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ANALYSIS OF THE FIELDS IN THREE DIMENSIONAL CASSEGRAIN SYSTEM

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Abstract—High frequency field expressions are derived around feed point of a three dimensional Cassegrain system using the Maslov's method. Maslov's method is a systematic procedure for predicting the field in the caustic region. It combines the simplicity of ray theory and generality of the transform method. Numerical computations are made for the analysis of field pattern around the caustic of a Cassegrain system.

1. INTRODUCTION

Asymptotic ray theory (ART) or the geometrical optics approximation is a powerful tool for analyzing general wave motion [1–4]. However, it predicts an erroneous infinite field in the vicinity of caustic, and we must seek an alternative representation for the field in such region. Maslov proposed a method to predict the field in the caustic region [5]. Maslov method combines the simplicity of asymptotic ray theory and the generality of the Fourier transform method. This is achieved by representing the geometrical optics fields in terms of mixed coordinates consisting of space coordinates and wave vector coordinates. That is by representing the field in terms of six coordinates. It may be noted that information of ray trajectories is included in both space coordinates $R = (x, y, z)$ and wave vector coordinates $P = (p_x, p_y, p_z)$. In this way, conventional ray expression may be considered as projection into space coordinates. Similarly one can replace a part of the components of (x, y, z) by corresponding components of (p_x, p_y, p_z), e.g., (x, p_y, p_z),

(x, y, p_z), (x, p_y, z) to describe a ray. The reason for considering the solution in mixed or hybrid domain is that, in general, the singularities in different domains do not coincide. This means that there exists always a domain which can give bounded solution.

The applications to Maslov's method to the problems in radio engineering are relatively few. Applications of Maslov's method in an inhomogeneous medium and continuation problems have been discussed by Kravtsov [6] and Gorman [7, 8]. The physical interpretation of the mathematics of Maslov's method and its relation to other ART methods have been discussed by Ziolkowski and Dechamps [9]. Hongo and co-workers applied Maslov's method to derive the high frequency solutions for field generated by a phase transformer and a cylindrical reflector [10, 11]. Aziz et al. recently utilized the Maslov's method to study the two dimensional Cassegrain system [12]. In present work, our interest is to apply the Maslov's method to three dimensional Cassegrain system.

Three dimensional Cassegrain system consists of two reflectors, that is, parabolic and hyperbolic reflectors. The aim of this paper is to derive the field distribution around the feed point using the Maslov's method when it is used as the receiving antenna. Before we discuss he analysis of the field in the caustic of a Cassegrain system we consider the field reflected by a single arbitrary reflector.

2. DERIVATION OF THE FIELD IN A SINGLE REFLECTOR

Consider the field reflected by an arbitrarily shaped single reflector, whose contour is described by

$$\zeta = f(\xi, \eta) \tag{1}$$

where (ξ, η, ζ) are the Cartesian coordinates on the surface of reflector. An incident plane wave is given by

$$E^i = \mathbf{i}_x \exp(-jk_0 z) \tag{2}$$

The wave vector of the reflected wave may be obtained using the relation $\mathbf{p}^r = \mathbf{p}^i - 2(\mathbf{p}^i \cdot \mathbf{n})\mathbf{n}$, which is derived from Snell's law, where $\mathbf{n}$ is the unit normal to the surface and is given by

$$\mathbf{n} = \sin\alpha\cos\beta\mathbf{i}_x + \sin\alpha\sin\beta\mathbf{i}_y - \cos\alpha\mathbf{i}_z \tag{3}$$

$$\begin{aligned}\sin\alpha &= \frac{\rho}{\sqrt{\rho^2+4f^2}}\\ \cos\alpha &= \frac{2f}{\sqrt{\rho^2+4f^2}}\\ \tan\beta &= \frac{\eta}{\xi}\end{aligned}$$

where $\rho = \sqrt{\xi^2+\eta^2}$. By using these relations we can derive wave vector $\mathbf{p}^r$ and is given below

$$\begin{aligned}\mathbf{p}^r &= -\sin 2\alpha\cos\beta\mathbf{i}_x - \sin 2\alpha\sin\beta\mathbf{i}_y + \cos 2\alpha\mathbf{i}_z\\ &= p_x^r\mathbf{i}_x + p_y^r\mathbf{i}_y + p_z^r\mathbf{i}_z\end{aligned} \tag{4}$$

Coordinates of point on the reflected ray are given by

$$x = \xi + p_x^r t, \qquad y = \eta + p_y^r, \qquad z = f(\xi,\eta) + p_z^r t \tag{5}$$

Initial value of the reflected wave may be obtained by Snell's law of refraction, that is

$$\mathbf{E}^r = -\mathbf{E}^i + 2(\mathbf{E}^i\cdot\mathbf{n})\mathbf{n} \tag{6}$$

and its rectangular components are given by

$$\begin{aligned}E_{x0} &= -\cos^2\alpha + \sin^2\alpha\cos 2\beta, \quad E_{y0} = \sin^2\alpha\sin 2\beta\\ E_{z0} &= -\sin 2\alpha\cos\beta\end{aligned} \tag{7}$$

The geometrical optics expression of the reflected ray is given by [13]

$$\underline{\mathrm{U}}(\mathbf{r}) = \underline{\mathrm{E}}_{r0}\left[\frac{D(t)}{D(0)}\right]^{-\frac{1}{2}}\exp\left\{-jk\left[S_0+t\right]\right\} \tag{8}$$

where $\underline{\mathrm{E}}_{r0}(\xi,\eta,\zeta)$ is the incident wave at the surface of the reflector. Parameter t represents the distance along the ray from reference point. S_0 represents the initial value of the phase function. $D(t)$ is the Jacobian of the transformation from the Cartesian to the ray coordinates, and it is given by [see Appendix A]

$$\begin{aligned}D(t) &= \frac{\partial(x,y,z)}{\partial(\xi,\eta,t)} = \begin{vmatrix} 1+\frac{\partial p_x}{\partial\xi}t & \frac{\partial p_y}{\partial\xi}t & \frac{\partial\zeta}{\partial\xi}+\frac{\partial p_z}{\partial\xi}t\\ \frac{\partial p_x}{\partial\eta}t & 1+\frac{\partial p_y}{\partial\eta}t & \frac{\partial\zeta}{\partial\eta}+\frac{\partial p_z}{\partial\eta}t\\ p_x & p_y & p_z\end{vmatrix}\\ &= Ut^2+Vt+W\end{aligned} \tag{9}$$

where

$$
\begin{aligned}
U &= \frac{\cos^4\alpha}{f^2} \\
V &= -2\frac{\cos^2\alpha}{f} \\
W &= 1
\end{aligned}
\tag{10}
$$

The expression for Jacobian is given by

$$D(t) = \frac{\partial(x, y, z)}{\partial(\xi, \eta, t)} = \frac{\cos^4\alpha}{f^2}t^2 - 2\frac{\cos^2\alpha}{f}t + 1 \tag{11}$$

$$J(t) = \frac{D(t)}{D(0)} = \frac{\cos^4\alpha}{f^2}t^2 - 2\frac{\cos^2\alpha}{f}t + 1 \tag{12}$$

The caustic along the ray may be obtained by satisfying the relation $D(t) = 0$. The relation yields following algebraic equation for t

$$
\begin{aligned}
Ut^2 + Vt + W &= 0 \\
\frac{\cos^4\alpha}{f^2}t^2 - 2\frac{\cos^2\alpha}{f}t + 1 &= 0 \\
\left(\frac{\cos^2\alpha}{f}t - 1\right)^2 &= 0
\end{aligned}
$$

The solution of algebraic equation is

$$t = \frac{f}{\cos^2\alpha}$$

More explicitly

$$
\begin{aligned}
x_c &= \xi - 2f\tan\alpha\cos\beta = 0 \\
y_c &= \eta - 2f\tan\alpha\sin\beta = 0 \\
z_c &= \zeta + f\frac{\cos 2\alpha}{\cos^2\alpha} = 0
\end{aligned}
\tag{13}
$$

At the point satisfying (13), the field becomes infinite.

According to the Maslov's method, the ray expression covering the caustics can be derived from the formula

$$
\begin{aligned}
\underline{\mathrm{U}}(\mathbf{r}) = \frac{k}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underline{\mathrm{E}}_{r0}\left[\frac{D(t)}{D(0)}\frac{\partial(p_x, p_y)}{\partial(x, y)}\right]^{-\frac{1}{2}} \\
\times \exp\Big\{-jk\Big[S_0 + t - x(p_x, p_y, z)p_x \\
- y(p_x, p_y, z)p_y + p_x x + p_y y\Big]\Big\}\, dp_x dp_x
\end{aligned}
\tag{14}
$$

where $S_0 = -\zeta$ is the initial phase. In (14), $x(p_x, p_y, z)p_x$, $y(p_x, p_y, z)p_y$ means that the coordinate x, y has been expressed in terms of mixed coordinates (p_x, p_y, z). The same is true for t and it is given by $t = \dfrac{z-\zeta}{p_z}$. The phase function $S(p_x, p_y)$ is given by

$$\begin{aligned} S(p_x, p_y) &= -\zeta + \frac{z-\zeta}{p_z} - \Big(\xi + p_x\Big)p_x - \Big(\eta + p_y\Big)p_y + p_x x + p_y y \\ &= -\zeta - \xi p_x - \eta p_y - \zeta p_z + p_x x + p_y y + p_z z \end{aligned} \tag{15}$$

We have

$$\xi = 2f\tan\alpha\cos\beta, \qquad \eta = 2f\tan\alpha\sin\beta, \qquad \zeta = c - f\frac{\cos 2\alpha}{\cos^2\alpha}$$

Introducing the polar coordinates

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$

yields the phase function

$$\begin{aligned} S(p_x, p_y) &= -\zeta + \frac{z-\zeta}{p_z} - \Big(\xi + p_x t\Big)p_x - \Big(\eta + p_y t\Big)p_y + p_x x + p_y y \\ &= 2f - r\sin\theta\sin 2\alpha\cos(\phi-\beta) + r\cos\theta\cos 2\alpha \end{aligned} \tag{16}$$

Amplitude of the integrand in (14) may be evaluated as

$$J(t)\frac{\partial(p_x, p_y)}{\partial(x, y)} = \frac{1}{D(0)}\frac{\partial(p_x, p_y, z)}{\partial(\xi, \eta, t)} \tag{17}$$

and are derived as follows

$$J(t)\frac{\partial(p_x, p_y)}{\partial(x, y)} = \frac{1}{D(0)}\begin{vmatrix} \dfrac{\partial p_x}{\partial\xi} & \dfrac{\partial p_y}{\partial\xi} & 0 \\ \dfrac{\partial p_x}{\partial\eta} & \dfrac{\partial p_y}{\partial\eta} & 0 \\ 0 & 0 & \dfrac{\partial z}{\partial t} \end{vmatrix} = \frac{\cos^4\alpha\cos^2 2\alpha}{f^2} \tag{18}$$

Substituting these results in (14), we have the field expression which is valid in the caustic region

$$\begin{aligned} \underline{\mathrm{U}}(\mathbf{r}) = \frac{k}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\underline{\mathrm{E}}_{r0}\left[\frac{\cos^4\alpha\cos^2 2\alpha}{f^2}\right]^{-\frac{1}{2}} \\ \times\exp\Big\{-jk\Big[2f - r\sin\theta\sin 2\alpha\cos(\phi-\beta) + r\cos\theta\cos 2\alpha\Big]\Big\}dp_x dp_y \end{aligned} \tag{19}$$

Conversion from cartesian coordinates (p_x, p_y) to ray coordinates (ξ, η), that is by the relation

$$\frac{\partial(p_x, p_y)}{\partial(\xi, \eta)} = \frac{\cos^4\alpha\cos 2\alpha}{f^2}$$

reduces above equation into the following form

$$\underset{\sim}{\mathrm{U}}(\mathbf{r}) = \frac{k}{2\pi}\int_S \underset{\sim}{\mathrm{E}}_{r0}\left[\frac{\cos^4\alpha\cos^2 2\alpha}{f^2}\right]^{-\frac{1}{2}}\left[\frac{\cos^4\alpha\cos 2\alpha}{f^2}\right] \times\exp\left\{-jk\Big[2f - r\sin\theta\sin 2\alpha\cos(\phi-\beta) + r\cos\theta\cos 2\alpha\Big]\right\} d\xi d\eta \tag{20}$$

Changing (ξ, η) to angular coordinates (α, β) by

$$\frac{\partial(\xi, \eta)}{\partial(\alpha, \beta)} = \frac{4f^2\sin\alpha}{\cos^3\alpha}$$

and using the polar coordinates (r, θ, ϕ) instead of the Cartesian coordinates (x, y, z) gives

$$\underset{\sim}{\mathrm{U}}(\mathbf{r}) = \frac{j2kf}{\pi}\int_0^H\int_0^{2\pi}\underset{\sim}{\mathrm{E}}_{r0}\tan\alpha \times\exp\left\{-jk\Big[2f - r\sin\theta\sin 2\alpha\cos(\phi-\beta) + r\cos\theta\cos 2\alpha\Big]\right\} d\alpha d\beta \tag{21}$$

where $H = \arctan(D/2f)$. It may be noted that D is height of the edge of the parabolic reflector from horizontal axis. The integration with respect to β can be performed by using the integral representation of Bessel function. The results are expressed as

$$\begin{aligned} U_x &= jkf\exp\{-j2kf\}\Big[P(r,\theta) + Q(r,\theta)\cos 2\phi\Big] \\ U_y &= jkf\exp\{-j2kf\}\,Q(r,\theta)\sin 2\phi \\ U_z &= jkf\exp\{-j2kf\}\,R(r,\theta)\cos\phi \end{aligned}$$

where

$$\begin{aligned} P(r,\theta) &= \int_0^{2H} J_0(kr\sin\theta\sin\tau)\exp\{-j(kr\cos\theta\cos\tau)\}\sin\tau d\tau \\ Q(r,\theta) &= \int_0^{2H}\frac{1-\cos\tau}{1+\cos\tau}J_2(kr\sin\theta\sin\tau)\exp\{-j(kr\cos\theta\cos\tau)\}\sin\tau d\tau \\ R(r,\theta) &= \int_0^{2H}\frac{\sin\tau}{1+\cos\tau}J_2(kr\sin\theta\sin\tau)\exp\{-j(kr\cos\theta\cos\tau)\}\sin\tau d\tau \end{aligned}$$

In a region far from the caustics, (21) can be evaluated approximately by applying the stationary phase method of integration and the result should agree with the GO expression derived. This serves as an important check of the validity of the expression (21). The stationary point is determined from

$$S(p_x, p_y) = 2f - r\sin\theta\sin 2\alpha\cos(\phi-\beta) + r\cos\theta\cos 2\alpha$$

The first derivative of the phase function is

$$\frac{\partial S}{\partial \beta} = \phi - \beta = 0 \tag{22}$$

$$\phi = \beta$$

$$\frac{\partial S}{\partial \alpha} = -2r(\cos\theta\sin 2\alpha + \cos\theta\sin 2\alpha) = 0 \tag{23}$$

$$\sin(2\alpha+\theta) = 0$$

$$\alpha = -\frac{\theta}{2} \tag{24}$$

From geometry of paraboloid we have value of r as

$$r = \frac{f}{\cos^2\alpha} - t \tag{25}$$

The second derivative of the phase function is

$$\frac{\partial^2 S}{\partial \alpha^2} = -4r\left\{\cos(2\alpha+\theta)\right\} = -4r$$

$$\frac{\partial^2 S}{\partial \alpha^2} = 4\left(t - \frac{f}{\cos^2\alpha}\right) \tag{26}$$

$$\frac{\partial^2 S}{\partial \beta^2} = r\sin\theta\sin 2\alpha = -r\sin^2\theta$$

$$= \left(t - \frac{f}{\cos^2\alpha}\right)\sin^2\theta \tag{27}$$

The result from (26) and (27) are found to agree with the GO solution.

3. RECEIVING CHARACTERISTIC OF THREE DIMENSIONAL CASSEGRAIN REFLECTOR

Cassegrain reflector consists of two reflectors, one is parabolic main reflector and another is hyperbolic sub reflector. This system has many advantages over a single parabolic reflector. We consider here

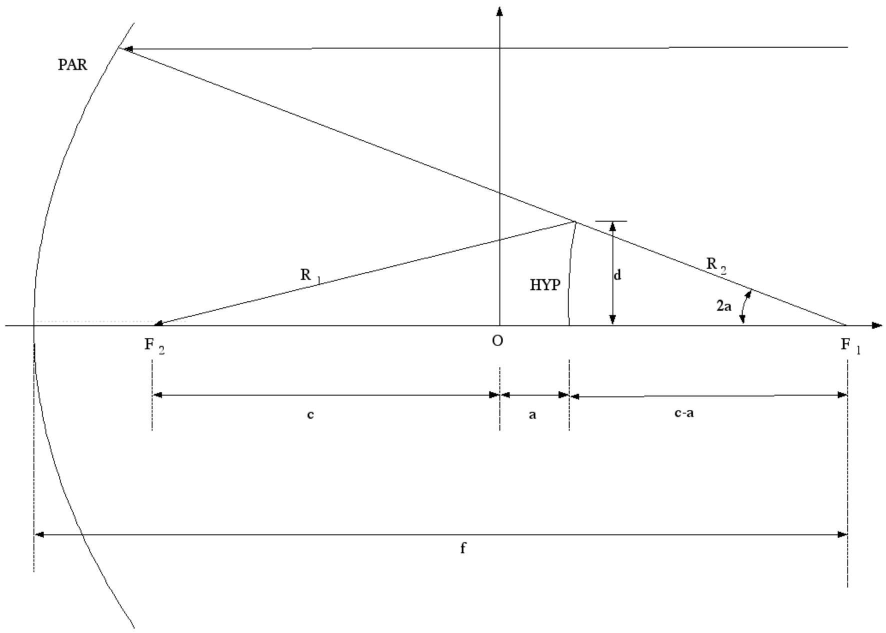


Figure 1. Cassegrain system.

a receiving characteristic of this system by applying Maslov's method. The equation of each surface is given by [see Fig. 1]

$$\zeta_1 = \frac{\xi_1^2 + \eta_1^2}{4f} - f + c = \frac{\rho_1^2}{4f} - f + c, \qquad \rho_1 = \xi_1^2 + \eta_1^2 \tag{28}$$

and that of the sub-reflector (hyperbola) is

$$\begin{aligned} \zeta_2 &= a\left[\frac{\xi_2^2 + \eta_2^2}{b^2} + 1\right]^{\frac{1}{2}} = \frac{a}{b}\left[\rho_2^2 + b^2\right]^{\frac{1}{2}} \\ \rho_2^2 &= \xi_2^2 + \eta_2^2, \quad c^2 = a^2 + b^2; \quad aR_2 = c\zeta_2 - a^2 \end{aligned} \tag{29}$$

where (ξ_1, η_1, ζ_1) and (ξ_2, η_2, ζ_2) are the Cartesian coordinates of the point on the parabolic and hyperbolic reflectors, respectively.

The unit normals of these surfaces are given by

$$\begin{aligned} \mathbf{n}_1 &= \sin\alpha\cos\beta\mathbf{i}_x + \sin\alpha\sin\beta\mathbf{i}_y - \cos\alpha\mathbf{i}_z \\ \sin\alpha &= \frac{\rho_1}{\sqrt{\rho_1^2 + 4f^2}} \end{aligned} \tag{30}$$

$$\begin{aligned}\cos\alpha &= \frac{2f}{\sqrt{\rho_1^2+4f^2}}\\ \tan\beta &= \frac{\eta_1}{\xi_1}\end{aligned}$$

and

$$\begin{aligned}\mathbf{n}_2 &= -\sin\gamma\cos\beta\mathbf{i}_x - \sin\gamma\sin\beta\mathbf{i}_y + \cos\gamma\mathbf{i}_z \qquad (31)\\ \sin\gamma &= \frac{a\rho_2}{b\sqrt{R_1R_2}}\\ \cos\gamma &= \frac{b\zeta_2}{a\sqrt{R_1R_2}}\\ \tan\beta &= \frac{\eta_2}{\xi_2}\end{aligned}$$

The wave reflected by the parabolic reflector will act as incident wave to the hyperbolic reflector and is given by

$$\begin{aligned}E_2^i &= \{\mathbf{i}_x(-\cos^2\alpha+\sin^2\alpha\cos 2\beta)+\mathbf{i}_y\sin^2\alpha\sin 2\beta-\mathbf{i}_z\sin^2\alpha\cos\beta\}\\ &\quad\times\exp\{jk\{x\sin 2\alpha\cos\beta+y\sin 2\alpha\sin\beta-z\cos 2\alpha\}\} \qquad (32)\end{aligned}$$

The wave vector of the wave reflected by the parabola is given by

$$\mathbf{p}_1^r = -\mathbf{i}_z+2(\mathbf{i}_z\cdot\mathbf{n}_1)\mathbf{n}_1 = -\sin 2\alpha\cos\beta\mathbf{i}_x-\sin 2\alpha\sin\beta\mathbf{i}_y+\cos 2\alpha\mathbf{i}_z \quad (33)$$

and the wave vector of the wave reflected by the hyperbola is

$$\mathbf{p}_2^r = -\sin(2\alpha-2\gamma)\cos\beta\mathbf{i}_x-\sin(2\alpha-2\gamma)\sin\beta\mathbf{i}_y-\cos(2\alpha-2\gamma)\mathbf{i}_z \quad (34)$$

Then the solution of the ray equation is given by

$$x=\xi_2+p_{2x}t, \qquad y=\eta_2+p_{2y}t, \qquad z=\zeta_2+p_{2z}t \qquad (35)$$

where (p_{2x}, p_{2y}, p_{2z}) are the Cartesian components of $\mathbf{p}_2^r$.

In the above equation (31) R_1 and R_2 are the distances from the point (ξ_2, η_2, ζ_2) to the focal points $z=-c$ and $z=c$, respectively with $c^2=a^2+b^2$. The Cartesian coordinates of the ray reflected by the hyperbolic reflector is given by

$$\begin{aligned}x &= \xi_2+p_{x2}t = \xi_1+p_{x1}t_1+p_{x2}t\\ y &= \eta_2+p_{2y}t = \eta_1+p_{y1}t_1+p_{y2}t\\ z &= \zeta_2+p_{z2}t = \zeta_1+p_{z1}t_1+p_{z2}t \qquad (36)\end{aligned}$$

where $t_1 = \sqrt{(\xi_1-\xi_2)^2+(\eta_1-\eta_2)^2+(\zeta_1-\zeta_2)^2}$. (p_{x1}, p_{y1}, p_{z1}) and (p_{x2}, p_{y2}, p_{z2}) are the rectangular components of $\mathbf{p}_1^r$ and $\mathbf{p}_2^r$, respectively. The initial value of the reflected wave is obtained by snell's law of refraction

$$E_2^r = -E_2^i + 2(E_2^i \cdot \mathbf{n}_2)\mathbf{n}_2$$

and its rectangular components are given by

$$\begin{aligned} U_{rx2} &= -\cos^2\alpha + \sin^2\alpha\cos 2\beta + 2Z\sin\gamma\cos\beta, \\ U_{ry2} &= \sin^2\alpha\sin 2\beta + 2Z\sin\gamma\sin\beta, \\ U_{rz2} &= -(\sin 2\alpha\cos\beta + 2Z\cos\gamma) \end{aligned}$$

where

$$\begin{aligned} Z &= -(\cos^2\alpha + \sin^2\alpha\cos 2\beta)\sin\gamma\cos\beta \\ &\quad - \sin^2\alpha\sin\gamma\sin 2\beta\sin\beta - \cos\beta\cos\gamma\sin 2\alpha \end{aligned}$$

We have also

$$\begin{aligned} \xi_2 &= \frac{b^2\sin\gamma\cos\beta}{\sqrt{a^2\cos^2\gamma - b^2\sin^2\gamma}}, \quad \eta_2 = \frac{b^2\sin\gamma\sin\beta}{\sqrt{a^2\cos^2\gamma - b^2\sin^2\gamma}} \\ \zeta_2 &= \frac{a^2\cos\gamma}{\sqrt{a^2\cos^2\gamma - b^2\sin^2\gamma}} \end{aligned}$$

Now consider the field after the reflection from the hyperbolic reflector. The transformation from the Cartesian coordinates (x, y, z) to the ray fixed coordinates (ξ_1, ζ_1, t) is given by [see Appendix B]

$$D(t) = Ut^2 + Vt + W \tag{37}$$

Thus the geometrical ray expression of the reflected wave is

$$E^r = E_0^r\left[\frac{U}{W}t^2 + \frac{V}{W}t + 1\right]^{-\frac{1}{2}}\exp\Big[-jk\Big(S_0 + t_1 + t\Big)\Big] \tag{38}$$

where E_0^r is the amplitude of the incident wave at the reflection point on the parabolic reflector and

$$\begin{aligned} S_0 &= -\zeta_1 = 2f\frac{\cos 2\alpha}{1+\cos 2\alpha} - c \\ t_1 &= \sqrt{(\xi_2-\xi_1)^2 + (\eta_2-\eta_1)^2 + (\zeta_2-\zeta_1)^2} \\ t &= \sqrt{(x-\xi_2)^2 + (y-\eta_2)^2 + (z-\zeta_2)^2} \end{aligned} \tag{39}$$

We introduce the polar coordinates

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta \tag{40}$$

It is readily seen that the GO expression of the reflected wave becomes infinity at the point F_2 as is expected. We can derive the refined expression which is valid at the focal point according to (13a). The value of $\left[J(t)\dfrac{\partial(p_{2x},p_{2y})}{\partial(x,y)}\right]^{-\frac{1}{2}}$ is given by [see Appendix B2]

$$\begin{aligned}\left[J(t)\frac{\partial(p_{2x},p_{2y})}{\partial(x,y)}\right]^{-\frac{1}{2}} &= \left[-\frac{1}{D(0)}\frac{SC^2}{f}\frac{\cos^2\alpha}{2f}\left\{1-\frac{2ac}{R_1(R_2+a)}\right.\right.\\ &\qquad \left.\left.\times\left[\Gamma_\xi\cos^2\beta+\Gamma_\eta\sin^2\beta\right]\right\}\right]^{-\frac{1}{2}}\end{aligned} \tag{41}$$

The phase function is given by

$$\begin{aligned}S &= S_0+t_1+\frac{z-\zeta_2}{p_{2z}}-\Big(\xi_2+p_{2x}t\Big)p_{2x}-\Big(\eta_2+p_{2y}t\Big)p_{2y}+p_{2x}x+p_{2y}y\\ &= S_0+t_1+\frac{z-\zeta_2}{p_{2z}}(1-p_{2x}^2-p_{2y}^2)-\xi_2p_{2x}-\eta_2p_{2y}+p_{2x}x+p_{2y}y\\ &= S_0+t_1-\xi_2p_{2x}-\eta_2p_{2y}-\zeta_2p_{2z}+p_{2x}x+p_{2y}y+p_{2z}z\end{aligned}$$

where S_0+t_1 is given by (39). The extra term is given by

$$\begin{aligned}S_{ex} &= -\xi_2p_{2x}-\eta_2p_{2y}-\zeta_2p_{2z}+p_{2x}x+p_{2y}y+p_{2z}z\\ &= S_{ex1}-r\sin\theta\sin(2\alpha-2\gamma)\cos(\beta-\phi)\end{aligned} \tag{42}$$

where

$$\begin{aligned}S_{ex1} &= \frac{a^2\cos\gamma\cos(2\alpha-2\gamma)}{\sqrt{a^2\cos^2\gamma-b^2\sin^2\gamma}}+\frac{b^2\sin\gamma\sin(2\alpha-2\gamma)}{\sqrt{a^2\cos^2\gamma-b^2\sin^2\gamma}}\\ &\quad -r\cos(2\alpha-2\gamma)\cos\theta\end{aligned}$$

Substituting (39) to (42) into (14), yields the following result

$$\begin{aligned}\underline{\mathrm{U}}(\mathbf{r}) &= \frac{k}{2\pi}\left[\int_{H1}^{H2}+\int_{-H2}^{-H1}\right]\int_0^{2\pi}\underline{\mathrm{E}}_{r0}\left[-\frac{1}{D(0)}\frac{SC^2}{f}\frac{\cos^2\alpha}{2f}\right.\\ &\quad\left.\times\left\{1-\frac{2ac}{R_1(R_2+a)}\Big[\Gamma_\xi\cos^2\beta+\Gamma_\eta\sin^2\beta\Big]\right\}\right]^{-\frac{1}{2}}\\ &\quad\times\exp\left\{-jk\Big[S_0+t_1+S_{ex}\Big]\right\}dP_{2x}dP_{2y}\end{aligned}$$

Changing (p_{2x}, p_{2y}) to (ξ_1, η_1) coordinates by

$$\frac{\partial(P_{2x}, p_{2y})}{\partial(\xi_1, \eta_1)} = \frac{SC}{f}\frac{\cos^2\alpha}{2f}\left\{1 - \frac{2ac}{R_1(R_2+a)}\left[\Gamma_\xi \cos^2\beta + \Gamma_\eta \sin^2\beta\right]\right\}$$

changes above equation into the following

$$\begin{aligned}\underline{\mathrm{U}}(\mathbf{r}) \; = \; & \frac{k}{2\pi}\left[\int_{H1}^{H2} + \int_{-H2}^{-H1}\right]\int_0^{2\pi} \underline{\mathrm{E}}_{r0}\left[-\frac{1}{D(0)}\frac{SC^2}{f}\frac{\cos^2\alpha}{2f}\right. \\ & \left.\times\left\{1 - \frac{2ac}{R_1(R_2+a)}\left[\Gamma_\xi \cos^2\beta + \Gamma_\eta \sin^2\beta\right]\right\}\right]^{-\frac{1}{2}} \\ & \times\frac{SC}{f}\frac{\cos^2\alpha}{2f}\left\{1 - \frac{2ac}{R_1(R_2+a)}\left[\Gamma_\xi \cos^2\beta + \Gamma_\eta \sin^2\beta\right]\right\} \\ & \times\exp\left\{-jk\left[S_0 + t_1 + S_{ex}\right]\right\} d\xi_1 d\eta_1\end{aligned}$$

where

$$D(0) = \frac{-R_2^2 \cos^2\alpha \cos(2\alpha - \gamma)\Gamma_\xi \Gamma_\eta}{f^2 \cos\gamma}$$

Changing (ξ_1, η_1) to angular coordinates (α, β) by $\frac{\partial(\xi_1,\eta_1)}{\partial(\alpha,\beta)} = R_2^2 \sin 4\alpha$ and using the polar coordinates (r, θ, ϕ) instead of the Cartesian coordinates (x, y, z), we have

$$\begin{aligned}\underline{\mathrm{U}}(\mathbf{r}) \; = \; & \frac{jk}{2\pi}\left[\int_{H1}^{H2} + \int_{-H2}^{-H1}\right]\int_0^{2\pi} \underline{\mathrm{E}}_{r0}\frac{R_2^3 \cos^3\alpha \sin 2\alpha}{f^2} \\ & \times\left[S\frac{\cos(2\alpha-\gamma)}{2\cos\gamma}\Gamma_\xi\Gamma_\eta\Gamma_\zeta\right]^{\frac{1}{2}} \\ & \times\exp\left\{-jk\left[S_0 + t_1 + S_{ex1} - r\sin\theta\sin(2\alpha - 2\gamma)\cos(\beta-\phi)\right]\right\}d\alpha d\beta\end{aligned} \tag{43}$$

where

$$\begin{aligned}\Gamma_\xi &= \frac{1 + \tan 2\alpha \tan\alpha \cos^2\beta}{1 + \tan 2\alpha \tan\gamma \cos^2\beta} \\ \Gamma_\eta &= \frac{1 + \tan 2\alpha \tan\alpha \sin^2\beta}{1 + \tan 2\alpha \tan\gamma \sin^2\beta} \\ \Gamma_\zeta &= 1 - \frac{2ac}{R_1(R_2+a)}\left[\Gamma_\xi \cos^2\beta + \Gamma_\eta \sin^2\beta\right] \\ S &= \sin(2\alpha - 2\gamma)\end{aligned}$$

The rectangular components of field are given by

$$U_{rx2} = \frac{jk}{2\pi}\left[\int_{H1}^{H2} + \int_{-H2}^{-H1}\right]\int_0^{2\pi}(-\cos^2\alpha+\sin^2\alpha\cos 2\beta+2Z\sin\gamma\cos\beta)$$
$$\times\frac{R_2^3\cos^3\alpha\sin 2\alpha}{f^2}\left[S\frac{\cos(2\alpha-\gamma)}{2\cos\gamma}\Gamma_\xi\Gamma_\eta\Gamma_\zeta\right]^{\frac{1}{2}}$$
$$\times\exp\Big\{-jk\Big[S_0+t_1+S_{ex1}-r\sin\theta\sin(2\alpha-2\gamma)\cos(\beta-\phi)\Big]\Big\}d\alpha d\beta \tag{44}$$

$$U_{ry2} = -\frac{jk}{2\pi}\left[\int_{H1}^{H2} + \int_{-H2}^{-H1}\right]\int_0^{2\pi}(\sin^2\alpha\sin 2\beta + 2Z\sin\gamma\sin\beta)$$
$$\times\frac{R_2^3\cos^3\alpha\sin 2\alpha}{f^2}\left[S\frac{\cos(2\alpha-\gamma)}{2\cos\gamma}\Gamma_\xi\Gamma_\eta\Gamma_\zeta\right]^{\frac{1}{2}}$$
$$\times\exp\Big\{-jk\Big[S_0+t_1+S_{ex1}-r\sin\theta\sin(2\alpha-2\gamma)\cos(\beta-\phi)\Big]\Big\}d\alpha d\beta \tag{45}$$

$$U_{rz2} = \frac{jk}{2\pi}\left[\int_{H1}^{H2} + \int_{-H2}^{-H1}\right]\int_0^{2\pi} -(\sin 2\alpha\cos\beta + 2Z\cos\gamma)$$
$$\times\frac{R_2^3\cos^3\alpha\sin 2\alpha}{f^2}\left[S\frac{\cos(2\alpha-\gamma)}{2\cos\gamma}\Gamma_\xi\Gamma_\eta\Gamma_\zeta\right]^{\frac{1}{2}}$$
$$\times\exp\Big\{-jk\Big[S_0+t_1+S_{ex1}-r\sin\theta\sin(2\alpha-2\gamma)\cos(\beta-\phi)\Big]\Big\}\,d\alpha d\beta \tag{46}$$

4. RESULT AND DISCUSSION

Field pattern around the caustic of a Cassegrain system are determined using equation (44) by performing the integration numerically. Figures 3 to 5 contain the plots of the field around focal region located between the reflectors, that point F_2 in Figure 1 and Figure 2. In Figure 3, it is assumed that $kf = 20$, $a = 5$, $b = 6$, $d = 4$ and $D = 16$. In Figure 4, it is assumed that $kf = 25$, $a = 8$, $b = 9$, $d = 6$ and $D = 18$ while Figure 5 deals with $kf = 50$, $a = 17$, $b = 18$, $d = 12$ and $D = 40$. The location of the caustic may be observed and verified easily. It may be noted that limits of the integrals in equation (44) are selected using the following relations [see Appendix D]

$$H_1 = \phi_\nu = 2\arctan\left(\frac{D}{2f}\right)$$

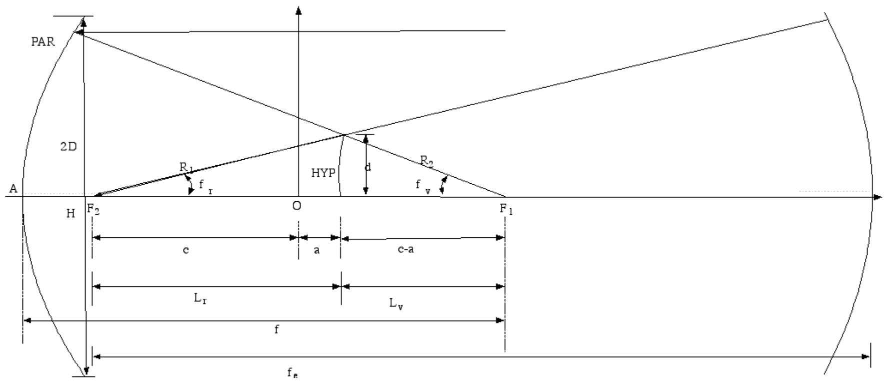


Figure 2. Parameters in Cassegrain system.

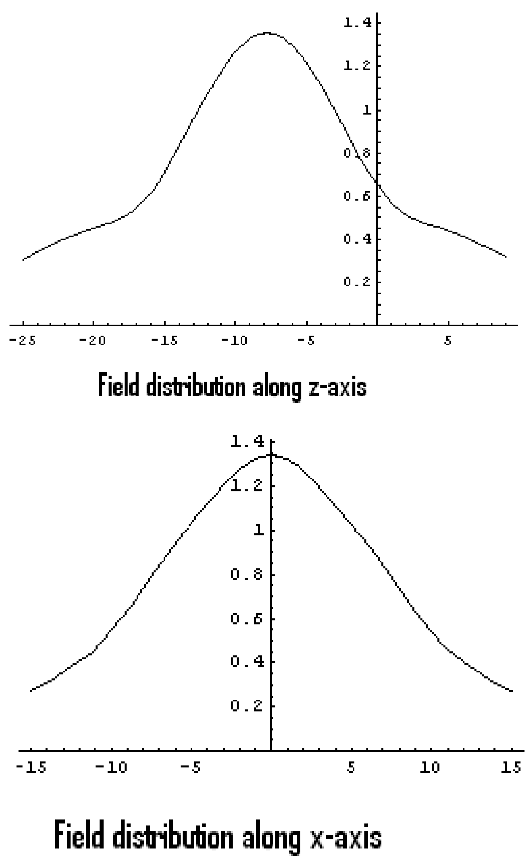
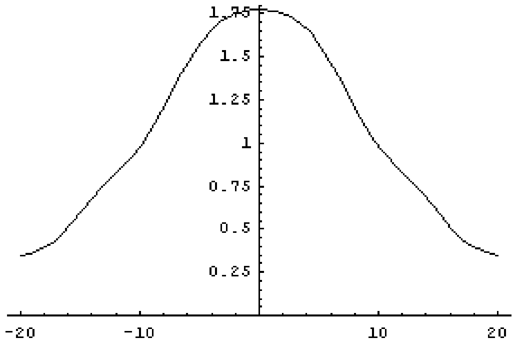
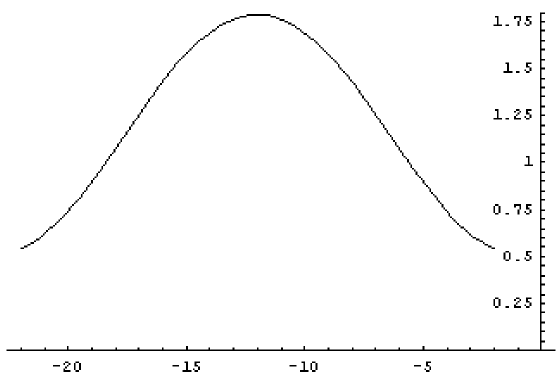


Figure 3. Field distribution around the focus.



Field distribution along x-axis at kf=25,a=8,b=9,d=6,D=18



Field distribution along z-axis at kf=25,a=8,b=9,d=6,D=18

Figure 4.

$$H_2 = \arctan\left(\frac{d}{2c}\right)$$

The results are compared with the results of an equivalent parabola, whose focal length is determined using the following relation [see Appendix D]

$$f_e = \left(\frac{c+a}{c-a}\right) f$$

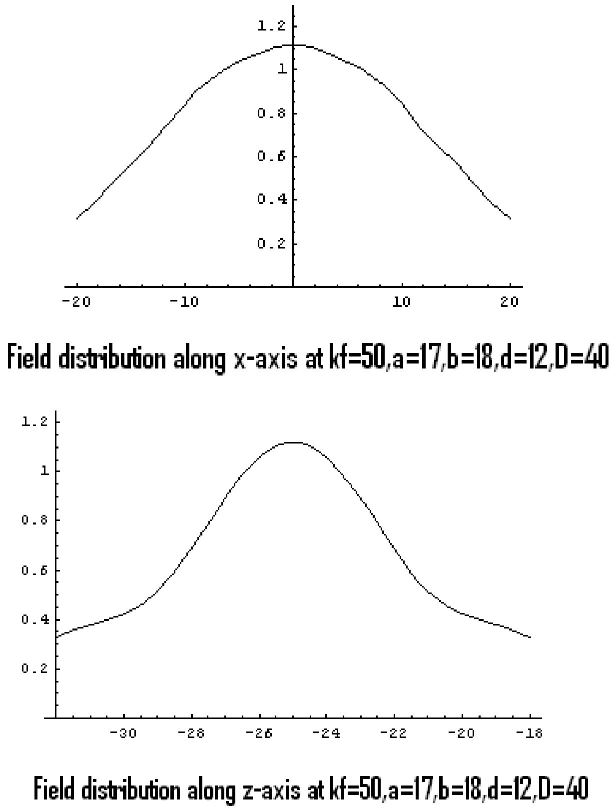


Figure 5.

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APPENDIX A. EVALUATION OF THE $D(T)$

$$D(t) = \frac{\partial(x,y,z)}{\partial(\xi,\eta,t)} = \begin{vmatrix} 1+\dfrac{\partial p_x}{\partial \xi}t & \dfrac{\partial p_y}{\partial \xi}t & \dfrac{\partial \zeta}{\partial \xi}+\dfrac{\partial p_z}{\partial \xi}t \\ \dfrac{\partial p_x}{\partial \eta}t & 1+\dfrac{\partial p_y}{\partial \eta}t & \dfrac{\partial \zeta}{\partial \eta}+\dfrac{\partial p_z}{\partial \eta}t \\ p_x & p_y & p_z \end{vmatrix}$$
$$= Ut^2 + Vt + W \tag{A1}$$

where U, V, W are

$$\begin{aligned}
U &= \left(\frac{\partial p_y}{\partial \xi}\frac{\partial p_z}{\partial \eta} - \frac{\partial p_z}{\partial \xi}\frac{\partial p_y}{\partial \eta}\right)p_x + \left(\frac{\partial p_x}{\partial \eta}\frac{\partial p_z}{\partial \xi} - \frac{\partial p_z}{\partial \eta}\frac{\partial p_x}{\partial \xi}\right)p_y \\
&\quad + \left(\frac{\partial p_x}{\partial \xi}\frac{\partial p_y}{\partial \eta} - \frac{\partial p_y}{\partial \xi}\frac{\partial p_x}{\partial \eta}\right)p_z \\
V &= \left(\frac{\partial p_y}{\partial \xi}\frac{\partial \zeta}{\partial \eta} - \frac{\partial \zeta}{\partial \xi}\frac{\partial p_y}{\partial \eta} - \frac{\partial p_z}{\partial \xi}\right)p_x + \left(\frac{\partial p_x}{\partial \eta}\frac{\partial \zeta}{\partial \xi} - \frac{\partial \zeta}{\partial \eta}\frac{\partial p_x}{\partial \xi} - \frac{\partial p_z}{\partial \eta}\right)p_y \\
&\quad + \left(\frac{\partial p_y}{\partial \eta} + \frac{\partial p_x}{\partial \xi}\right)p_z \\
W &= -\left(\frac{\partial \zeta}{\partial \xi}p_x + \frac{\partial \zeta}{\partial \eta}p_y\right) + p_z
\end{aligned} \tag{A2}$$

We may rewrite the values of U, V and W by using the following relations

$$\begin{aligned}
\frac{\partial p_x}{\partial \xi} &= -2C\Lambda_\xi \cos\beta - \frac{S}{2f}\cot\alpha\sin^2\beta \\
\frac{\partial p_y}{\partial \xi} &= -2C\Lambda_\xi \sin\beta + \frac{S}{2f}\cot\alpha\cos\beta\sin\beta \\
\frac{\partial p_x}{\partial \eta_1} &= -2C\Lambda_\eta \cos\beta + \frac{S}{2f}\cot\alpha\sin\beta\cos\beta \\
\frac{\partial p_y}{\partial \eta} &= -2C\Lambda_\eta \sin\beta - \frac{S}{2f}\cot\alpha\cos^2\beta \\
\frac{\partial p_z}{\partial \xi_1} &= -2S\Lambda_\xi \\
\frac{\partial p_x}{\partial \eta_1} &= -2S\Lambda_\eta
\end{aligned} \tag{A3}$$

where

$$C = \cos 2\alpha, \qquad S = \sin 2\alpha, \qquad \Lambda_\xi = \frac{\partial \alpha}{\partial \xi}, \qquad \Lambda_\eta = \frac{\partial \alpha}{\partial \eta} \tag{A4}$$

The new expressions for U, V and W are given by

$$\begin{aligned}
U &= -2S^2\cos\beta\left\{\left[-2C\Lambda_\xi\sin\beta + \frac{S}{2f}\cot\alpha\cos\beta\sin\beta\right]\Lambda_\eta\right. \\
&\quad \left. - \left[-2C\Lambda_\eta\sin\beta - \frac{S}{2f}\cot\alpha\cos^2\beta\right]\Lambda_\xi\right\} \\
&\quad -2S^2\sin\beta\left\{\left[-2C\Lambda_\eta\cos\beta + \frac{S}{2f}\cot\alpha\sin\beta\cos\beta\right]\Lambda_\xi\right.
\end{aligned}$$

$$
\begin{aligned}
& - \left[-2C\Lambda\cos\beta - \frac{S}{2f}\cot\alpha\sin^2\beta\right]\Lambda_\eta\Bigg\} \\
& -C\Bigg\{\left[-2C\Lambda_\xi\cos\beta - \frac{S}{2f}\cot\alpha\sin^2\beta\right] \\
& \times\left[-2C\Lambda_\eta\sin\beta - \frac{S}{2f}\cot\alpha\cos^2\beta\right] \\
& -\left[-2C\Lambda_\xi\sin\beta + \frac{S}{2f}\cot\alpha\cos\beta\sin\beta\right] \\
& \times\left[-2C\Lambda_\eta\cos\beta + \frac{S}{2f}\cot\alpha\sin\beta\cos\beta\right]\Bigg\}\Big(-C\Big) \\
= {} & -\frac{S}{f}\cot\alpha\Big[\Lambda_\xi\cos\beta + \Lambda_\eta\sin\beta\Big] \\
= {} & \frac{\cos^4\alpha}{f^2} \\
V = {} & \Bigg\{\tan\alpha\sin\beta\left[-2C\Lambda_\xi\sin\beta + \frac{S}{2f}\cot\alpha\cos\beta\sin\beta\right] \\
& -\tan\alpha\cos\beta\left[-2C\Lambda_\eta\sin\beta - \frac{S}{2f}\cot\alpha\cos^2\beta\right] - 2S\Lambda_\xi\Bigg\}\Big(-S\cos\beta\Big) \\
& +\Bigg\{\tan\alpha\cos\beta\left[-2C\Lambda_\eta\cos\beta + \frac{S}{2f}\cot\alpha\cos\beta\sin\beta\right] \\
& -\tan\alpha\sin\beta\left[-2C\Lambda_\xi\cos\beta - \frac{S}{2f}\cot\alpha\sin^2\beta\right] - 2S\Lambda_\eta\Bigg\}\Big(-S\sin\beta\Big) \\
& +C\left[-2C\Lambda_\eta\sin\beta - \frac{S}{2f}\cot\alpha\cos^2\beta\right] \\
& +C\left[-2C\Lambda_\xi\cos\beta - \frac{S}{2f}\cot\alpha\sin^2\beta\right] \\
= {} & -\frac{S^2}{2f}a - \frac{CS}{2f}\cot\alpha - 2\Lambda_\xi\cos\beta - 2\Lambda_\eta\sin\beta \\
= {} & -2\frac{\cos^2\alpha}{f} \\
W = {} & 1 \qquad \text{(A5)}
\end{aligned}
$$

APPENDIX B. EVALUATION OF THE JACOBIAN

B.1. Evaluation $J(t) = \frac{D(t)}{D(0)}$

First we evaluate the Jacobian which give the amplitude of the ray. From (8), this is given by

$$\begin{aligned} D(t) &= \frac{\partial(x,y,z)}{\partial(\xi_1,\eta_1,t)} = \begin{vmatrix} \frac{\partial\xi_2}{\partial\xi_1}+\frac{\partial p_{2x}}{\partial\xi_1}t & \frac{\partial p_{2y}}{\partial\xi_1}t & \frac{\partial\zeta_2}{\partial\xi_1}+\frac{\partial p_{2z}}{\partial\xi_1}t \\ \frac{\partial p_{2x}}{\partial\eta_1}t & \frac{\partial\eta_2}{\partial\eta_1}+\frac{\partial p_{2y}}{\partial\eta_1}t & \frac{\partial\zeta_2}{\partial\eta_1}+\frac{\partial p_{2z}}{\partial\eta_1}t \\ p_{2x} & p_{2y} & p_{2z} \end{vmatrix} \\ &= Ut^2+Vt+W \end{aligned} \tag{B1}$$

where U, V, W are

$$\begin{aligned} U &= \left(\frac{\partial p_{2y}}{\partial\xi_1}\frac{\partial p_{2z}}{\partial\eta_1}-\frac{\partial p_{2z}}{\partial\xi_1}\frac{\partial p_{2y}}{\partial\eta_1}\right)p_{2x}+\left(\frac{\partial p_{2x}}{\partial\eta_1}\frac{\partial p_{2z}}{\partial\xi_1}-\frac{\partial p_{2z}}{\partial\eta_1}\frac{\partial p_{2x}}{\partial\xi_1}\right)p_{2y} \\ &\quad+\left(\frac{\partial p_{2x}}{\partial\xi_1}\frac{\partial p_{2y}}{\partial\eta_1}-\frac{\partial p_{2y}}{\partial\xi_1}\frac{\partial p_{2x}}{\partial\eta_1}\right)p_{2z} \\ V &= \left(\frac{\partial p_{2y}}{\partial\xi_1}\frac{\partial\zeta_2}{\partial\eta_1}-\frac{\partial\zeta_2}{\partial\xi_1}\frac{\partial p_{2y}}{\partial\eta_1}-\frac{\partial\eta_2}{\partial\eta_1}\frac{\partial p_{2z}}{\partial\xi_1}\right)p_{2x} \\ &\quad+\left(\frac{\partial p_{2x}}{\partial\eta_1}\frac{\partial\zeta_2}{\partial\xi_1}-\frac{\partial\zeta_2}{\partial\eta_1}\frac{\partial p_{2x}}{\partial xi_1}-\frac{\partial\xi_2}{\partial\xi_1}\frac{\partial p_{2z}}{\partial\eta_1}\right)p_{2y} \\ &\quad+\left(\frac{\partial\xi_2}{\partial\xi_1}\frac{\partial p_{2y}}{\partial\eta_1}+\frac{\partial\eta_2}{\partial\eta_1}\frac{\partial p_{2x}}{\partial\xi_1}\right)p_{2z} \\ W &= -\left(\frac{\partial\zeta_2}{\partial\xi_1}\frac{\partial\eta_2}{\partial\eta_1}p_{2x}+\frac{\partial\xi_2}{\partial\xi_1}\frac{\partial\zeta_2}{\partial\eta_1}p_{2y}\right)+\frac{\partial\xi_2}{\partial\xi_1}\frac{\partial\eta_2}{\partial\eta_1}p_{2z} \end{aligned} \tag{B2}$$

We rewrite the values of U, V and W by using the relations

$$\begin{aligned} \frac{\partial p_{2x}}{\partial\xi_1} &= -2C\Lambda_\xi\cos\beta-\frac{S}{2f}\cot\alpha\sin^2\beta \\ \frac{\partial p_{2y}}{\partial\xi_1} &= -2C\Lambda_\xi\sin\beta+\frac{S}{2f}\cot\alpha\cos\beta\sin\beta \\ \frac{\partial p_{2x}}{\partial\eta_1} &= -2C\Lambda_\eta\cos\beta+\frac{S}{2f}\cot\alpha\sin\beta\cos\beta \\ \frac{\partial p_{2y}}{\partial\eta_1} &= -2C\Lambda_\eta\sin\beta-\frac{S}{2f}\cot\alpha\cos^2\beta \\ \frac{\partial p_{2z}}{\partial\xi_1} &= 2S\Lambda_\xi \end{aligned} \tag{B3}$$

$$\frac{\partial p_{2z}}{\partial \eta_1} = 2S\Lambda_\eta$$

where

$$C = \cos(2\alpha - 2\gamma), \qquad S = \sin(2\alpha - 2\gamma)$$
$$\Lambda_\xi = \frac{\partial(\alpha - \gamma)}{\partial \xi_1}, \qquad \Lambda_\xi = \frac{\partial(\alpha - \gamma)}{\partial \xi_1} \tag{B4}$$

The results are given by

$$\begin{aligned}
U = & -2S^2 \cos\beta \left\{ \left[-2C\Lambda_\xi \sin\beta + \frac{S}{2f} \cot\alpha \cos\beta \sin\beta \right] \Lambda_\eta \right. \\
& \left. - \left[-2C\Lambda_\eta \sin\beta - \frac{S}{2f} \cot\alpha \cos^2\beta \right] \Lambda_\xi \right\} \\
& -2S^2 \sin\beta \left\{ \left[-2C\Lambda_\eta \cos\beta + \frac{S}{2f} \cot\alpha \sin\beta \cos\beta \right] \Lambda_\xi \right. \\
& \left. - \left[-2C\Lambda \cos\beta - \frac{S}{2f} \cot\alpha \sin^2\beta \right] \Lambda_\eta \right\} \\
& -C \left\{ \left[-2C\Lambda_\xi \cos\beta - \frac{S}{2f} \cot\alpha \sin^2\beta \right] \right. \\
& \times \left[-2C\Lambda_\eta \sin\beta - \frac{S}{2f} \cot\alpha \cos^2\beta \right] \\
& - \left[-2C\Lambda_\xi \sin\beta + \frac{S}{2f} \cot\alpha \cos\beta \sin\beta \right] \\
& \left. \times \left[-2C\Lambda_\eta \cos\beta + \frac{S}{2f} \cot\alpha \sin\beta \cos\beta \right] \right\} \Big(-C \Big) \\
= & -\frac{S}{f} \cot\alpha \Big[\Lambda_\xi \cos\beta + \Lambda_\eta \sin\beta \Big] \\
V = & \left\{ \tan\gamma \sin\beta \left[-2C\Lambda_\xi \sin\beta + \frac{S}{2f} \cot\alpha \cos\beta \sin\beta \right] \frac{\partial \eta_2}{\partial \eta_1} \right. \\
& \left. - \tan\gamma \cos\beta \left[-2C\Lambda_\eta \sin\beta - \frac{S}{2f} \cot\alpha \cos^2\beta \right] \frac{\partial \xi_2}{\partial \xi_1} - 2S\Lambda_\xi \frac{\partial \eta_2}{\partial \eta_1} \right\} \\
& \times \Big(-S \cos\beta \Big) \\
& + \left\{ \tan\gamma \cos\beta \left[-2C\Lambda_\eta \cos\beta + \frac{S}{2f} \cot\alpha \cos\beta \sin\beta \right] \frac{\partial \xi_2}{\partial \xi_1} \right. \\
& - \tan\gamma \sin\beta \left[-2C\Lambda_\xi \cos\beta - \frac{S}{2f} \cot\alpha \sin^2\beta \right] \frac{\partial \eta_2}{\partial \eta_1} \\
& \left. - 2S\Lambda_\eta \frac{\partial \xi_2}{\partial \xi_1} \right\} \Big(-S \sin\beta \Big)
\end{aligned}$$

$$\begin{aligned}
&-C\frac{\partial\xi_2}{\partial\xi_1}\left[-2C\Lambda_\eta\sin\beta-\frac{S}{2f}\cot\alpha\cos^2\beta\right]\\
&-C\frac{\partial\eta_2}{\partial\eta_1}\left[-2C\Lambda_\xi\cos\beta-\frac{S}{2f}\cot\alpha\sin^2\beta\right]\\
=&-\frac{S^2}{2f}\frac{\partial\xi_2}{\partial\xi_1}\tan\gamma\cot\alpha\cos^2\beta-\frac{S^2}{2f}\frac{\partial\eta_2}{\partial\eta_1}\tan\gamma\cot\alpha\sin^2\beta\\
&+\frac{\partial\xi_1}{\partial\xi_1}\left[2\Lambda_\eta\sin\beta+\frac{CS}{2f}\cot\alpha\cos^2\beta\right]\\
&+\frac{\partial\eta_2}{\partial\eta_1}\left[2\Lambda_\xi\cos\beta+\frac{CS}{2f}\cot\alpha\sin^2\beta\right]\\
W=&-\frac{\partial\xi_2}{\partial\xi_1}\frac{\partial\eta_2}{\partial\eta_1}\frac{\cos(2\alpha-\gamma)}{\cos\gamma}
\end{aligned}\tag{B5}$$

B.2. Evaluation $J(t)\frac{\partial(p_{2x},p_{2y})}{\partial(x,y)}$

Next we evaluate the value of

$$W=J(t)\frac{\partial(p_{2x},p_{2y})}{\partial(x,y)}=\frac{1}{D(0)}\frac{\partial(p_{2x},p_{2y},z)}{\partial(\xi_1,\eta_1,t)}\tag{B6}$$

The can be derived as follows.

$$\begin{aligned}
W=&\frac{1}{D(0)}\begin{vmatrix}\frac{\partial p_{2x}}{\partial\xi_1} & \frac{\partial p_{2y}}{\partial\xi_1} & 0\\ \frac{\partial p_{2x}}{\partial\eta_1} & \frac{\partial p_{2y}}{\partial\eta_1} & 0\\ 0 & 0 & \frac{\partial z}{\partial t}\end{vmatrix}\\
=&-\frac{C}{D(0)}\left\{\left[-2C\Lambda_\xi\cos\beta-\frac{S}{2f}\cot\alpha\sin^2\beta\right]\right.\\
&\times\left[-2C\Lambda_\eta\sin\beta-\frac{S}{2f}\cot\alpha\cos^2\beta\right]\\
&-\left[-2C\Lambda_\xi\sin\beta+\frac{S}{2f}\cot\alpha\cos\beta\sin\beta\right]\\
&\left.\times\left[-2C\Lambda_\eta\cos\beta+\frac{S}{2f}\cot\alpha\sin\beta\cos\beta\right]\right\}\\
=&-\frac{1}{D(0)}\frac{SC^2}{f}\Big[\Lambda_\xi\cos\beta+\Lambda_\eta\sin\beta\Big]\\
=&-\frac{1}{D(0)}\frac{SC^2}{f}\frac{\cos^2\alpha}{2f}\left\{1-\frac{2ac}{R_1(R_2+a)}\Big[\Gamma_\xi\cos^2\beta+\Gamma_\eta\sin^2\beta\Big]\right\}
\end{aligned}\tag{B7}$$

APPENDIX C. EVALUATION OF SOME DERIVATIVES

C.1. Evaluation of the Derivative $\frac{\partial \xi_2}{\partial \xi_1}$

The ray reflected by the parabola is expressed by

$$\xi_2 - \xi_1 = -\tan 2\alpha \cos\beta(\zeta_2 - \zeta_1) \tag{C1}$$

Taking the derivative with respect to ξ_1 yields

$$\begin{aligned}\frac{\partial \xi_2}{\partial \xi_1} - 1 \;=\; &-\left[\frac{2\cos\beta}{\cos^2 2\alpha}\frac{\partial \alpha}{\partial \xi_1} - \tan 2\alpha \sin\beta \frac{\partial \beta}{\partial \xi_1}\right](\zeta_2 - \zeta_1)\\ &-\tan 2\alpha \cos\beta\left(\frac{\partial \eta_2}{\partial \xi_2}\frac{\partial \xi_2}{\partial \xi_1} - \frac{\partial \zeta_1}{\partial \xi_1}\right)\end{aligned} \tag{C2}$$

If we use the relation $\frac{\partial \zeta_2}{\partial \xi_2} = \tan\gamma\cos\beta$, we have

$$\begin{aligned}&\left[1 + \tan 2\alpha \tan\gamma \cos^2\beta\right]\frac{\partial \xi_2}{\partial \xi_1} = 1 + \tan 2\alpha \tan\alpha \cos^2\beta\\ &-\left[\frac{2\cos\beta}{\cos^2 2\alpha}\frac{\partial \alpha}{\partial \xi_1} - \tan 2\alpha \sin\beta \frac{\partial \beta}{\partial \xi_1}\right](\zeta_2 - \zeta_1)\\ =\;& 1 + \tan 2\alpha \tan\alpha \cos^2\beta - \frac{\cos^2\alpha}{f\cos 2\alpha}\left[1 + \frac{2\sin^2\alpha}{\cos 2\alpha}\cos^2\beta\right](\zeta_2 - \zeta_1)\\ =\;& 1 + \tan 2\alpha \tan\alpha \cos^2\beta - \frac{\cos^2\alpha}{f\cos 2\alpha}\left[1 + \frac{2\sin^2\alpha}{\cos 2\alpha}\cos^2\beta\right]\\ &\times\left[-R_2\cos 2\alpha + f\frac{\cos 2\alpha}{\cos^2\alpha}\right]\\ =\;& \frac{R_2\cos^2\alpha}{f}\left[1 + \frac{2\sin^2\alpha}{\cos 2\alpha}\cos^2\beta\right]\end{aligned} \tag{C3}$$

Therefore we obtain

$$\begin{aligned}\frac{\partial \xi_2}{\partial \xi_1} \;=\;& \frac{R_2\cos^2\alpha}{f}\left[1 + \tan 2\alpha \tan\alpha \cos^2\beta\right]\left[1 + \tan 2\alpha \tan\gamma \cos^2\beta\right]^{-1}\\ \equiv\;& \frac{R_2\cos^2\alpha}{f}\Gamma_\xi\end{aligned} \tag{C4}$$

C.2. Evaluation of the Derivative $\frac{\partial \eta_2}{\partial \eta_1}$

The ray reflected by the parabola is expressed by

$$\eta_2 - \eta_1 = -\tan 2\alpha \sin\beta(\zeta_2 - \zeta_1) \tag{C5}$$

Taking the derivative with respect to η_1 yields

$$\frac{\partial\eta_2}{\partial\eta_1} - 1 = -\left[\frac{2\sin\beta}{\cos^2 2\alpha}\frac{\partial\alpha}{\partial\eta_1} + \tan 2\alpha\cos\beta\frac{\partial\beta}{\partial\eta_1}\right](\zeta_2-\zeta_1)$$
$$-\tan 2\alpha\sin\beta\left(\frac{\partial\zeta_2}{\partial\eta_2}\frac{\partial\eta_2}{\partial\eta_1} - \frac{\partial\zeta_1}{\partial\eta_1}\right) \tag{C6}$$

If we use the relation $\dfrac{\partial\zeta_2}{\partial\eta_2} = \tan\gamma\sin\beta$, we have

$$\left[1+\tan 2\alpha\tan\gamma\sin^2\beta\right]\frac{\partial\eta_2}{\partial\eta_1} = 1+\tan 2\alpha\tan\alpha\sin^2\beta$$
$$-\left[\frac{2\sin\beta}{\cos^2 2\alpha}\frac{\partial\alpha}{\partial\eta_1} - \tan 2\alpha\sin\beta\frac{\partial\beta}{\partial\eta_1}\right](\zeta_2-\zeta_1)$$
$$= 1+\tan 2\alpha\tan\alpha\cos^2\beta - \frac{\cos^2\alpha}{f\cos 2\alpha}\left[1+\frac{2\sin^2\alpha}{\cos 2\alpha}\cos^2\beta\right](\zeta_2-\zeta_1)$$
$$= 1+\tan 2\alpha\tan\alpha\cos^2\beta - \frac{\cos^2\alpha}{f\cos 2\alpha}\left[1+\frac{2\sin^2\alpha}{\cos 2\alpha}\sin^2\beta\right]$$
$$\times\left[-R_2\cos 2\alpha + f\frac{\cos 2\alpha}{\cos^2\alpha}\right]$$
$$= \frac{R_2\cos^2\alpha}{f}\left[1+\frac{2\sin^2\alpha}{\cos 2\alpha}\sin^2\beta\right] \tag{C7}$$

Therefore we obtain

$$\frac{\partial\eta_2}{\partial\eta_1} = \frac{R_2\cos^2\alpha}{f}\left[1+\tan 2\alpha\tan\alpha\sin^2\beta\right]\left[1+\tan 2\alpha\tan\gamma\sin^2\beta\right]^{-1}$$
$$= \frac{R_2\cos^2\alpha}{f}\Gamma_\eta \tag{C8}$$

C.3. Evaluation of the Derivative $\frac{\partial(\alpha-\gamma)}{\partial\xi_1}$ and $\frac{\partial(\alpha-\gamma)}{\partial\eta_1}$

Since

$$\frac{\partial\alpha}{\partial\xi_1} = \frac{\cos^2\alpha}{2f}\cos\beta, \qquad \frac{\partial\gamma}{\partial\xi_1} = \frac{\partial\gamma}{\partial\xi_2}\frac{\partial\xi_2}{\partial\xi_1} = \cos^2\gamma\cos\beta\frac{a^4}{b^2\zeta_2^3} \tag{C9}$$

we have

$$\frac{\partial(\alpha-\gamma)}{\partial\xi_1} = \left[\frac{\cos^2\alpha}{2f} - \cos^2\gamma\frac{a^4}{b^2\zeta_2^3}\frac{\partial\xi_2}{\partial\xi_1}\right]\cos\beta$$

$$= \left[\frac{\cos^2\alpha}{2f} - \frac{ac}{R_1 R_2(R_2+a)}\frac{\partial\xi_2}{\partial\xi_1}\right]\cos\beta$$
$$= \frac{\cos^2\alpha}{2f}\left[1 - \frac{2ac}{R_2(R_2+a)}\Gamma_\xi\right]\cos\beta \tag{C10}$$

Similarly we have

$$\frac{\partial(\alpha-\gamma)}{\partial\eta_1} = \frac{\cos^2\alpha}{2f}\left[1 - \frac{2ac}{R_2(R_2+a)}\Gamma_\eta\right]\sin\beta \tag{C11}$$

C.4. Verification of the Relation $E = \frac{\sin(2\alpha-\gamma)}{\sin\gamma} = \frac{c}{a}$

By using the relations

$$\begin{aligned} E &= \frac{\sin(2\alpha-\gamma)}{\sin\gamma} = \sin 2\alpha\cot\gamma - \cos 2\alpha \\ \cos 2\alpha &= \frac{c-\zeta_2}{R_2}, \quad \sin 2\alpha = \frac{\rho_2}{R_2}; \qquad \cot\gamma = \frac{b^2\zeta_2}{a^2\rho_2} \end{aligned} \tag{C12}$$

we obtain

$$E = \frac{\rho_2}{R_2}\frac{b^2\zeta_2}{a^2\rho_2} - \frac{c-\zeta_2}{R_2} = \frac{b^2\zeta_2}{R_2 a^2} - \frac{c-\zeta_2}{R_2} = \frac{c(c\zeta_2 - a^2)}{a^2 R_2} = \frac{c}{a} \tag{C13}$$

APPENDIX D. PARAMETERS OF CASSEGRAIN ANTENNA

D.1. $\tan\frac{\phi_v}{2} = \frac{D}{2f}$ and $\tan\frac{\phi_r}{2} = \frac{D}{2f_e}$

The equation of the parabolic reflector is given by (16). Hence we have

$$OA = -f + c, \qquad AH = \frac{D^2}{4f}, \qquad FH = f - \frac{D^2}{4f} \tag{D1}$$

and

$$\tan\phi_v = \frac{D}{f - \dfrac{D^2}{4f}} = \frac{2\cdot\dfrac{D}{2f}}{1 - \left(\dfrac{D}{2f}\right)^2} = \frac{2\tan\dfrac{\phi_v}{2}}{1 - \tan^2\dfrac{\phi_v}{2}} \tag{D2}$$

From (D2) we have $\tan\frac{\phi_v}{2} = \frac{D}{2f}$. It may be noted that D is height of the edge of the parabolic reflector from horizontal axis.

Similarly we have

$$\tan\frac{\phi_r}{2} = \frac{D}{2f_e} \tag{D3}$$

where f_e is the focal length of the equivalent parabola.

D.2. $e = \frac{\sin\frac{1}{2}(\phi_v+\phi_r)}{\sin\frac{1}{2}(\phi_v-\phi_r)}$

From the similarities of the triangles we have

$$\frac{f_e}{f} = \frac{L_r}{L_v} = \frac{c+a}{c-a} = \frac{e+1}{e-1} = \frac{\tan\frac{1}{2}\phi_v}{\tan\frac{1}{2}\phi_r} \tag{D4}$$

The last term is obtained from the results in (D1). From the above equation e is obtained as

$$e = \frac{\tan\frac{1}{2}\phi_v + \tan\frac{1}{2}\phi_r}{\tan\frac{1}{2}\phi_v - \tan\frac{1}{2}\phi_r} = \frac{\sin\frac{1}{2}(\phi_v+\phi_r)}{\sin\frac{1}{2}(\phi_v-\phi_r)} = \frac{L_r+L_v}{L_r-L_v} \tag{D5}$$

and

$$1-\frac{1}{e} = \frac{2L_v}{L_r+L_v} = 1-\frac{\sin\frac{1}{2}(\phi_v+\phi_r)}{\sin\frac{1}{2}(\phi_v-\phi_r)} \tag{D6}$$

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