(\omega)$.

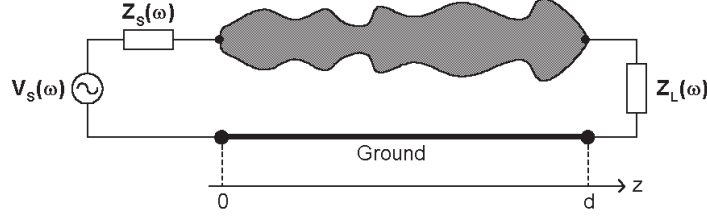


Figure 1. A typical NTL with length d , terminated by arbitrary loads.

The differential equations describing lossy and dispersive NTLs in the frequency domain are given by

$$\frac{dV(z)}{dz} = -Z(z)I(z) \quad (1)$$

$$\frac{dI(z)}{dz} = -Y(z)V(z) \quad (2)$$

in which

$$Z(z) = R(z) + j\omega L(z) \quad (3)$$

$$Y(z) = G(z) + j\omega C(z) \quad (4)$$

In (3)–(4), R , L , G and C are frequency dependent distributed primary parameters of transmission lines. The secondary parameters of transmission lines (the characteristic impedance and the propagation coefficient) will be as follows

$$Z_c(z) = \sqrt{\frac{Z(z)}{Y(z)}} = \sqrt{\frac{R(z) + j\omega L(z)}{G(z) + j\omega C(z)}} \quad (5)$$

$$\begin{aligned} \gamma_c(z) &= \alpha_c(z) + j\beta_c(z) = \sqrt{Z(z)Y(z)} \\ &= \sqrt{[R(z) + j\omega L(z)][G(z) + j\omega C(z)]} \end{aligned} \quad (6)$$

Furthermore, the terminal conditions for the loaded NTLs are as follows

$$V(0) + Z_S I(0) = V_S \quad (7)$$

$$V(d) - Z_L I(d) = 0 \quad (8)$$

One sees from (1)–(8) that, solving analytically the equations of general NTLs is a difficult problem.

3. THE EQUIVALENT SOURCES METHOD

In this section, the analysis of arbitrary loaded NTLs using the method of equivalent sources is introduced. First, the average of the primary parameters are defined as follows

$$\bar{Z} = \frac{1}{d} \int_0^d Z(z) dz \quad (9)$$

$$\bar{Y} = \frac{1}{d} \int_0^d Y(z) dz \quad (10)$$

The differential equations (1) and (2) can be converted to the following equations

$$\frac{dV(z)}{dz} = -\bar{Z}I(z) + V_F(z) \quad (11)$$

$$\frac{dI(z)}{dz} = -\bar{Y}V(z) + I_F(z) \quad (12)$$

The equations (11) and (12) are related to uniform transmission lines, which have been excited by distributed equivalent sources defined by

$$V_F(z) = -(Z(z) - \bar{Z})I(z) \quad (13)$$

$$I_F(z) = -(Y(z) - \bar{Y})V(z) \quad (14)$$

Combining (11) and (12) with each other, gives the following differential equations

$$\frac{d^2V(z)}{dz^2} - \bar{\gamma}^2 V(z) = \frac{dV_F(z)}{dz} - \bar{Z}I_F(z) \quad (15)$$

$$I(z) = \frac{1}{\bar{Z}} \left(V_F(z) - \frac{dV(z)}{dz} \right) \quad (16)$$

where

$$\bar{\gamma} = \sqrt{\bar{Z}\bar{Y}} \quad (17)$$

As it has been shown in the Appendix, the voltage and current distributions are obtained from (15) and (16) as follows

$$\begin{aligned} V(z) = & V^+ \exp(-\bar{\gamma}z) + V^- \exp(\bar{\gamma}z) \\ & + \frac{1}{2\bar{\gamma}} \exp(\bar{\gamma}z) \int_0^z \exp(-\bar{\gamma}z') \left(\frac{dV_F(z')}{dz'} - \bar{Z}I_F(z') \right) dz' \\ & - \frac{1}{2\bar{\gamma}} \exp(-\bar{\gamma}z) \int_0^z \exp(\bar{\gamma}z') \left(\frac{dV_F(z')}{dz'} - \bar{Z}I_F(z') \right) dz' \quad (18) \end{aligned}$$

$$\begin{aligned} I(z) = & \frac{1}{\bar{Z}} V_F(z) + \frac{V^+}{\bar{Z}_c} \exp(-\bar{\gamma}z) - \frac{V^-}{\bar{Z}_c} \exp(\bar{\gamma}z) \\ & - \frac{1}{2\bar{Z}} \exp(\bar{\gamma}z) \int_0^z \exp(-\bar{\gamma}z') \left(\frac{dV_F(z')}{dz'} - \bar{Z}I_F(z') \right) dz' \\ & - \frac{1}{2\bar{Z}} \exp(-\bar{\gamma}z) \int_0^z \exp(\bar{\gamma}z') \left(\frac{dV_F(z')}{dz'} - \bar{Z}I_F(z') \right) dz' \quad (19) \end{aligned}$$

The constants V^+ and V^- in (18) and (19) are obtained from the terminal conditions (7) and (8) as follows

$$\begin{aligned} V^+ = & \frac{1}{1 - \bar{\Gamma}_S \bar{\Gamma}_L \exp(-2\bar{\gamma}d)} \\ & \times \left[\frac{\bar{Z}_c}{Z_S + \bar{Z}_c} V_S - \frac{1}{\bar{\gamma}} \frac{Z_S}{Z_S + \bar{Z}_c} V_F(0) + \bar{\Gamma}_S \frac{\exp(-\bar{\gamma}d)}{\bar{\gamma}} \frac{Z_L}{Z_L + \bar{Z}_c} V_F(d) \right. \\ & - \bar{\Gamma}_S \frac{1}{2\bar{\gamma}} \int_0^d \exp(-\bar{\gamma}z') \left(\frac{dV_F(z')}{dz'} - \bar{Z}I_F(z') \right) dz' \\ & \left. - \bar{\Gamma}_S \bar{\Gamma}_L \frac{\exp(-2\bar{\gamma}d)}{2\bar{\gamma}} \int_0^d \exp(\bar{\gamma}z') \left(\frac{dV_F(z')}{dz'} - \bar{Z}I_F(z') \right) dz' \right] \quad (20) \\ V^- = & \frac{1}{1 - \bar{\Gamma}_S \bar{\Gamma}_L \exp(-2\bar{\gamma}d)} \\ & \times \left[\bar{\Gamma}_L \exp(-2\bar{\gamma}d) \frac{\bar{Z}_c}{Z_S + \bar{Z}_c} V_S - \bar{\Gamma}_L \frac{\exp(-2\bar{\gamma}d)}{\bar{\gamma}} \frac{Z_S}{Z_S + \bar{Z}_c} V_F(0) \right. \end{aligned}$$

$$\begin{aligned}
& + \frac{\exp(-\bar{\gamma}d)}{\bar{\gamma}} \frac{Z_L}{Z_L + \bar{Z}_c} V_F(d) \\
& - \frac{1}{2\bar{\gamma}} \int_0^d \exp(-\bar{\gamma}z') \left(\frac{dV_F(z')}{dz'} - \bar{Z} I_F(z') \right) dz' \\
& - \frac{\exp(-2\bar{\gamma}d)}{2\bar{\gamma}} \bar{\Gamma}_L \int_0^d \exp(\bar{\gamma}z') \left(\frac{dV_F(z')}{dz'} - \bar{Z} I_F(z') \right) dz' \Big] \quad (21)
\end{aligned}$$

where

$$\bar{\Gamma}_S = \frac{Z_S - \bar{Z}_c}{Z_S + \bar{Z}_c} \quad (22)$$

$$\bar{\Gamma}_L = \frac{Z_L - \bar{Z}_c}{Z_L + \bar{Z}_c} \quad (23)$$

in which

$$\bar{Z}_c = \sqrt{\frac{\bar{Z}}{\bar{Y}}} \quad (24)$$

4. AN ITERATIVE APPROACH

The voltage and current distributions obtained as (18)–(21) require the distributed equivalent sources defined in (13) and (14). On the other hand, the equivalent sources require the voltage and current distributions. To overcome this problem, we can use an iterative method. At first iteration, we consider the equivalent sources be zero.

$$V_F^{(1)}(z) = I_F^{(1)}(z) = 0 \quad (25)$$

The voltage and current distributions at first iteration are obtained from (18)–(21).

$$V^{(1)}(z) = V^{+(1)} \exp(-\bar{\gamma}z) + V^{-(1)} \exp(\bar{\gamma}z) \quad (26)$$

$$I^{(1)}(z) = \frac{1}{\bar{Z}_c} \left(V^{+(1)} \exp(-\bar{\gamma}z) - V^{-(1)} \exp(\bar{\gamma}z) \right) \quad (27)$$

in which

$$V^{+(1)} = \frac{\exp(2\bar{\gamma}d)}{\bar{\Gamma}_L} V^{-(1)} = V_S \frac{\bar{Z}_c}{Z_S + \bar{Z}_c} \frac{1}{1 - \bar{\Gamma}_L \bar{\Gamma}_S \exp(-2\bar{\gamma}d)} \quad (28)$$

Then, the equivalent sources are corrected at the second iteration using (13) and (14). Consequently, using (18)–(21) and (13)–(14) alternately, the voltage and current distributions are obtained with a low error.

The integrals existed in (9)–(10) and (18)–(21) are exactly calculated if the primary parameters are known at all points, continuously. However, these integrals can be approximately calculated if the primary parameters are known only at some points. In this case, which is more practical, we can assume that the primary parameters vary between two adjacent points stepwise, linearly or in another manner.

5. EXAMPLE AND RESULTS

In this section, a comprehensive example is presented to study the validity of the introduced method. Consider a lossless and exponential NTL with the following primary parameters.

$$L(z) = L_0 \exp(kz/d) \quad (29)$$

$$C(z) = C_0 \exp(-kz/d) \quad (30)$$

$$R(z) = G(z) = 0 \quad (31)$$

This type of transmission line will have the following secondary parameters defined in (5)–(6)

$$Z_c(z) = \sqrt{L_0/C_0} \exp(kz/d) \quad (32)$$

$$\gamma_c = j\beta_c = j\omega \sqrt{L_0 C_0} \quad (33)$$

Also, the average of the primary parameters defined in (9) and (10) will be as follows

$$\bar{Z} = j\omega L_0 \frac{\exp(k) - 1}{k} \quad (34)$$

$$\bar{Y} = j\omega C_0 \frac{1 - \exp(-k)}{k} \quad (35)$$

The equivalent sources at the second iteration can be obtained using (26)–(28) and (13)–(14).

$$\begin{aligned} V_F^{(2)}(z) = & j\omega L_0 \frac{V^{+(1)}}{\bar{Z}_c} \left(\frac{\exp(k) - 1}{k} - \exp(kz/d) \right) \\ & \times (\exp(-\bar{\gamma}z) - \bar{\Gamma}_L \exp(\bar{\gamma}(z - 2d))) \end{aligned} \quad (36)$$

$$\begin{aligned} I_F^{(2)}(z) = & j\omega C_0 V^{+(1)} \left(\frac{1 - \exp(-k)}{k} - \exp(-kz/d) \right) \\ & \times (\exp(-\bar{\gamma}z) + \bar{\Gamma}_L \exp(\bar{\gamma}(z - 2d))) \end{aligned} \quad (37)$$

Also, the voltage and current distributions at second iteration are obtained by substituting (36)–(37) in (18)–(21) as follows

$$V^{(2)}(z) = V^{+(2)} \exp(-\bar{\gamma}z) + V^{-(2)} \exp(\bar{\gamma}z) + \frac{1}{2\bar{\gamma}} \exp(\bar{\gamma}z)(A(z) - A(0)) - \frac{1}{2\bar{\gamma}} \exp(-\bar{\gamma}z)(B(z) - B(0)) \quad (38)$$

$$I^{(2)}(z) = \frac{1}{\bar{Z}} V_F^{(2)}(z) + \frac{V^{+(2)}}{\bar{Z}_c} \exp(-\bar{\gamma}z) - \frac{V^{-(2)}}{\bar{Z}_c} \exp(\bar{\gamma}z) - \frac{1}{2\bar{Z}} \exp(\bar{\gamma}z)(A(z) - A(0)) - \frac{1}{2\bar{Z}} \exp(-\bar{\gamma}z)(B(z) - B(0)) \quad (39)$$

where

$$V^{+(2)} = \frac{1}{1 - \bar{\Gamma}_S \bar{\Gamma}_L \exp(-2\bar{\gamma}d)} \times \left(\frac{\bar{Z}_c}{Z_S + \bar{Z}_c} V_S - \frac{1}{\bar{\gamma}} \frac{Z_S}{Z_S + \bar{Z}_c} V_F^{(2)}(0) + \bar{\Gamma}_S \frac{\exp(-\bar{\gamma}d)}{\bar{\gamma}} \frac{Z_L}{Z_L + \bar{Z}_c} V_F^{(2)}(d) - \bar{\Gamma}_S \frac{1}{2\bar{\gamma}} (A(d) - A(0)) - \bar{\Gamma}_S \bar{\Gamma}_L \frac{\exp(-2\bar{\gamma}d)}{2\bar{\gamma}} (B(d) - B(0)) \right) \quad (40)$$

$$V^{-(2)} = \frac{1}{1 - \bar{\Gamma}_S \bar{\Gamma}_L \exp(-2\bar{\gamma}d)} \times \left(\bar{\Gamma}_L \exp(-2\bar{\gamma}d) \frac{\bar{Z}_c}{Z_S + \bar{Z}_c} V_S - \bar{\Gamma}_L \frac{\exp(-2\bar{\gamma}d)}{\bar{\gamma}} \frac{Z_S}{Z_S + \bar{Z}_c} V_F^{(2)}(0) + \frac{\exp(-\bar{\gamma}d)}{\bar{\gamma}} \frac{Z_L}{Z_L + \bar{Z}_c} V_F^{(2)}(d) - \frac{1}{2\bar{\gamma}} (A(d) - A(0)) - \frac{\exp(-2\bar{\gamma}d)}{2\bar{\gamma}} \bar{\Gamma}_L (B(d) - B(0)) \right) \quad (41)$$

in which $A(z)$ and $B(z)$ are two functions defined as

$$A(z) = j\omega L_0 \frac{V^{+(1)}}{\bar{Z}_c} \left[-\frac{\bar{\gamma} - k/d}{2\bar{\gamma} - k/d} \exp((k/d - 2\bar{\gamma})z) + \bar{\Gamma}_L \frac{\bar{\gamma} + k/d}{k/d} \exp(kz/d - 2\bar{\gamma}d) - \bar{\gamma} \frac{\exp(k) - 1}{k} \left(\bar{\Gamma}_L \exp(-2\bar{\gamma}d) - \frac{1}{2\bar{\gamma}} \exp(-2\bar{\gamma}z) \right) \right] - j\omega C_0 \bar{Z} V^{+(1)} \left[\frac{1 - \exp(-k)}{k} \left(\bar{\Gamma}_L \exp(-2\bar{\gamma}d) z - \frac{1}{2\bar{\gamma}} \exp(-2\bar{\gamma}z) \right) \right]$$

$$+ \frac{1}{k/d + 2\bar{\gamma}} \exp(-(k/d + 2\bar{\gamma})z) + \bar{\Gamma}_L \exp(-2\bar{\gamma}d) \frac{1}{k/d} \exp(-kz/d) \Big] \quad (42)$$

$$\begin{aligned} B(z) = & j\omega L_0 \frac{V^{+(1)}}{\bar{Z}_c} \left[\frac{\bar{\gamma} - k/d}{k/d} \exp(kz/d) \right. \\ & + \bar{\Gamma}_L \frac{\bar{\gamma} + k/d}{2\bar{\gamma} + k/d} \exp((k/d + 2\bar{\gamma})z - 2\bar{\gamma}d) \\ & - \bar{\gamma} \frac{\exp(k) - 1}{k} \left(z + \bar{\Gamma}_L \frac{1}{2\bar{\gamma}} \exp(2\bar{\gamma}(z - d)) \right) \Big] \\ & - j\omega C_0 \bar{Z} V^{+(1)} \left[\frac{1 - \exp(-k)}{k} \left(z + \bar{\Gamma}_L \frac{1}{2\bar{\gamma}} \exp(2\bar{\gamma}(z - d)) \right) \right. \\ & \left. + \frac{1}{k/d} \exp(-kz/d) - \bar{\Gamma}_L \exp(-2\bar{\gamma}d) \frac{1}{2\bar{\gamma} - k/d} \exp((2\bar{\gamma} - k/d)z) \right] \end{aligned} \quad (43)$$

Now, assume that $Z_c(0) = \sqrt{L_0/C_0} = 50 \Omega$, $\beta = \omega\sqrt{L_0C_0} = \omega/c$ (c is the velocity of the light), $k = 1$, $d = 20$ cm, $f = 1.0$ GHz, $Z_S = Z_c(0) = 50 \Omega$, $Z_L = Z_c(d) = 136 \Omega$ and $V_S = 1.0$ V. Figures 2–3, compare the magnitude of the voltage and current distributions obtained from exact solutions [11] and from the introduced method in its first and second iterations. One observes a good agreement between the exact solutions and the solutions obtained from the proposed method at the second iteration. Also, Figures 4–5 illustrate the magnitude of the input reflection coefficient, given by

$$\Gamma_{in} = \frac{V(0) - 50I(0)}{V(0) + 50I(0)} \quad (44)$$

versus the frequency and for $d = 10$ cm and 20 cm, respectively. It is concluded from Figs. 2–5 that the accuracy of the obtained solutions is increased as the iterations are increased. Also, as the source frequency or the length of the lines increases, the accuracy of the method is decreased. Hence, as the length of the lines or the source frequency increases, the necessary iterations have to be increased. From the above example, one may satisfy that the introduced method is very efficient and can be applicable to all arbitrary lossy and dispersive NTLs, whose primary parameters are known at all or even at some points along their length.

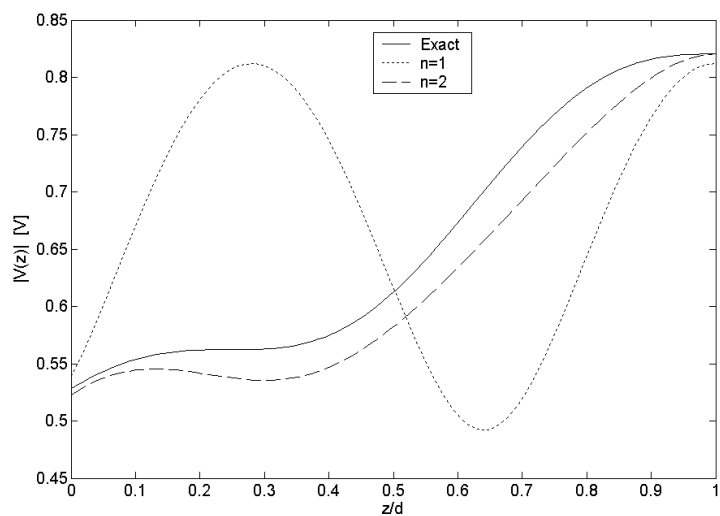


Figure 2. The magnitude of the voltage distribution for $d = 20$ cm at frequency $f = 1.0$ GHz.

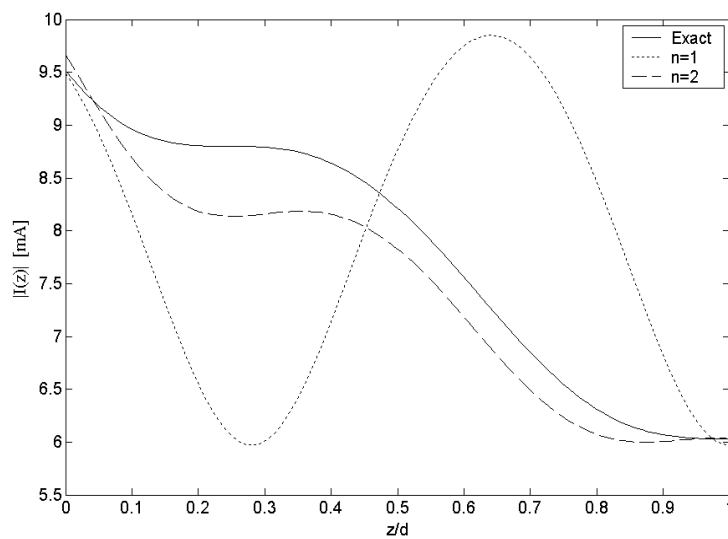


Figure 3. The magnitude of the current distribution for $d = 20$ cm at frequency $f = 1.0$ GHz.

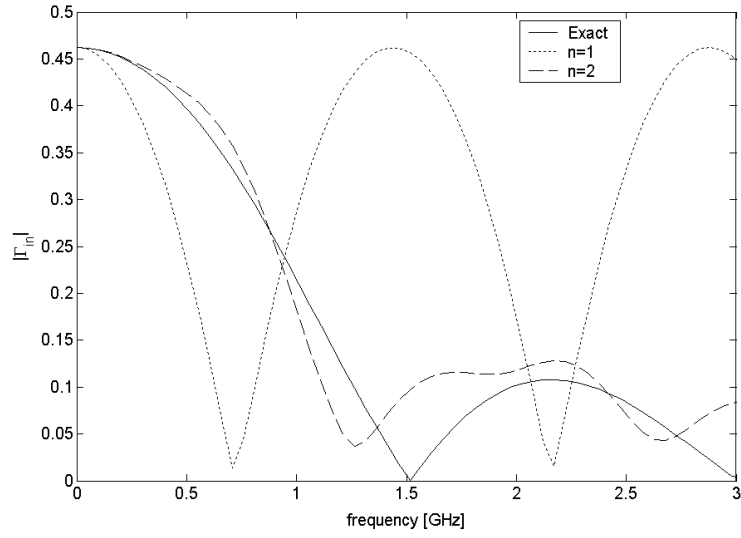


Figure 4. The magnitude of the input reflection coefficient for $d = 10$ cm.

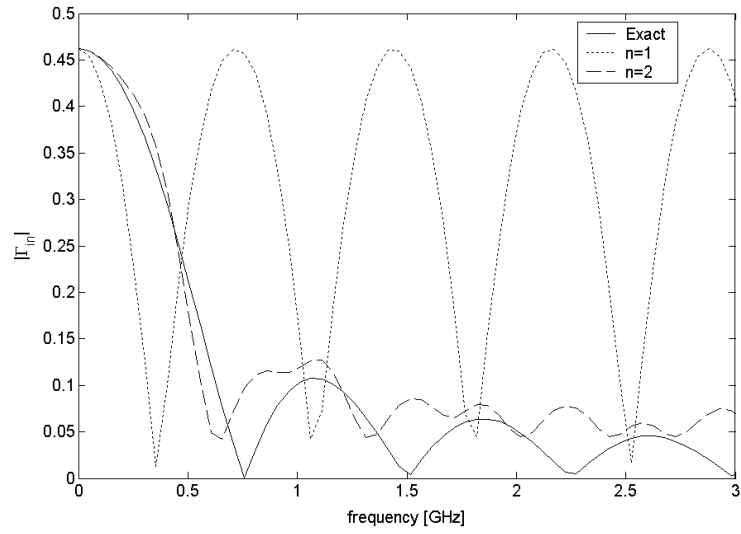


Figure 5. The magnitude of the input reflection coefficient for $d = 20$ cm.

6. CONCLUSIONS

A new method was introduced to analyze arbitrary Nonuniform Transmission Lines (NTLs). In this method, the equations of nonuniform transmission lines are converted to the equations of uniform transmission lines, which have been excited by distributed equivalent sources. Then, the voltage and current distributions are obtained using an iterative method. The validity of the method was verified using a comprehensive example. It was seen that this method is applicable to all arbitrary NTLs, whose primary parameters are known at all or even at some points along their length. Also, as the source frequency or the length of the lines increases, the accuracy of the method is decreased. Hence, as the length of the lines or the source frequency increases, the necessary iterations have to be increased.

APPENDIX A.

To solve (15), consider the following differential equation

$$\frac{d^2V(z)}{dz^2} - \bar{\gamma}^2V(z) = F(z) \quad (\text{A1})$$

The solution of (A1) can be written as follows

$$V(z) = V^+ \exp(-\bar{\gamma}z) + V^- \exp(\bar{\gamma}z) + \int_{-\infty}^{\infty} h(z-z')F(z')dz' \quad (\text{A2})$$

where $h(z)$ is the solution of (A1), when $F(z)$ is an impulse function, i.e.,

$$\frac{d^2h(z)}{dz^2} - \bar{\gamma}^2h(z) = \delta(z) \quad (\text{A3})$$

One can see that

$$h(z) = \frac{1}{2\bar{\gamma}}(\exp(\bar{\gamma}z) - \exp(-\bar{\gamma}z))u(z) \quad (\text{A4})$$

Substituting (A4) to (A2), gives us

$$\begin{aligned} V(z) = & V^+ \exp(-\bar{\gamma}z) + V^- \exp(\bar{\gamma}z) + \frac{1}{2\bar{\gamma}} \int_0^z \exp(\bar{\gamma}(z-z'))F(z')dz' \\ & - \frac{1}{2\bar{\gamma}} \int_0^z \exp(-\bar{\gamma}(z-z'))F(z')dz' \end{aligned} \quad (\text{A5})$$

Therefore, (18) is obtained as the solution of (15).

REFERENCES

1. Ghose, R. N., "Exponential transmission lines as resonators and transformers," *IRE Trans. Micro. Theory and Tech.*, Vol. 5, No. 3, 213–217, Jul. 1957.
2. Collin, R. E., *Foundations for Microwave Engineering*, McGraw-Hill, New York, 1996.
3. Tang, C. C. H., "Delay equalization by tapered cutoff waveguides," *IEEE Trans. Micro. Theory and Tech.*, Vol. 12, No. 6, 608–615, Nov. 1964.
4. Roberts, P. P. and G. E. Town, "Design of microwave filters by inverse scattering," *IEEE Trans. Micro. Theory and Tech.*, Vol. 43, No. 4, 739–743, Apr. 1995.
5. Burkhart, S. C. and R. B. Wilcox, "Arbitrary pulse shape synthesis via nonuniform transmission lines," *IEEE Trans. Micro. Theory and Tech.*, Vol. 38, No. 10, 1514–1518, Oct. 1990.
6. Hayden, L. A. and V. K. Tripathi, "Nonuniform coupled microstrip transversal filters for analog signal processing," *IEEE Trans. Micro. Theory and Tech.*, Vol. 39, No. 1, 47–53, Jan. 1991.
7. Oufi, E. A., A. S. AlFuhaid, and M. M. Saied, "Transient analysis of lossless single-phase nonuniform transmission lines," *IEEE Trans. Power Delivery*, Vol. 9, No. 7, 1694–1700, Jul. 1994.
8. Lu, K., "An efficient method for analysis of arbitrary nonuniform transmission lines," *IEEE Trans. Micro. Theory and Tech.*, Vol. 45, No. 1, 9–14, Jan. 1997.
9. Cheldavi, A., M. Kamarei, and S. Safavi-Naeini, "First order approximation of the exact solution of arbitrary nonuniform transmission lines: application in high speed integrated circuits," *IEICE Trans. Electron.*, Vol. E82-C, No. 12, 2248–2254, Dec. 1999.
10. Khalaj-Amirhosseini, M., "Analysis of coupled or single nonuniform transmission lines using step-by-step numerical integration," *Progress In Electromagnetics Research*, PIER 58, 187–198, 2006.
11. Khalaj-Amirhosseini, M., "Analysis of nonuniform transmission lines using Taylor's series expansion," *International Journal of RF and Microwave Computer-Aided Engineering*, Vol. 16, No. 5, 536–544, Sep. 2006.
12. Khalaj-Amirhosseini, M., "Analysis of coupled or single nonuniform transmission lines using Taylor's series expansion," *Progress In Electromagnetics Research*, PIER 60, 107–117, 2006.
13. Khalaj-Amirhosseini, M., "Analysis of nonuniform transmission lines using Fourier series expansion," *International Journal of RF*

and Microwave Computer-Aided Engineering, Vol. 17, No. 4, Jul. 2007.

14. Khalaj-Amirhosseini, M., "Analysis of periodic and aperiodic coupled nonuniform transmission lines using the Fourier series expansion," *Progress In Electromagnetics Research*, PIER 65, 15–26, 2006.