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ANALYSIS OF LONGITUDINALLY INHOMOGENEOUS WAVEGUIDES USING THE METHOD OF MOMENTS

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Abstract—A new method is introduced to analyze arbitrary Longitudinally Inhomogeneous Waveguides (LIWs). In this method, the integral equations of the LIWs, converted from their differential equations, are solved using the method of moments (MOM). It is assumed that the electric permittivity function is known at all or only at some points along the length of LIWs. The validity of the method is verified using a comprehensive example.

1. INTRODUCTION

Longitudinally Inhomogeneous Waveguides (LIWs) can be used in microwaves as phase changers, matching transformers and filters [1–3], specially for high power applications. The differential equations describing LIWs have non-constant coefficients and so except for a few special cases no analytical solution exists for them. Of course, the most straightforward method to analyze LIWs is subdividing the filled region into many thin uniform layers [3, 4]. Analysis of arbitrary LIWs using Taylor's and the Fourier series expansion of the permittivity function has been introduced in [5] and [6], respectively. In this paper, a new method is introduced to analyze arbitrary LIWs, also. First, the differential equations of LIWs are converted into integral equations. Then the integral equations are solved using the method of moments (MOM). Rectangular pulse function expansion with point matching (Collocation) is explained for this purpose. The validity of the method is verified using a comprehensive example. This method is applicable to all arbitrary LIWs, whose the electric permittivity function is known at all or only some points along their length.

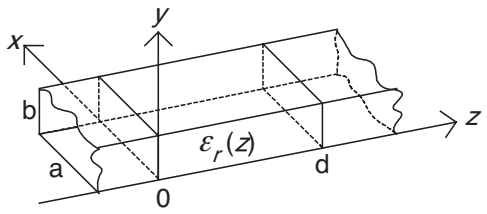

Figure 1. A typical LIW.

2. THE EQUATIONS OF LIWS

In this section, the frequency domain equations of the LIWs are reviewed. Figure 1 shows a typical LIW with dimensions of a and b, filled by an inhomogeneous lossy dielectric with complex electric permittivity function $\varepsilon_r(z)$ and length d. It is assumed that a TE_{10} mode with electric filed strength E^i propagates towards the positive z direction. With this assumption, we have

$$E_y(x,z) = \sin(\pi x/a)E_y(z) \tag{1}$$
$$H_x(x,z) = \sin(\pi x/a)H_x(z) \tag{2}$$

The differential equations describing LIWs are given by

$$\frac{dE_y(z)}{dz} = j\omega\mu_0 H_x(z) \tag{3}$$
$$\frac{dH_x(z)}{dz} = j\omega\varepsilon_0\left(\varepsilon_r(z) - (f_c/f)^2\right)E_y(z) \tag{4}$$

where

$$f_c = \frac{c}{2a} \tag{5}$$

in which c is the velocity of the light, is the cutoff frequency of the hollow waveguide. Furthermore, the terminal conditions are as follows

$$E_y(0) + Z_S H_x(0) = 2E^i \tag{6}$$
$$E_y(d) - Z_L H_x(d) = 0 \tag{7}$$

where

$$Z_S = Z_L = Z_{TE} = \begin{cases} -\dfrac{\eta_0}{\sqrt{1-(f_c/f)^2}}, & f > f_c \\ -j\dfrac{\eta_0}{\sqrt{(f_c/f)^2-1}}, & f < f_c \end{cases} \tag{8}$$

is the waveguide impedance, in which $\eta_0 = \sqrt{\mu_0/\varepsilon_0}$ is the wave impedance of the free space.

3. INTEGRAL EQUATIONS FOR LIWS

In this section, the differential equations of LIWs are converted into integral equations. The Equations (3)–(4) are converted to the following integral equations

$$E_y(z) = j\omega\mu_0 \int_0^z H_x(z')dz' + C_1 \tag{9}$$

$$H_x(z) = j\omega\varepsilon_0 \int_0^z \left(\varepsilon_r(z') - (f_c/f)^2\right) E_y(z')dz' + C_2 \tag{10}$$

In (9) and (10), C_1 and C_2 are two coefficients, which can be found according to the boundary conditions (6)–(7). After finding these coefficients, we will have the following integral equations

$$E_y(z) = E^i - \frac{1}{2}\int_0^d \left(j\omega\mu_0 H_x(z') - j\omega\varepsilon_0 Z_{TE}\left(\varepsilon_r(z') - (f_c/f)^2\right)E_y(z')\right) dz' + j\omega\mu_0 \int_0^z H_x(z')dz' \tag{11}$$

$$H_x(z) = \frac{1}{Z_{TE}}E^i + \frac{1}{2Z_{TE}} \int_0^d \left(j\omega\mu_0 H_x(z') - j\omega\varepsilon_0 Z_{TE}\left(\varepsilon_r(z') - (f_c/f)^2\right)E_y(z')\right) dz' + j\omega\varepsilon_0 \int_0^z \left(\varepsilon_r(z') - (f_c/f)^2\right)E_y(z')dz' \tag{12}$$

3.1. The Method of Moments

To solve the integral Equations (11) and (12), we can use the method of moments (MOM). For this purpose, the transverse electric and

magnetic fields are written as the sum of N basis functions as follows

$$E_y(z) \cong \sum_{n=1}^{N} E_n f_n(z) \tag{13}$$

$$H_x(z) \cong \sum_{n=1}^{N} H_n g_n(z) \tag{14}$$

where E_n and H_n are the electric and magnetic coefficients, respectively, which have to be found. Substituting (13) and (14) into (11) and (12), gives us the following matrix relation at the arbitrary surface z.

$$\begin{bmatrix} A_1(z) & \cdots & A_N(z) & B_1(z) & \cdots & B_N(z) \\ C_1(z) & \cdots & C_N(z) & D_1(z) & \cdots & D_N(z) \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix} = \begin{bmatrix} 1 \\ Z_{TE}^{-1} \end{bmatrix} E^i \tag{15}$$

In (15), $\mathbf{E} = [\ E_1 \quad E_2 \quad \cdots \quad E_N]^{\ T}$ and $\mathbf{H} = [\ H_1 \quad H_2 \quad \cdots \quad H_N]^{\ T}$ and also A_n, B_n, C_n and D_n $(n = 1, 2, \ldots, N)$ are four functions given by

$$A_n(z) = f_n(z) - \frac{1}{2} j\omega\varepsilon_0 Z_{TE} \int_0^d f_n(z') \left(\varepsilon_r(z') - (f_c/f)^2\right) dz' \tag{16}$$

$$B_n(z) = -j\omega\mu_0 \int_0^z g_n(z')dz' + \frac{1}{2} j\omega\mu_0 \int_0^d g_n(z')dz' \tag{17}$$

$$\begin{aligned} C_n(z) &= -j\omega\varepsilon_0 \int_0^z f_n(z') \left(\varepsilon_r(z') - (f_c/f)^2\right) dz' \\ &\quad + \frac{1}{2} j\omega\varepsilon_0 \int_0^d f_n(z') \left(\varepsilon_r(z') - (f_c/f)^2\right) dz' \end{aligned} \tag{18}$$

$$D_n(z) = g_n(z) - \frac{1}{2Z_{TE}} j\omega\mu_0 \int_0^d g_n(z')dz' \tag{19}$$

To find the electric and magnetic coefficients in (15), one can use both collocation (rectangular pulse function expansion with point matching) and Galerkin's method. To use collocation, the LIW is partitioned into N partitions with thickness $\Delta z = d/N$ and the basis functions are

chosen as follows

$$f_n(z) = g_n(z) = \begin{cases} 1, & (n-1)\Delta z < z < n\Delta z \\ 0, & \text{elsewhere} \end{cases} \tag{20}$$

for $n = 1, 2, \ldots, N$. Now, setting z equal to $z_m = (m - 1/2)\Delta z$ in (15) for $m = 1, 2, \ldots, N$ (the centers of the partitions), yields the following matrix equation

$$\begin{bmatrix} A_1(z_1) & \cdots & A_N(z_1) & B_1(z_1) & \cdots & B_N(z_1) \\ C_1(z_1) & \cdots & C_N(z_1) & D_1(z_1) & \cdots & D_N(z_1) \\ \vdots & & \vdots & \vdots & & \vdots \\ A_1(z_N) & \cdots & A_N(z_N) & B_1(z_N) & \cdots & B_N(z_N) \\ C_1(z_N) & \cdots & C_N(z_N) & D_1(z_N) & \cdots & D_N(z_N) \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix} = \begin{bmatrix} 1 \\ Z_{TE}^{-1} \\ \vdots \\ 1 \\ Z_{TE}^{-1} \end{bmatrix} E^i \tag{21}$$

in which for m and $n = 1, 2, \ldots, N$, we will have

$$A_n(z_m) = \delta(n-m) - \frac{1}{2} j\omega\varepsilon_0 Z_{TE} \int_{(n-1)\Delta z}^{n\Delta z} \left(\varepsilon_r(z') - (f_c/f)^2\right) dz' \tag{22}$$

$$B_n(z_m) = -j\omega\mu_0 \left(U(m-n) - \frac{1}{2}\right) \Delta z \tag{23}$$

$$C_n(z_m) = -j\omega\varepsilon_0 \int_{(n-1)\Delta z}^{(n-1+U(m-n))\Delta z} \left(\varepsilon_r(z') - (f_c/f)^2\right) dz' + \frac{1}{2} j\omega\varepsilon_0 \int_{(n-1)\Delta z}^{n\Delta z} \left(\varepsilon_r(z') - (f_c/f)^2\right) dz' \tag{24}$$

$$D_n(z_m) = \delta(n-m) - \frac{1}{2Z_{TE}} j\omega\mu_0 \Delta z \tag{25}$$

Here δ is the Kronecker delta function and also

$$U(m-n) \triangleq \begin{cases} 1 & m > n \\ 1/2 & m = n \\ 0 & m < n \end{cases} \tag{26}$$

The relation (21) gives us the electric and magnetic coefficients by a simple matrix inversion process.

The integrals existed in (22) and (24) will be exactly calculated if the electric permittivity function is known at all points, continuously.

However, these integrals can be approximately calculated if the electric permittivity function is known only at N points. In this case, which is more practical, we can assume that the electric permittivity function varies between two adjacent points stepwise, linearly or in other manners. For stepwise and linearly variations, we have

$$\int_{(n-1)\Delta z}^{k\Delta z} X(z')dz' \cong$$

$$\begin{cases} X((n-1)\Delta z)(k-n+1)\Delta z, & \text{Stepwise Variation} \\ [(1-0.5(k-n+1))X((n-1)\Delta z) & \\ +0.5(k-n+1)X(n\Delta z)]\,(k-n+1)\Delta z, & \text{Linearly Variation} \end{cases} \quad (27)$$

4. EXAMPLE AND RESULTS

In this section, a special type of LIW is analyzed using the proposed method. Consider a WRG-90 waveguide ($a = 0.9\,\text{in}$ and $b = 0.4\,\text{in}$) filled by a lossless and exponential dielectric with the following electric permittivity function

$$\varepsilon_r(z) = \varepsilon_{r0}\exp(kz/d) \quad (28)$$

Now, assume that $\varepsilon_{r0} = 1 - j0$, $d = 2\,\text{cm}$ and $k = 1$. A TE_{10} mode wave with frequency $f = 10\,\text{GHz}$ and the electric field strength $E^i = 1.0\,\text{V/m}$ propagates in the assumed LIW. We use stepwise variation between two adjacent points. Figs. 2–3, compare the magnitude and the angle of the electric field obtained from the exact solutions (in the Appendix) and from the introduced method considering $N = 10$ and $N = 20$ partitions. One observes a good agreement between the exact solutions and the solutions obtained from the proposed method. Also, Figs. 4–5 shows the relative error of the obtained S_{11} and S_{21} parameters in four cases of ($k = 1$, $f = 10\,\text{GHz}$, $d = 2\,\text{cm}$), ($k = 0.5$, $f = 10\,\text{GHz}$, $d = 2\,\text{cm}$), ($k = 1$, $f = 8\,\text{GHz}$, $d = 2\,\text{cm}$) and ($k = 1$, $f = 10\,\text{GHz}$, $d = 1.8\,\text{cm}$). These two scattering parameters can be obtained from the following relations

$$S_{11} = \frac{E_y(0) - E^i}{E^i} \quad (29)$$

$$S_{21} = S_{12} = \frac{E_y(d)}{E^i} \quad (30)$$

It is seen as the number of partitions increases, the error is decreased. Also, as the excitation frequency, the length of dielectric or the

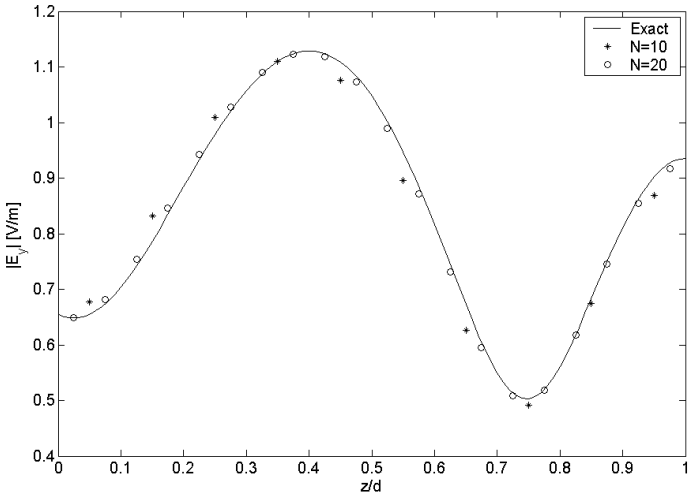

Figure 2. The magnitude of the electric field versus z for $k = 1$ and $f = 10\,\text{GHz}$.

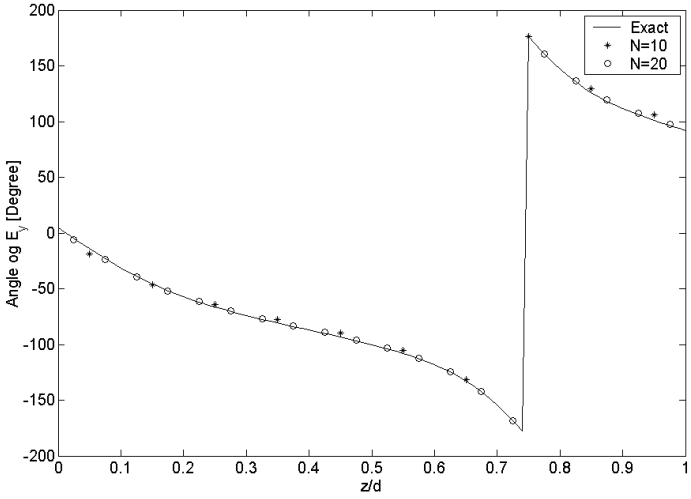

Figure 3. The angle of the electric field versus z for $k = 1$ and $f = 10\,\text{GHz}$.

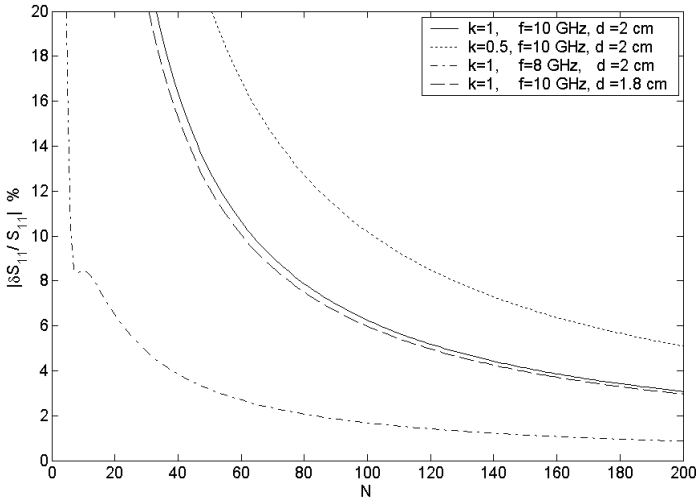

Figure 4. The relative error of the obtained S_{11} parameter in four cases.

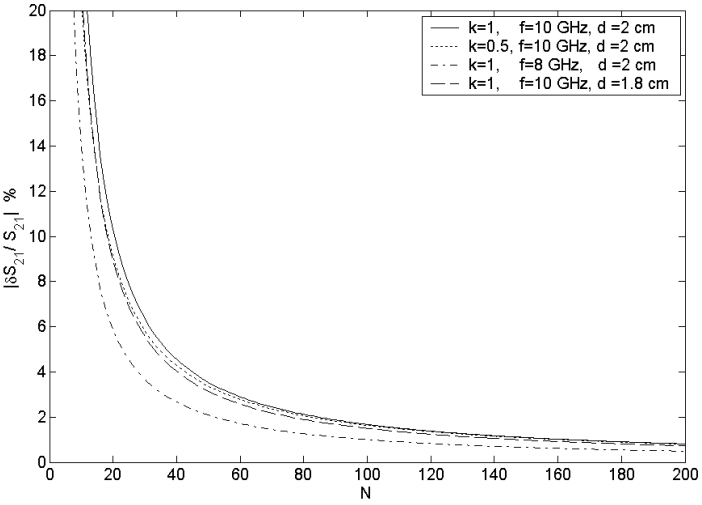

Figure 5. The relative error of the obtained S_{11} parameter in four cases.

variation of the permittivity function (k) decrease, the accuracy of the method is increased. From the above example, one may satisfy that the introduced method is applicable to all arbitrary LIWs, whose electric permittivity function is known at all or even only at some points along their length.

5. CONCLUSION

A new method was introduced for frequency domain analysis of arbitrary Longitudinally Inhomogeneous Waveguides (LIWs). In this method, the integral equations of the LIWs, converted from their differential equations, are solved using the method of moments (MOM). Rectangular pulse function expansion with point matching (Collocation) is explained for this purpose. The validity of the method was verified using a comprehensive example. It was seen that, as the number of partitions increases, the accuracy of the obtained solution is increased. Furthermore, as the excitation frequency, the length of dielectric and the variation of the permittivity function decrease, the accuracy of the method is increased. This method is applicable to all arbitrary LIWs, whose electric permittivity function is known at all or even at some points along their length.

APPENDIX A.

The exact electric field of the lossless exponential LIWs is determined. Using (28) in (3) and (4), the following second order differential equation is obtained.

$$\frac{d^2E_y(z)}{dz^2} + k_0^2\left(\varepsilon_{r0}\exp(kz/d) - (f_c/f)^2\right)E_y(z) = 0 \qquad \text{(A1)}$$

The solution of (A1) gets us the electric and magnetic fields as follows

$$\begin{aligned} E_y(z) &= A_1 J\left[\frac{2k_0d}{k}\frac{f_c}{f}, \frac{2k_0d\sqrt{\varepsilon_{r0}}}{k}\exp\left(k\frac{z}{2d}\right)\right] \\ &\quad + A_2 J\left[-\frac{2dk_0}{k}\frac{f_c}{f}, \frac{2k_0d\sqrt{\varepsilon_{r0}}}{k}\exp\left(k\frac{z}{2d}\right)\right] \qquad \text{(A2)} \\ H_x(z) &= \frac{-j\sqrt{\varepsilon_{r0}}}{\eta_0}\exp\left(k\frac{z}{2d}\right) \\ &\quad \times\left\{A_1 J'\left[\frac{2k_0d}{k}\frac{f_c}{f}, \frac{2k_0d\sqrt{\varepsilon_{r0}}}{k}\exp\left(k\frac{z}{2d}\right)\right]\right. \\ &\quad \left. + A_2 J'\left[-\frac{2k_0d}{k}\frac{f_c}{f}, \frac{2k_0d\sqrt{\varepsilon_{r0}}}{k}\exp\left(k\frac{z}{2d}\right)\right]\right\} \qquad \text{(A3)} \end{aligned}$$

where the function $J[\alpha,\beta]$ is the Bessel function J of order α and argument β and the primes indicate the first derivative of the function with respect to its argument. From (A2) and (A3) and the boundary conditions (6) and (7), we have

$$a_1 A_1 + a_2 A_2 = 2E^i \tag{A4}$$
$$a_3 A_1 + a_4 A_2 = 0 \tag{A5}$$

in which

$$a_1 = J\left[\frac{2k_0 d}{k}\frac{f_c}{f}, \frac{2k_0 d\sqrt{\varepsilon_{r0}}}{k}\right] + \frac{j\sqrt{\varepsilon_{r0}}}{\sqrt{1-(f_c/f)^2}} J'\left[\frac{2k_0 d}{k}\frac{f_c}{f}, \frac{2k_0 d\sqrt{\varepsilon_{r0}}}{k}\right] \tag{A6}$$

$$a_2 = J\left[\frac{-2k_0 d}{k}\frac{f_c}{f}, \frac{2k_0 d\sqrt{\varepsilon_{r0}}}{k}\right] + \frac{j\sqrt{\varepsilon_{r0}}}{\sqrt{1-(f_c/f)^2}} J'\left[\frac{-2k_0 d}{k}\frac{f_c}{f}, \frac{2k_0 d\sqrt{\varepsilon_{r0}}}{k}\right] \tag{A7}$$

$$a_3 = J\left[\frac{2k_0 d}{k}\frac{f_c}{f}, \frac{2k_0 d\sqrt{\varepsilon_{r0}}}{k}\exp(k/2)\right] - \frac{j\sqrt{\varepsilon_{r0}}}{\sqrt{1-(f_c/f)^2}} J'\left[\frac{2k_0 d}{k}\frac{f_c}{f}, \frac{2k_0 d\sqrt{\varepsilon_{r0}}}{k}\exp(k/2)\right]\exp(k/2) \tag{A8}$$

$$a_4 = J\left[\frac{-2k_0 d}{k}\frac{f_c}{f}, \frac{2k_0 d\sqrt{\varepsilon_{r0}}}{k}\exp(k/2)\right] - \frac{j\sqrt{\varepsilon_{r0}}}{\sqrt{1-(f_c/f)^2}} J'\left[\frac{-2k_0 d}{k}\frac{f_c}{f}, \frac{2k_0 d\sqrt{\varepsilon_{r0}}}{k}\exp(k/2)\right]\exp(k/2) \tag{A9}$$

The unknown coefficients A_1 and A_2 are determined using (A4) and (A5) as follows

$$A_1 = \frac{2a_4}{a_1 a_4 - a_2 a_3} E^i \tag{A10}$$
$$A_2 = \frac{-2a_3}{a_1 a_4 - a_2 a_3} E^i \tag{A11}$$

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