

ON THE EQUIVALENT RADIUS OF A RADIATING SLOT IN IMPEDANCE CALCULATIONS

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Abstract—By means of evaluating impedance of a slot by spectral domain analysis, the radius of the equivalent cylindrical dipole is found in terms of slot parameters. The analysis proceeds from the integral equation for the surface current density induced on a planar strip. Explicit expressions for real and imaginary parts of impedances are derived in visible and invisible regions respectively.

1. INTRODUCTION

Waveguide fed slot arrays are used in many radar and communication systems. Considerable investigations have already been carried out on the evaluation of admittance of radiating slot using variational [1–3] as well as moment method [4] formulations. The specific advantage of variational formulation is that it enables evaluation of the parameters of the complex equivalent circuit [1, 3]. As a result the entire problem reduces to a simple circuit analysis problem, which is necessary for slot radiator using multilayered junctions [5]. In many cases [14–16] it is convenient to replace the slot by an equivalent planar or

cylindrical dipole for analysis [17, 18]. Hallen [10] has found the radius of the equivalent cylindrical dipole using static field considerations. Warne et al. [13] evaluated the equivalent radius for narrow slots in terms of the solution to the Hallen's integral equations [10, 19, 20] using Galerkin method with piecewise sinusoidal function. Rhodes [8] has established the equivalence between a planar and cylindrical dipole using spectral domain approach. This paper aims at developing a simple analytical procedure for determining the radius of the equivalent cylindrical dipole for a radiating slot which is based upon the impedance considerations.

The expression for the complex radiated power in the numerator of the variational formula appears in the form of an integral basically in the spatial domain. The formula contains half-space dyadic Green's function [2]. A close form expression for the quadruple integral has been derived [2] for slots resonating near half wavelength making considerable simplifying approximations.

The same integral has also been evaluated by transforming it to the spatial domain [6–8] employing Fourier Transform of Maxwell's equation. The complete input impedance i.e., real and imaginary parts have been found by analytical extension of the Poynting vector method by evaluating the integrals in the visible ($k_x^2 + k_z^2 \leq k^2$) and invisible ($k_x^2 + k_z^2 \geq k^2$) regions of the entire wavenumber plane.

In the present work, complex radiated power from the aperture is determined from the Fourier Transform of the electric field distribution in the aperture. This Fourier Transform gives the angular spectrum of the plane waves. The plane wave spectrum in visible and invisible regions determines the radiated and reactive power stored in the aperture respectively. The aperture admittance is obtained by the analytical expression of Poynting vector method and from the expression of aperture admittance; equivalent planar dipole impedance is calculated using Bookers relation. By comparison of these expressions to that of thin cylindrical dipole antenna obtained by induced emf method [12], radius of the equivalent cylindrical dipole is obtained.

2. ANALYSIS

Consider a slot fed by a waveguide as shown in Fig. 1. The complex power radiated from the slot on one side of the surface is given by

$$\mathcal{P} = \frac{1}{2} \iint_S \vec{E} \times \vec{H}^* \cdot \hat{u}_y dz dx \quad (1)$$

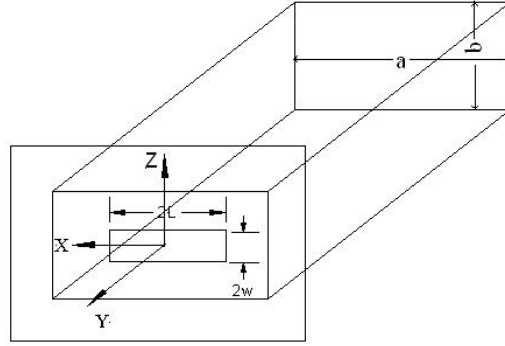


Figure 1. A slot of length $2L$ and width $2w$ fed by a rectangular waveguide.

The magnetic field \vec{H} has been expressed in the form of a double integral in terms of half-space dyadic Green's function [2]. The integration of the quadruple integral, so obtained, has been reduced to a closed form expression containing summation of sine and cosine integrals using simplifying approximation. The electric field \vec{E} in the aperture plane of the slot of Fig. 1 can be regarded as equivalent to a magnetic current $\vec{\mathcal{M}}$ [1]. The relation between $\vec{\mathcal{M}}$ and \vec{E} is of the form

$$\vec{\mathcal{M}} = \vec{E} \times \hat{n} \quad (2)$$

From the image theory, it is found that a magnetic current on the surface of an infinite ground plane results in magnetic having strength twice that given by Eq. (2). Hence total effective magnetic current $\vec{\mathcal{M}}_t$ is given by

$$\vec{\mathcal{M}}_t = 2\vec{\mathcal{M}} = 2\vec{E} \times \hat{n} \quad (3)$$

Self reaction of the magnetic current is

$$\langle a, a \rangle_m = - \iint_s \vec{H}^a \cdot \vec{\mathcal{M}}_t ds = -2 \iint_s \vec{H}^a \cdot \vec{E} \times \hat{u}_z ds = 2 \iint_s \vec{E} \cdot \vec{H}^a ds \quad (4)$$

where \vec{H}^a is the magnetic field in the aperture plane produced by magnetic current.

If $\vec{\mathcal{E}}$ and $\vec{\mathcal{H}}$ are the electric and magnetic fields in the spectral domain, they satisfy the following equations, which are Fourier

transforms of Maxwell's equations [6, 7, 21]

$$\vec{\mathcal{H}}(k_x, k_z) = \frac{1}{\omega \varepsilon} \vec{\mathcal{E}}(k_x, k_z) \times \vec{k} = \frac{1}{\omega \mu} \vec{\mathcal{H}}(k_x, k_z) \times \vec{k} \quad (5)$$

where

$$\vec{k} \cdot \vec{\mathcal{E}} = 0 \text{ and } \vec{k} \cdot \vec{\mathcal{H}} = 1 \quad (6)$$

Using Parseval's relation, the double integral in (1) is expressed in the spectral domain as [6, 7]

$$\mathcal{P} = \frac{1}{2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \vec{\mathcal{E}} \times \vec{\mathcal{H}} \cdot \hat{u}_y dk_x dk_z \quad (7)$$

From (2)–(6), it is found that

$$\mathcal{P} = \frac{1}{2k\eta} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{k^2 - k_z^2}{k_y^*} |\vec{\mathcal{E}}|^2 dk_x dk_z \quad (8)$$

where

$$\vec{\mathcal{E}} = \hat{u}_x \mathcal{E}_x = \frac{1}{2\pi} \iint_S E_x(x, z) \cdot e^{j(k_x x + k_z z)} dx dz \quad (9)$$

Since the electric field in the aperture plane of the slot of Fig. 1 is x -directed, it is reasonable to assume that the electric field in the aperture plane of the slot of Fig. 1 can be represented by an expression of the form [1]

$$E_x(x, z, 0) = E_0 \cos\left(\frac{\pi x}{2L}\right) \quad (10)$$

Substituting (10) in (9)

$$\mathcal{E}_x = \frac{E_0 \cdot 2w}{\pi} \cdot \frac{(\pi/2L) \cos(k_z L)}{(\pi/2L)^2 - k_z^2} \cdot \frac{\sin(k_x w)}{k_x w} \quad (11)$$

The electric field in the aperture plane of the slot has been expressed as the summation of an infinite number of plane waves having propagation constants k_x , k_y , k_z in the x , y , and z directions respectively. The propagation constants of the plane waves satisfy the relation

$$k_x^2 + k_y^2 + k_z^2 = k^2 \quad (12)$$

From (8), (11) and (12), it is found that the complex radiated power satisfies the relation

$$\mathcal{P} = \frac{(E_0 w)^2}{4k\eta L^2} \cdot \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[\frac{\sin(k_x w)}{k_x w} \right]^2 \times \frac{\cos k_z L}{\left[\left(\frac{\pi}{2L} \right)^2 - k_z^2 \right]^2} \frac{k^2 - k_z^2}{\sqrt{k^2 - k_x^2 - k_z^2}} dk_x dk_z \quad (13)$$

\mathcal{P} is complex and may be represented as $\mathcal{P} = \mathcal{P}_r + j\mathcal{P}_j$, where \mathcal{P}_r is the real part and is obtained for $k_x^2 + k_z^2 \leq k^2$ and \mathcal{P}_j is the imaginary part and is obtained for $k_x^2 + k_z^2 \geq k^2$.

The aperture admittance is obtained on dividing the expression (13) for \mathcal{P} , the complex radiated power by the square of the voltage at the centre of the slot, when its length is near $\lambda/2$.

$$\Delta V = E_0 2w \quad (14)$$

Hence, Dividing (13) by the square of (14), the aperture admittance Y_a is obtained as

$$Y_a = \frac{8wL}{\pi^4 k\eta} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{k^2 - k_z^2}{\sqrt{k^2 - k_x^2 - k_z^2}} \left[\frac{\sin(k_x w)}{k_x w} \right]^2 \times \frac{\cos^2(k_z L)}{\left\{ 1 - \left(\frac{2Lk_z}{\pi} \right)^2 \right\}^2} dk_x dk_z \quad (15)$$

For the purpose of integration, the following substitution are made

$$k_x = t \cos \phi, \quad k_z = t \sin \phi \Rightarrow dk_x dk_z = t dt d\phi \quad (16)$$

Further, for integration in the visible region, the following substitutions are made

$$t = k \sin \theta, \quad dt = k \cos \theta d\theta \quad (17)$$

where the limits of t for the visible region are $t = 0$ and $t = k$. From (15)–(17), it is found that

$$\begin{aligned} G_a = & \frac{8wLk^2}{\pi^4 \eta} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} (1 - \sin^2 \theta \sin^2 \phi) \cdot \frac{\sin^2(wk \sin \theta \cos \phi)}{(wk \sin \theta \cos \phi)^2} \\ & \times \frac{\cos^2(kL \sin \theta \sin \phi)}{\left[1 - \left(\frac{2kL \sin \theta \sin \phi}{\pi} \right)^2 \right]^2} \sin \theta d\theta d\phi \end{aligned} \quad (18)$$

For integration in the invisible region, the following substitution is made

$$t = k \cosh \theta, \quad dt = k \sinh \theta d\theta \quad (19)$$

where the limits of integration are from $t = k$ to $t = \infty$. From (14), (15) and (19), the imaginary part assumes the form

$$\begin{aligned} B_a = & \frac{8wLk^2}{\pi^4\eta} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} (1 - \cosh^2 \theta \sin^2 \phi) \frac{\sin^2(wk \cosh \theta \sin \phi)}{(wk \cosh \theta \sin \phi)^2} \\ & \times \frac{\cos^2(kL \cosh \theta \sin \phi)}{\left[1 - \left(\frac{2kL \cosh \theta \sin \phi}{\pi}\right)^2\right]^2} \cosh \theta d\theta d\phi \end{aligned} \quad (20)$$

The double integrals in (18) and (20) are evaluated using the Simpson's formula for double integration. However if $2L \sim \lambda/2$ i.e., in the immediate vicinity of the $\lambda/2$ it is possible to derive closed form expressions for these integrals with the help of some approximations [6–8]. With these simplifications the expressions in (18), (20) reduces to,

$$\begin{aligned} G_a = & \frac{w}{30a\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n (wk)^{2n}}{(2n+2)(2n+1)(n!)^2} \\ & \cdot \left[Cin(2kL) + \left(Cin(2kL) - \frac{1}{2}Cin(4kL) \right) \cos(2kL) \right. \\ & \left. - \left(Si(2kL) - \frac{1}{2}Si(4kL) \right) \sin(2kL) \right] \end{aligned} \quad (21)$$

and

$$\begin{aligned} B_a = & \frac{w}{240a\pi^2} \cdot \left[Si(2kL) + \left(Si(2kL) - \frac{1}{2}Si(4kL) \right) \cos(2kL) \right. \\ & \left. - \left(Cin(2kL) - \frac{1}{2}Cin(4kL) - \ln\left(\frac{e^{3/2}L}{2w}\right) \right) \sin(2kL) \right] \end{aligned} \quad (22)$$

Using Booker's relation the impedance for the complimentary planar

dipole differs only by proportionality constant.

$$\begin{aligned}
 R = & \frac{35530.L}{30a\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n (wk)^{2n}}{(2n+2)(2n+1)(n!)^2} \\
 & \cdot \left[Cin(2kL) + \left(Cin(2kL) - \frac{1}{2}Cin(4kL) \right) \cos(2kL) \right. \\
 & \left. - \left(Si(2kL) - \frac{1}{2}Si(4kL) \right) \sin(2kL) \right] \quad (23)
 \end{aligned}$$

and

$$\begin{aligned}
 X = & \frac{35530.L}{240a\pi^2} \cdot \left[Si(2kL) + \left(Si(2kL) - \frac{1}{2}Si(4kL) \right) \cos(2kL) \right. \\
 & \left. - \left(Cin(2kL) - \frac{1}{2}Cin(4kL) - \ln \left(\frac{e^{3/2}L}{2w} \right) \right) \sin(2kL) \right] \quad (24)
 \end{aligned}$$

Comparing the closed form expressions of input reactance obtained here with that of cylindrical dipole antenna [11] with a radius r , it is noted that both expressions give exactly the same result if $r = \frac{2w}{e^{3/2}} \approx \frac{2w}{4.48}$. This result is quite near to result obtained based on static field considerations by Hallen [10], $r = 2w/4$ and is in exact agreement to the one predicted by Rhodes [8].

3. NUMERICAL RESULTS AND DISCUSSION

The results for the radiation resistance and input reactance of a planar dipole antenna of width $2w$ and length $2L$ computed from the numerical evaluation of integration in (18) and (20) where standard X-band waveguide parameters have been taken is shown in Fig. 2. It is worthwhile to point out that both the integrals are evaluated as a function of $2L/\lambda$ for $f = 9.37$ GHz with $2w/\lambda$ as parameter. The values obtained are in perfect agreement with those obtained for a dipole antenna using induced emf method [12]. Further, for $2L = \lambda/2$, one of the factors, assumes 0/0 form when ϕ is odd multiple of $\pi/2$, for $\theta = \pi/2$ in (18) and $\theta = 0$ in (20). In the process of the evaluation of the integrals, the limiting value of 0/0 form is evaluated using L Hospital's theorem.

From Eq. (25) it is concluded that the equivalent radius of the cylindrical dipole antenna is $2w/e^{3/2}$. This equivalent radius obtained from impedance considerations is different from but close to (within 10 percent), the equivalent radius $2w/4$ obtained from static

field considerations [10]. It is usually assumed that the static field equivalence will lead to correct impedance but it is demonstrated that the static field equivalence may not apply exactly to impedance calculations. Further it provides a way to account for the finite slot thickness while equivalent dipole radius being calculated by variational method. The effect of slot thickness can be accounted using transmission line representation of the equivalent network parameters and the comparable dipole of similar length but correction factor included in the radius may be chosen to represent the slot with finite thickness.

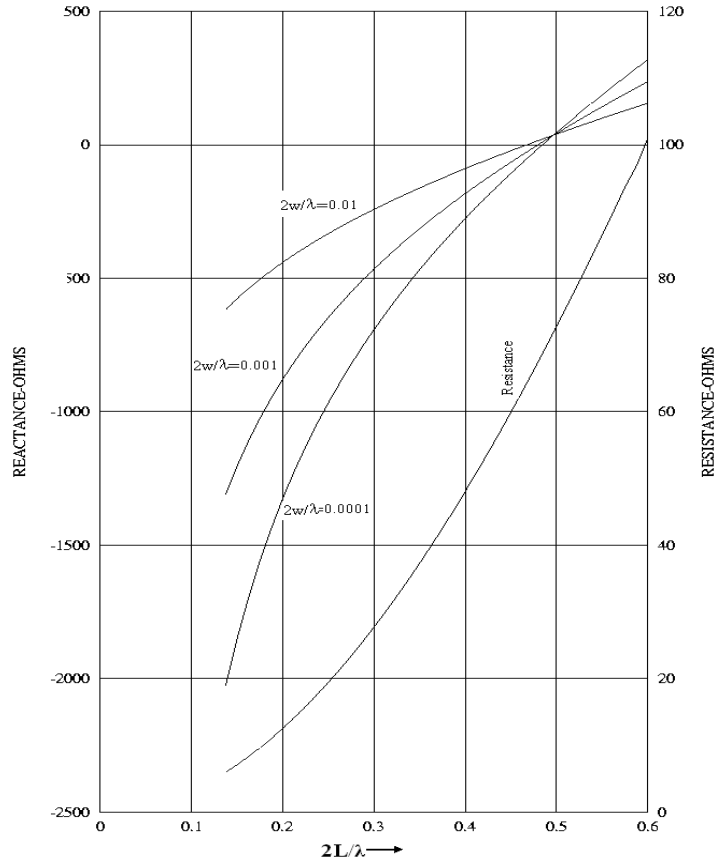


Figure 2. The radiation resistance and input reactance of a planar dipole antenna of width $2w$ and length $2L$ computed from the numerical evaluation of integration in (18) and (20).

4. CONCLUSION

Integral equations have been developed. These equations have been solved employing spectral domain approach. Numerical results for the radiation resistance and input reactance of a planar dipole antenna of width $2w$ and length $2L$ computed from the numerical evaluation of integration in (18) and (20) for standard X-band waveguide parameters have been presented. A method to account for the finite slot thickness in equivalent dipole radius is discussed. To arrive at a closed form expression for thickness correction the problem is being investigated further.

ACKNOWLEDGMENT

Ashish Kumar Tiwari is grateful to Shri P. R. Hannurkar at RRCAT, Indore, for his interest in the problem and useful suggestions. B. N. Das thanks Indian National Science Academy, New Delhi for financial support under INSA Honorary Scientist Scheme.

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