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ELECTROMAGNETIC SCATTERING APPROXIMATIONS REVISITED

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Abstract—Various electromagnetic scattering approximations beyond the Born assumption have been published during the recent years. This paper introduces a simple framework of analyses and investigates in a systematic way the fundamentals of the proposed theories. Our main focus is to demonstrate the link and similarities between the different scattering approximations employing a common physical basis. Based on analogies established we try to bridge the apparent gap between existing theories as well as introducing possible extensions and refinements.

1. INTRODUCTION

Analytical solutions of electromagnetic (EM) scattering problems exist for special cases [1, 2]. However, often numerical methods must be employed based on differential-equation [3] or integral-equation techniques. In this paper we concentrate on the latter approach. One solution strategy is to solve the exact problem based on high-performance algorithms like the Conjugate Gradient-Fast Fourier Transform (CG-FFT) method [4] and the stabilized Bi-conjugate Gradient FFT method (BCGS-FFT) [5]. Alternatively, one may apply other iterative algorithms like the multilevel fast multipole algorithm (MLFMA) and represent the Green's function by plane-wave or spectral representations [6]. Also the method of moments (MoM) can be employed, where the anomalous region is divided into subdomains with the electromagnetic parameters within each such subdomain approximated by so-called basis functions [7]. Another approach is to employ a combined global and local (GL) technique to model the

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electromagnetic field [8]. To handle mixed cases of both dielectric and conductive objects, hybrid techniques have been proposed [9]. However, in many cases the computational time can still be prohibitive high (for example in the case of using repeated modelling as part of inversion) and one must seek for efficient approximate solutions. Several approximations to the EM scattering problem have therefore been proposed in the past. Among these are the extended Born approximation (EBA) [10], the local non-linear (LN) approximation [10], the quasi-analytical (QA) approximation [11], the quasi-linear (QL) approximation [12] and the Diagonal Tensor Approximation (DTA) [13]. In order to handle more complex media including larger contrasts and possible anisotropy, also higher-order versions of these methods have been introduced. Such iterative methods can be based on the scattering equation of the iterative dissipative method (MIDM) [14, 15] like the modified Born (MB) approximation [16] or alternative formulations like the High-Order Generalized Extended Born Approximation (Ho-GEBA) [17]. The purpose of this paper is not to analyze the accuracy of these different scattering techniques for various practical settings. Such results are already available from the literature in a vast amount. However, there is still a need to demonstrate in a simple manner how these different scattering theories relate to each other. In this paper we therefore introduce a common framework of analyses based on the electric field equation to efficiently investigate the link and similarities between the various EM scattering theories. Based on analogies established we try to bridge the apparent gap between existing theories as well as proposing possible extensions and refinements. Most of the cited works above are concentrating on conductive anomalies, but resistive scatterers are of equal importance since during the recent years the use of low-frequency EM methods (e.g., Sea Bed Logging) to detect high-resistive hydrocarbon layers have evolved rapidly [18].

2. BASIC EQUATIONS AND PROBLEM DEFINITION

We consider the case of a 3-D electric scatterer embedded in a background model, and specialize to the conductive case (the dielectric case follows by analogy). In Fig. 1, a horizontally layered (1-D) background model is shown as an example. However, analytical dyadic Green's functions can be obtained for various backgrounds ranging from a homogeneous isotropic [19] and a 1-D isotropic [20] to an electrically gyrotropic medium [21]. After illuminating the anomalous region with a time-harmonic electric field, the total field measured at an arbitrary receiver location $\vec{r}$ is governed by the electric field equation

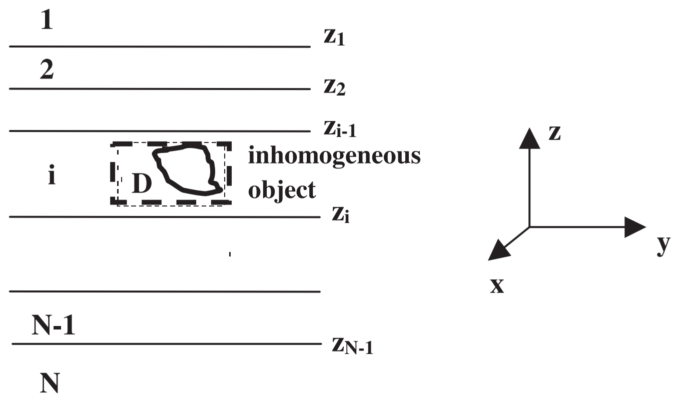

Figure 1. 3-D scatterer embedded in a layered model (vertical slice).

[19]

$$\bar{E}(\vec{r}) = \bar{E}_b(\vec{r}) + \int_D \overline{\overline{G_e}}(\vec{r}, \vec{r}_0) \cdot \Delta\overline{\overline{\sigma}}(\vec{r}_0) \cdot \bar{E}(\vec{r}_0) d\vec{r}_0 \tag{1}$$

In Eq. (1) $\bar{E}$ and $\bar{E}_b$ are respectively the total and background electric field vectors and $\overline{\overline{G}}_e$ is the dyadic electric Green's function. $\overline{\overline{G}}_e$ is represented by a 3×3 matrix with its three columns corresponding to a dipole oriented in x, y and z-directions, respectively. The second order tensor $\Delta\overline{\overline{\sigma}}$ represents the contrast function of the inhomogeneous object defined as

$$\Delta\overline{\overline{\sigma}} = \overline{\overline{\sigma}} - \sigma_b \overline{\overline{I}} \tag{2}$$

where σ_b is the background conductivity and $\overline{\overline{I}}$ is the unity dyad. Note that if the dielectric constants are not neglected as here, the quantities in Eq. (2) should be replaced by $\overline{\overline{\sigma}} \rightarrow \overline{\overline{\sigma}}' - i\omega\varepsilon_r\varepsilon_0\overline{\overline{I}}$ and $\sigma_b \rightarrow \sigma_b' - i\omega\varepsilon_{rb}\varepsilon_0$. The region of integration D in Eq. (1) is a region surrounding the anomalous object. In a practical numerical implementation this region will be subdivided into 3-D cells of a chosen shape. Since Eq. (1) is equally valid for measurement points outside as well as inside D, we consider first the problem of computing the total electric inside the anomalous region for all points $\vec{r} \in D$. In this paper we will only discuss approximate methods in order to speed up the computations. After having determined the electric field inside the anomalous region, the field response at any receiver position can be computed using Eq. (1).

Introduce now the integral operator $\mathbf{G_D}$ defined as:

$$\mathbf{G_D}(\bar{x}) = \int\limits_D \overline{\overline{G_e}}(\vec{r}, \vec{r}_0) \cdot \bar{x}(\vec{r}_0) d\vec{r}_0 \tag{3}$$

Combination of Eqs. (1) and (3) gives the compact form of the field equation:

$$\bar{E}(\vec{r}\,) = \bar{E}_b(\vec{r}\,) + \mathbf{G_D}(\Delta\overline{\overline{\sigma}} \cdot \bar{E}) \tag{4}$$

Finally, by introducing the scattered electric field vector: $\bar{E}_s(\vec{r}\,) = \bar{E}(\vec{r}\,) - \bar{E}_b(\vec{r}\,)$, Eq. (4) can be recast as

$$\bar{E}_s(\vec{r}\,) = \mathbf{G_D}(\Delta\overline{\overline{\sigma}} \cdot \bar{E}_b) + \mathbf{G_D}(\Delta\overline{\overline{\sigma}} \cdot \bar{E}_s) = \bar{E}_B(\vec{r}\,) + \mathbf{G_D}(\Delta\overline{\overline{\sigma}} \cdot \bar{E}_s) \tag{5}$$

where the first term on the right-hand side is identified as the (lowest-order) scattered Born field [22].

In the following we employ Eq. (5) as the physical basis when we discuss and analyze various possible EM scattering approximations. First we consider non-iterative methods and investigate possible approximations of the scattered field. Both source-independent and data-dependent techniques will be treated. In cases of more complex anomalies iterative techniques can be useful. In the last part of this paper such methods will be the topic.

3. NON-ITERATIVE SOLUTIONS

3.1. Source-independent Scattering Tensor

We start with the fundamental approximation

$$\mathbf{G_D}(\Delta\overline{\overline{\sigma}} \cdot \bar{E}_s) \cong \mathbf{G_D}(\Delta\overline{\overline{\sigma}}) \cdot \bar{E}_s \tag{6}$$

In Eq. (6) the scattered field is treated apparently as spatially invariant inside the scatterer (or anomalous region D). However, this assumption can be somewhat relaxed due to the known singular behaviour of the dyadic Green's function. Hence, fast decay of the amplitude of the dyadic Green's function allows for spatially varying scattered fields inside D. However, if the fields varies significantly the approximation in Eq. (6) will be more dubious.

By combining Eqs. (5) and (6) we arrive at

$$\left[\overline{\overline{I}} - \mathbf{G_D}(\Delta\overline{\overline{\sigma}})\right] \cdot \bar{E}_s(\vec{r}\,) = \bar{E}_B(\vec{r}\,) \tag{7}$$

or alternatively

$$\bar{E}_s(\vec{r}\,) = \overline{\overline{\Gamma}}_{QA} \cdot \bar{E}_B(\vec{r}\,) \tag{8}$$

where $\overline{\overline{\Gamma}}_{QA}$ is a scattering tensor defined explicitly as

$$\overline{\overline{\Gamma}}_{QA}(\vec{r}\,) = \left[\overline{\overline{I}} - \mathbf{G_D}(\Delta\overline{\overline{\sigma}})\right]^{-1} \tag{9}$$

which is seen to be **source-independent**. Since $\overline{\overline{\Gamma}}_{QA}$, in general, contains non-diagonal values possible cross-polarization is included (at least to a certain extent). This approximation is known as the Quasi-Analytical (QA) approximation [11].

We now revisit Eq. (8) and approximate it further:

$$\begin{aligned} \bar{E}_s(\vec{r}\,) &= \overline{\overline{\Gamma}}_{QA} \cdot \bar{E}_B(\vec{r}\,) = \overline{\overline{\Gamma}}_{QA} \cdot \mathbf{G_D}(\Delta\overline{\overline{\sigma}} \cdot \bar{E}_b) \\ &\cong \overline{\overline{\Gamma}}_{QA} \cdot \mathbf{G_D}(\Delta\overline{\overline{\sigma}}) \cdot \bar{E}_b(\vec{r}\,) \equiv \overline{\overline{\Gamma}}_{LN} \cdot \bar{E}_b(\vec{r}\,) \end{aligned} \tag{10}$$

which is known in the literature as the local non-linear (LN) approximation [10]. Alternatively, if the total field is considered this approximation is known as the Extended Born Approximation (EBA) [10]:

$$\begin{aligned} \bar{E}(\vec{r}\,) &= \bar{E}_b(\vec{r}\,) + \bar{E}_s(\vec{r}\,) \cong \bar{E}_b(\vec{r}\,) + \overline{\overline{\Gamma}}_{LN} \cdot \bar{E}_b(\vec{r}\,) \\ &= (\overline{\overline{I}} + \overline{\overline{\Gamma}}_{LN}) \cdot \bar{E}_b(\vec{r}\,) \equiv \overline{\overline{\Gamma}}_{EBA} \cdot \bar{E}_b(\vec{r}\,) \end{aligned} \tag{11}$$

with

$$\overline{\overline{\Gamma}}_{EBA} = (\overline{\overline{I}} + \overline{\overline{\Gamma}}_{LN}) = \left[\overline{\overline{I}} - \mathbf{G_D}(\Delta\overline{\overline{\sigma}})\right]^{-1} \equiv \overline{\overline{\Gamma}}_{QA} \tag{12}$$

From Eq. (10) it follows that LN also restricts the behaviour of the background field. Hence, LN (and consequently also EBA) will work poorer than QA in case of close sources. Also, since the scattered field is approximated based on the behaviour of the background field (cf. Eqs. (10) and (11)), LN and EBA will not handle EM coupling in an electrically anisotropic medium well.

If the scattering contrast is **small** we have in the limit $\mathbf{G_D}(\Delta\overline{\overline{\sigma}}) \to 0$ as $\Delta\overline{\overline{\sigma}} \to 0$. It follows now from Eqs. (10) and (11) (e.g., limit of LN and EBA):

$$\bar{E}(\vec{r}\,) = \bar{E}_b(\vec{r}\,), \quad \bar{E}_s(\vec{r}\,) = 0 \tag{13}$$

which is equivalent with the zero-order Born approximation [22]. Likewise, Eq. (8) gives now (limit of QA):

$$\bar{E}_s(\vec{r}\,) = \bar{E}_B(\vec{r}\,) \tag{14}$$

which is equivalent to a first-order Born approximation [22]. Eqs. (13) and (14) support earlier observations that QA is in general more accurate than EBA/LN.

Note that specializing to a **high-resistive** (e.g., low-conductive) anomalous region gives in the very limit that $\mathbf{G_D}(\Delta\overline{\overline{\sigma}}) \to -\sigma_b \mathbf{G_D}(\overline{\overline{I}})$. Hence, the various scattering tensors are now virtually not influenced by the conductivity of the actual scatterer.

3.2. Source Dependent and Diagonal Scattering Tensor

In this section special cases of the scattering tensor neglecting cross-polarization are considered. Since the scattering tensor now will be **diagonal**, transversal isotropic (TI) type of media can (at least to a certain extent) be handled. If the contrast function $\Delta\overline{\overline{\sigma}}$ has significant non-diagonal values, higher-order solutions should be considered.

Approximation # 1: QA-type

This derivation can be considered as a 'hybrid' QA-method since it also makes use of the same type of approximation as in Eq. (10), e.g., we start by assuming

$$\bar{E}_B(\vec{r}\,) = \mathbf{G_D}(\Delta\overline{\overline{\sigma}} \cdot \bar{E}_b) \cong \mathbf{G_D}(\Delta\overline{\overline{\sigma}}) \cdot \bar{E}_b(\vec{r}\,) \tag{15}$$

Adding the background field on both sides of Eq. (15) and reorganizing gives the revised equation:

$$\begin{aligned} \left(\overline{\overline{I}} - \mathbf{G_D}(\Delta\overline{\overline{\sigma}})\right) \cdot \bar{E}_b \;&=\; \bar{E}_b - \bar{E}_B \Rightarrow \overline{\overline{\Gamma}}_{QA} \cdot \left(\bar{E}_b - \bar{E}_B\right) \\ &=\; \bar{E}_b \Leftrightarrow [A_b - A_B]\,\bar{\xi}_{QA} = \bar{E}_b \end{aligned} \tag{16}$$

where the definition in Eq. (9) has been employed as well. Eq. (16) also assumes a diagonal scattering tensor $\overline{\overline{\Gamma}}_{QA}$ and introduces its alternative representation employing a column vector $\bar{\xi}_{QA}$. Finally, the matrixes A and A_b are given explicitly by

$$A_b = \begin{bmatrix} E_{b,x} & 0 & 0 \\ 0 & E_{b,y} & 0 \\ 0 & 0 & E_{b,z} \end{bmatrix}, \quad A_B = \begin{bmatrix} E_{B,x} & 0 & 0 \\ 0 & E_{B,y} & 0 \\ 0 & 0 & E_{B,z} \end{bmatrix} \tag{17}$$

In Eq. (17) $E_{b,i}$ and $E_{B,i}$ $(i = x, y, z)$ represent the three components of the background field and the scattered Born field, respectively, measured along the Cartesian model axes. Based on Eq. (16), the scattered field is now approximated as:

$$\begin{aligned}\bar{E}_s(\vec{r}) &\cong \overline{\overline{\Gamma}}_{QA} \cdot \bar{E}_B(\vec{r}) = A_B \cdot \bar{\xi}_{QA} = A_B \left[A_b - A_B\right]^{-1} \cdot \bar{E}_b(\vec{r}) \\ &\equiv A_b[A_b - A_B]^{-1} \cdot \bar{E}_B(\vec{r}) \end{aligned} \tag{18}$$

which can be considered as a special case of the QA-approximation.

For the **source-independent scattering tensor** case, we have already pointed out that QA is a more accurate approximation than LN/EBA. However, unlike LN/EBA the QA method also needs the additional computation of the scattered Born field inside the scatterer (cf. Eq. (18)). Since the **data-dependent scattering tensor** based on the diagonal approximation in Eq. (16) also includes computation of the scattered Born field, QA should always be preferred to LN/EBA in this case (diagonal versions of the scattering tensors $\overline{\overline{\Gamma}}_{LN}$ and $\overline{\overline{\Gamma}}_{EBA}$ are closely related to $\overline{\overline{\Gamma}}_{QA}$ as follows from Eq. (12)).

Approximation # 2: QL-type

Alternatively, one may employ a more heuristic approach to establish a diagonal and data-dependent scattering tensor. By analogy with the LN-result in Eq. (10) postulate now

$$\bar{E}_s(\vec{r}) = \overline{\overline{\chi}}_{QL} \cdot \bar{E}_b(\vec{r}) = A_b \cdot \bar{\xi}_{QL} \tag{19}$$

where $\overline{\overline{\chi}}_{QL}$ is a **diagonal** scattering tensor (also called reflectivity tensor in the literature) yet to be determined. In Eq. (19) we have also replaced the scattering tensor by its alternative column vector form (matrix A_b as defined in Eq. (17)). Combining Eqs. (5) and (19) gives:

$$A_b \cdot \bar{\xi}_{QL} = \mathbf{G_D}(\Delta\overline{\overline{\sigma}} \cdot A_b \cdot \bar{\xi}_{QL}) + \bar{E}_B \cong \mathbf{G_D}(\Delta\overline{\overline{\sigma}} \cdot A_b) \cdot \bar{\xi}_{QL} + \bar{E}_B \tag{20}$$

where a slow variation of $\bar{\xi}_{QL}$inside the anomalous region D has been assumed. Its actual values are determined as those minimizing the following norm [12]:

$$\left\|B_b \cdot \bar{\xi}_{QL} - \bar{E}_B\right\| = \min \tag{21}$$

with

$$B_b(\vec{r}) = A_b(\vec{r}) - \mathbf{G_D}(\Delta\overline{\overline{\sigma}} \cdot A_b) \tag{22}$$

By considering a grid of receiver points $\vec{r}_j$, $j = 1, 2, \ldots J$ representing a smaller or larger subdomain inside D, solving Eq. (21) is equivalent to finding a least-squares (LS) solution of the over-determined system:

$$M_{QL} \cdot \bar{\xi}_{QL} = \bar{d}_{QL} \tag{23}$$

where

$$M_{QL} = \begin{bmatrix} B_{b,1} \\ B_{b,2} \\ \cdot \\ \cdot \\ B_{b,J} \end{bmatrix}, \quad \bar{d}_{QL} = \begin{bmatrix} \bar{E}_{B,1} \\ \bar{E}_{B,2} \\ \cdot \\ \cdot \\ \bar{E}_{B,J} \end{bmatrix} \tag{24}$$

and $B_{b,j} \equiv B_b(\vec{r} = \vec{r}_j)$, $\bar{E}_{B,j} \equiv \bar{E}_B(\vec{r} = \vec{r}_j)$.

The LS-solution implies a source-dependent scattering tensor on the form

$$\bar{\xi}_{QL} = \left[M^{*T}_{QL} \cdot M_{QL}\right]^{-1} \cdot M^{*T}_{QL} \cdot \bar{d}_{QL} \tag{25}$$

or alternatively

$$\bar{\xi}_{QL} = \left[\sum_{j=1}^{J} B^{*T}_{b,j} \cdot B_{b,j}\right]^{-1} \cdot \left[\sum_{j=1}^{J} B^{*T}_{b,j} \cdot \bar{E}_{B,j}\right] \tag{26}$$

In Eqs. (25) and (26) the star means taking the complex conjugate and Tmeans the transpose.

Combination of Eqs. (19) and (26) gives now the scattered field. This type of approximation is known as the Quasi-Linear (QL) approximation [12]. Unlike QA, no approximations about the background field are introduced (besides the common assumption of no cross-polarization) and the QL-technique is therefore more accurate. However, the computational effort is also larger as can be seen directly from Eq. (26). In the original work [12] the QL-method was introduced as a (least-squares) minimum-norm type of solution which can be applied to a general model. However, in more general cases involving non-diagonal scattering tensors, the computational burden will be significant. In this paper we have therefore chosen to limit our discussion to the most computational attractive versions of QL.

In the QL-method the scattered field is approximated based on the behaviour of the background field as shown in Eq. (19). In this paper we propose a possible further improvement in case of close sources and/or anisotropy by replacing Eq. (19) with (by analogy with Eq. (8)):

$$\bar{E}_s(\vec{r}\,) = \bar{\bar{\chi}}'_{QL} \cdot \bar{E}_B(\vec{r}\,) = A_B \cdot \bar{\xi}'_{QL} \tag{27}$$

and again solve for the reflectivity tensor by assuming diagonal form and a least-squares solution. It is straightforward to show that the solution now reads:

$$\bar{\xi}'_{QL} = \left[\sum_{j=1}^{J} B_{B,j}^{*T} \cdot B_{B,j}\right]^{-1} \cdot \left[\sum_{j=1}^{J} B_{B,j}^{*T} \cdot \bar{E}_{B,j}\right] \tag{28}$$

with the matrix B_B defined as:

$$B_B(\vec{r}) = A_B(\vec{r}) - \mathbf{G_D}(\Delta\overline{\overline{\sigma}} \cdot A_B) \tag{29}$$

This modification represents the same type of computational burden since the scattered Born field $\bar{E}_B$ has to be computed in the original QL-method as well (cf. Eq. (26)).

In cases where one component of the background field is zero inside the anomalous region, this implies a zero value of the corresponding scattered Born field component (cf. Eq. (15)). Consequently, approximating the scattered field employing Eqs. (19) or (27) also implies a **zero value of the corresponding scattered field component.** Such a limitation imposed on the scattered field can give inaccurate estimates for complex models. To handle such cases one can make use of similar ideas as employed within well-log modelling and assume that the scattered field is linear proportional to the absolute value of the background field [23]. Hence, Eq. (19) is replaced by:

$$\bar{E}_s(\vec{r}) = \overline{\overline{\chi}}''_{QL} \cdot \left|\bar{E}_b(\vec{r})\right| = A \cdot \bar{\xi}''_{QL} \tag{30}$$

with the matrix A defined as:

$$A = \begin{bmatrix} |E_b| & 0 & 0 \\ 0 & |E_b| & 0 \\ 0 & 0 & |E_b| \end{bmatrix} = |E_b| \overline{\overline{I}} \tag{31}$$

and $\left|\bar{E}_b\right|$ being the magnitude of the background field.

It is straightforward to show that the solution now reads:

$$\bar{\xi}''_{QL} = \left[\sum_{j=1}^{J} B_j^{*T} \cdot B_j\right]^{-1} \cdot \left[\sum_{j=1}^{J} B_j^{*T} \cdot \bar{E}_{B,j}\right] \tag{32}$$

where the matrix B takes the form:

$$B(\vec{r}) = |E_b(\vec{r})| \overline{\overline{I}} - \mathbf{G_D}(\Delta\overline{\overline{\sigma}} \cdot |E_b(\vec{r})| \overline{\overline{I}}) \tag{33}$$

Approximation # 3: DTA-type

In this paragraph we will consider an approximation technique which is closely related to the QL-method, but avoids the least-squares formulation. Hence, this time we assume that Eq. (20) is 'exact', i.e., we write (renaming the reflectivity tensor and making use of Eq. (22)):

$$A_b \cdot \bar{\xi}_{DTA} = \mathbf{G_D}(\Delta\overline{\overline{\sigma}} \cdot A_b) \cdot \bar{\xi}_{DTA} + \bar{E}_B \Rightarrow B_b \cdot \bar{\xi}_{DTA} = \bar{E}_B \tag{34}$$

which gives the following expression for the scattered field assuming direct solution (by analogy with Eq. (19)):

$$\bar{E}_s = A_b \cdot \bar{\xi}_{DTA} = A_b B_b^{-1} \cdot \bar{E}_B \tag{35}$$

Equation (35) is known in the literature as the Diagonal Tensor Approximation (DTA) [13]. Alternatively, if Eq. (34) had been solved employing normal-equation form (pseudo-inverse) we had obtained by analogy with QL:

$$\bar{\xi}_{DTA,LS} = \left[B_b^{*T} B_b\right]^{-1} \left[B_b^{*T} \bar{E}_B\right] \tag{36}$$

which is to be compared with Eq. (26).

Note that DTA is closely related to QA, but has the potential of being slightly more accurate. This can be demonstrated as follows. Assuming a **diagonal** dyadic Greens function as well as contrast function implies that

$$\mathbf{G_D}(\Delta\overline{\overline{\sigma}} A_b) = A_B \tag{37}$$

which when combined with Eq. (34) and compared with Eq. (18) gives

$$(\bar{E}_s =) A_b \cdot \bar{\xi}_{DTA} = A_B \cdot \bar{\xi}_{QA} \tag{38a}$$

since matrices A_b and A_B are diagonal and

$$A_B \left[A_b - A_B\right]^{-1} \cdot \bar{E}_b(\vec{r}\,) \equiv A_b \left[A_b - A_B\right]^{-1} \cdot \bar{E}_B(\vec{r}\,) \tag{38b}$$

Hence, QA and DTA give now **identical** results for the scattered field approximation.

By analogy with the extended QL-case (cf. Eq. (27)) we propose here a possible further improvement for close sources and/or anisotropy by assuming

$$\bar{E}_s(\vec{r}\,) = \overline{\overline{\chi}}'_{DTA} \cdot \bar{E}_B(\vec{r}\,) = A_B \cdot \bar{\xi}'_{DTA} \tag{39}$$

and solve for the scattering tensor again assuming diagonal form. The solution can now be explicitly written as:

$$\bar{\xi}'_{DTA} = (B_B)^{-1} \cdot \bar{E}_B \tag{40}$$

with the matrix B_B defined in Eq. (29). There is virtually no increase in the computational effort since the scattered Born field $\bar{E}_B$ must be determined also in the original version of DTA.

For completeness we also discuss the case of avoiding non-physical scattering fields in case of zero-valued components of the background field. Hence, by analogy with Eq. (30) we postulate

$$\bar{E}_s(\vec{r}\,) = \overline{\overline{\chi}}''_{DTA} \cdot \left|\bar{E}_b(\vec{r}\,)\right| = A \cdot \bar{\xi}''_{DTA} \tag{41}$$

It is again straightforward to show that the solution now reads:

$$\bar{\xi}''_{DTA} = (B)^{-1} \cdot \bar{E}_B \tag{42}$$

with the matrix B defined by Eq. (33).

If we assume **slowly varying magnitude** of the background field, the matrix B can be approximated as (also making use of Eq. (9))

$$\begin{aligned} B(\vec{r}\,) &= |E_b(\vec{r}\,)|\,\overline{\overline{I}} - \mathbf{G_D}(\Delta\overline{\overline{\sigma}} \cdot |E_b(\vec{r}\,)|\,\overline{\overline{I}}) \cong |E_b(\vec{r}\,)| \left[\overline{\overline{I}} - \mathbf{G_D}(\Delta\overline{\overline{\sigma}})\right] \\ &\equiv |E_b(\vec{r}\,)|\,\overline{\overline{\Gamma}}^{-1}_{QA} \end{aligned} \tag{43}$$

Finally, by combining Eqs. (41)–(43) we obtain

$$\bar{E}_s = A \cdot \bar{\xi}''_{DTA} \cong \frac{1}{|E_b(\vec{r}\,)|} A \overline{\overline{\Gamma}}_{QA} \cdot \bar{E}_B = \frac{1}{|E_b(\vec{r}\,)|} A \cdot \bar{E}_{s,QA} = \bar{E}_{s,QA} \tag{44}$$

where $\bar{E}_{s,QA}$ represents the scattered-field approximation employing the (source-independent) QA-method as given by Eq. (8). Hence, Eq. (44) represents a link between the source-independent and source-dependent scattering tensor theories. We can also obtain similar insight by assuming a **slowly varying background field** in Eq. (34):

$$\begin{aligned} &A_b \cdot \bar{\xi}_{DTA} = \mathbf{G_D}(\Delta\overline{\overline{\sigma}} \cdot A_b) \cdot \bar{\xi}_{DTA} + \bar{E}_B \cong \mathbf{G_D}(\Delta\overline{\overline{\sigma}}) \cdot A_b \cdot \bar{\xi}_{DTA} + \bar{E}_B \Rightarrow \\ &\overline{\overline{\Gamma}}^{-1}_{QA} \cdot A_b \cdot \bar{\xi}_{DTA} = \bar{E}_B \end{aligned} \tag{45}$$

By analogy with Eq. (35) the scattered field is now approximated as:

$$\bar{E}_s = A_b \cdot \bar{\xi}_{DTA} = A_b \cdot A_b^{-1} \cdot \overline{\overline{\Gamma}}_{QA} \cdot \bar{E}_B = \bar{E}_{s,QA} \tag{46}$$

in correspondence with Eq. (44). Eqs. (44) and (46) do show that a source-independent and a source-dependent scattering theory meet when the background-field (e.g., source-field) do not vary (or slowly varies), which is a reasonable result indeed.

3.3. Source-dependent Scattering Factor

In this paragraph we simplify the source-dependent scattering tensor further and replace it with a scalar factor. Such a case corresponds to both an isotropic and fairly weak scatterer with no severe EM coupling. We consider this limiting case for all the three main approximations discussed in the previous section.

Approximation # 1: QA-type

The limiting case of Eq. (15) will be

$$\bar{E}_B(\vec{r}\,) \cong \varsigma(\vec{r}\,)\bar{E}_b(\vec{r}\,) \tag{47}$$

According to Eq. (47) a reasonable expression for the scalar scattering factor will be

$$\varsigma(\vec{r}\,) = \frac{\bar{E}_B \cdot \bar{E}_b^*}{\bar{E}_b \cdot \bar{E}_b^*} \tag{48}$$

By analogy with Eqs. (8) and (9) we now have

$$\bar{E}_s(\vec{r}\,) = \frac{1}{1-\varsigma(\vec{r}\,)}\bar{E}_B(\vec{r}\,) \equiv \lambda_{QA}(\vec{r}\,)\bar{E}_B(\vec{r}\,) \tag{49}$$

where

$$\lambda_{QA}(\vec{r}\,) = \frac{\bar{E}_b \cdot \bar{E}_b^*}{\bar{E}_b \cdot \bar{E}_b^* - \bar{E}_B \cdot \bar{E}_b^*} = \frac{\bar{E}_b \cdot \bar{E}_b^*}{\Delta\bar{E} \cdot \bar{E}_b^*} = \frac{\Delta\bar{E}^* \cdot \bar{E}_b}{\Delta\bar{E}^* \cdot \Delta\bar{E}},\ \Delta\bar{E} = \bar{E}_b - \bar{E}_B \tag{50}$$

Since Eq. (50) also includes computation of the scattered Born field, QA should always be preferred to EBA/LN in this case (same arguments as for the diagonal-tensor case).

Approximation # 2: QL-type

The limiting case of Eq. (20) can be written as (assuming slow variation of scattering factor)

$$\lambda_{QL}(\vec{r}\,) \cdot \bar{E}_b(\vec{r}\,) \cong \mathbf{G_D}(\Delta\overline{\overline{\sigma}} \cdot \bar{E}_b) \cdot \lambda_{QL} + \bar{E}_B(\vec{r}\,) = \bar{E}_B(\vec{r}\,) \cdot \lambda_{QL} + \bar{E}_B(\vec{r}\,) \tag{51}$$

Since the scattering tensor is approximated by a scalar (implying an isotropic and weak scatterer), the contrast function $\Delta\overline{\overline{\sigma}}$ should also be close to a scalar by consistency.

By considering a grid of receiver points $\vec{r}_j,\ j = 1, 2, \ldots J$ representing a (smaller or larger) **subdomain** inside D, we can determine λ_{QL} as the linearized fit in a least-squares sense of the 'observations':

$$\bar{E}_B(\vec{r}_j) = \lambda_{QL} \cdot \left(\bar{E}_b(\vec{r}_j) - \bar{E}_B(\vec{r}_j)\right) = \lambda_{QL} \cdot \Delta\bar{E}(\vec{r}_j) \tag{52}$$

with the solution [12]

$$\lambda_{QL} = \frac{\sum_{j=1}^{J} \Delta\bar{E}^*(\vec{r}_j) \cdot \bar{E}_B(\vec{r}_j)}{\sum_{j=1}^{J} \Delta\bar{E}^*(\vec{r}_j) \cdot \Delta\bar{E}(\vec{r}_j)} \tag{53}$$

Based on the solution of Eq. (53) the scattered field inside the anomalous region is approximated as

$$\bar{E}_s(\vec{r}) = \lambda_{QL}(\vec{r})\bar{E}_b(\vec{r}) \tag{54}$$

By analogy with earlier discussions a possible further improvement of the QL-approximation in case of close sources can be to alternatively assume

$$\bar{E}_s(\vec{r}) = \lambda'_{QL}(\vec{r})\bar{E}_B(\vec{r}) \tag{55}$$

which corresponds to the LS-solution

$$\lambda'_{QL} = \frac{\sum_{j=1}^{J} \Delta\bar{E}_n^*(\vec{r}_j) \cdot \bar{E}_B(\vec{r}_j)}{\sum_{j=1}^{J} \Delta\bar{E}_n^*(\vec{r}_j) \cdot \Delta\bar{E}_n(\vec{r}_j)} \tag{56}$$

where the quantity $\Delta\bar{E}_n$ is given by the expression:

$$\Delta\bar{E}_n(\vec{r}) = \bar{E}_B(\vec{r}) - \mathbf{G_D}(\Delta\bar{\bar{\sigma}} \cdot \bar{E}_B) = \bar{E}_B(\vec{r}) - \bar{E}'_B(\vec{r}) \tag{57}$$

and E'_B is the scattered field corresponding to the second-order Born approximation [22].

Approximation # 3: DTA-type

In order to make the analysis as complete as possible we also investigate here the limiting case of DTA. Hence, we assume that Eq. (34) is 'exact' (renaming the scattering tensor):

$$\lambda_{DTA}(\vec{r}) \cdot \bar{E}_b(\vec{r}) = \bar{E}_B(\vec{r}) \cdot \lambda_{DTA} + \bar{E}_B(\vec{r}) \tag{58}$$

which has the solution

$$\lambda_{DTA}(\vec{r}\,) = \frac{\bar{E}_B(\vec{r}\,)}{\bar{E}_b(\vec{r}\,) - \bar{E}_B(\vec{r}\,)} = \frac{\bar{E}_B(\vec{r}\,)}{\Delta\bar{E}(\vec{r}\,)} = \frac{\Delta\bar{E}^*(\vec{r}\,)\cdot\bar{E}_B(\vec{r}\,)}{\Delta\bar{E}^*(\vec{r}\,)\cdot\Delta\bar{E}(\vec{r}\,)} \tag{59}$$

and when compared with Eq. (53) shows that unlike the QL-approach DTA involves no averaging. Also by comparing Eqs. (50) and (59) we see that DTA and QA now **give exactly the same result**, i.e.,

$$(\bar{E}_s =)\lambda_{DTA}\bar{E}_b = \lambda_{QA}\bar{E}_B \tag{60}$$

For completeness and by analogy with Eq. (55) we look for a possible improvement (especially for close sources) by postulating

$$\bar{E}_s(\vec{r}\,) = \lambda'_{DTA}(\vec{r}\,)\bar{E}_B(\vec{r}\,) \tag{61}$$

which (by analogy with Eq. (56)) gives the solution

$$\lambda'_{DTA}(\vec{r}\,) = \frac{\Delta\bar{E}_n^*(\vec{r}\,)\cdot\bar{E}_B(\vec{r}\,)}{\Delta\bar{E}_n^*(\vec{r}\,)\cdot\Delta\bar{E}_n(\vec{r}\,)} \tag{62}$$

with the quantity $\Delta\bar{E}_n$ defined in Eq. (57) and interpreted as the residual Born field (e.g., difference between the first-order and second-order Born approximation of the scattered field).

4. ITERATIVE SOLUTIONS

4.1. MIDM-equation

For a medium with significant conductivity contrasts as well as strong anisotropy higher-order schemes can be useful. The value of the lowest-order term (initial value) can then be determined employing the solutions described in the previous section (e.g., non-iterative solutions). As discussed by many authors [14–16], constructing iterative versions of Eq. (5) will not in general ensure convergence. Especially for cases involving larger conductivity contrasts, the operator $\mathbf{G_D}[.]$ is no longer stable. The idea is then to introduce a so-called contrastive operator according to the definition:

$$\mathbf{G_D^c}(\bar{x}) = \sqrt{\sigma_b}\,\mathbf{G_D}(2\sqrt{\sigma_b}\bar{x}) + \bar{x} \tag{63}$$

which is characterized by

$$\|\mathbf{G_D^c}(\bar{x})\| \le \|\bar{x}\| \tag{64}$$

Combination of Eqs. (5) and (63) gives the scattering equation of the iterative dissipative method (MIDM) [14, 15]

$$\overline{\overline{\alpha}} \cdot \bar{E}_s(\vec{r}\,) = \mathbf{G_D^c}(\overline{\overline{\beta}} \cdot \overline{\overline{\alpha}} \cdot \bar{E}_s) + \sqrt{\sigma_b}\bar{E}_B(\vec{r}\,) = \mathbf{O_D^c}(\overline{\overline{\alpha}} \cdot \bar{E}_s) + \sqrt{\sigma_b}\bar{E}_B(\vec{r}\,) \tag{65}$$

where

$$\overline{\overline{\alpha}} = \frac{\Delta\overline{\overline{\sigma}} + 2\sigma_b}{2\sqrt{\sigma_b}}, \quad \overline{\overline{\beta}} = \frac{\Delta\overline{\overline{\sigma}}}{2\sigma_b\overline{\overline{I}} + \Delta\overline{\overline{\sigma}}} \tag{66}$$

In Eq. (65) the operator $\mathbf{O_D^c}$is also well behaved since it follows directly from Eq. (66) that we always have $\left\|\overline{\overline{\beta}}\right\| < 1$. Based on the method of successive iterations (von Neumann series) the iterative version of Eq. (65) can be constructed as [14, 15]:

$$\begin{aligned}\overline{\overline{\alpha}}\cdot\bar{E}_s^{(n)}(\vec{r}\,) \;=\; & \mathbf{O_D^c}\left(\overline{\overline{\alpha}}\cdot\bar{E}_s^{(n-1)}\right)+\sqrt{\sigma_b}\bar{E}_B(\vec{r}\,)=\mathbf{G_D^c}\left(\overline{\overline{\beta}}\cdot\overline{\overline{\alpha}}\cdot\bar{E}_s^{(n-1)}\right)\\ & +\sqrt{\sigma_b}\bar{E}_B(\vec{r}\,), n = 1, 2, 3\ldots\end{aligned} \tag{67}$$

When Eq. (67) is applied in the context discussed in this paper, only **few iterations** are considered. (Note however that Eq. (67), even though being convergent, does not represent the most efficient solution to the exact problem and Krylov subspace iteration methods should then be employed [4, 5, 24]). Different higher-order approximations employing Eq. (67) are obtained depending on the choice of the initial value:

- Higher-order Modified Born (MB) [16]:

$$\bar{E}_s^{(0)} = 0 \tag{68a}$$

- Higher-order QA:

$$\bar{E}_s^{(0)}(\vec{r}\,) = \overline{\overline{\Gamma}}_{QA} \cdot \bar{E}_B(\vec{r}\,) \tag{68b}$$

- Higher-order EBA:

$$\bar{E}_s^{(0)}(\vec{r}\,) = \overline{\overline{\Gamma}}_{EBA} \cdot \bar{E}_b(\vec{r}\,) \tag{68c}$$

- Higher-order DTA:

$$\bar{E}_s^{(0)}(\vec{r}\,) = \overline{\overline{\chi}}_{DTA} \cdot \bar{E}_b(\vec{r}\,) \tag{68d}$$

- Higher-order QL:

$$\bar{E}_s^{(0)}(\vec{r}\,) = \overline{\overline{\chi}}_{QL} \cdot \bar{E}_b(\vec{r}\,) \tag{68e}$$

Note that in case of QA and EBA both source-independent and data-dependent (e.g., diagonal) versions of the scattering tensor may be employed. To further enhance the accuracy of DTA and QL (and possible reduce number of iterations), one may use the extensions proposed in this paper.

Combination of Eqs. (63) and (67) gives the alternative writing of the iterative MIDM-equation:

$$\begin{aligned}&\overline{\overline{\alpha}} \cdot \bar{E}_s^{(n)}(\vec{r}\,) = \overline{\overline{\alpha}} \cdot \bar{E}_s^{(n-1)}(\vec{r}\,)\\ &+\sqrt{\sigma_b}\left[\mathbf{G_D}\left(\Delta\overline{\overline{\sigma}} \cdot \bar{E}_s^{(n-1)}\right)+\bar{E}_B(\vec{r}\,) - \bar{E}_s^{(n-1)}(\vec{r}\,)\right],\ n = 1, 2, 3\ldots\end{aligned} \tag{69}$$

where all terms inside the brackets will cancel in case of an exact solution (follows directly from Eq. (5)).

4.2. High-order Hybrid Scheme

When employing Eq. (69) to compute approximate scattered field solutions, the idea is that only few iterations should be used. However, for complex models several iterations can be needed to arrive at a good convergence. The question is then if Eq. (69) can be modified in a way so that faster convergence is obtained in case of few iterations. This alternative approach should still honour the physics of the problem. We split now the operator $\mathbf{G_D}$ in a 'singular' and a 'non-singular' part and rewrite Eq. (5) as follows:

$$\bar{E}_s(\vec{r}\,) = \bar{E}_B(\vec{r}\,) + \mathbf{G_D}(\Delta\overline{\overline{\sigma}} \cdot \bar{E}_s) - \mathbf{G_{D_s}}(\Delta\overline{\overline{\sigma}} \cdot \bar{E}_s) + \mathbf{G_{D_s}}(\Delta\overline{\overline{\sigma}} \cdot \bar{E}_s) \tag{70}$$

with D_s representing a sub domain of the total volume D including the singularity in the dyadic Green's function when $\vec{r} \to \vec{r}_0$. This subdomain can be chosen very locally and consequently represent a rather small computational effort. Assume now that the electric field is stationary within this sub domain, so that Eq. (70) can be approximated as follows

$$\begin{aligned}&\left[\overline{\overline{I}} - \mathbf{G_{D_s}}(\Delta\overline{\overline{\sigma}})\right] \cdot \bar{E}_s(\vec{r}\,) \cong \bar{E}_B(\vec{r}\,) + \mathbf{G_D}(\Delta\overline{\overline{\sigma}} \cdot \bar{E}_s) - \mathbf{G_{D_s}}(\Delta\overline{\overline{\sigma}}) \cdot \bar{E}_s(\vec{r}\,) =\\ &\left[\overline{\overline{I}} - \mathbf{G_{D_s}}(\Delta\overline{\overline{\sigma}})\right] \cdot \bar{E}_s(\vec{r}\,) + \left[\mathbf{G_D}(\Delta\overline{\overline{\sigma}} \cdot \bar{E}_s) + \bar{E}_B(\vec{r}\,) - \bar{E}_s(\vec{r}\,)\right]\end{aligned} \tag{71}$$

and further rearranged as:

$$\overline{\overline{\vartheta}} \cdot \bar{E}_s = \overline{\overline{\vartheta}} \cdot \bar{E}_s + \left[\mathbf{G_D}(\Delta\overline{\overline{\sigma}} \cdot \bar{E}_s) + \bar{E}_B - \bar{E}_s\right] \tag{72}$$

with

$$\overline{\overline{\vartheta}} = \left[\overline{\overline{I}} - \mathbf{G_{D_s}}(\Delta\overline{\overline{\sigma}})\right] \tag{73}$$

For a homogeneous and isotropic medium, $\overline{\overline{\vartheta}}$ can be efficiently computed employing analytical techniques [25]. It follows from Eqs. (69) and(72) that $\overline{\overline{\vartheta}}$ can be regarded as the counterpart of $\overline{\overline{\alpha}}/\sqrt{\sigma_b}$. Representing D_s by a small sphere [25] gives $G_{D_s}(\Delta\overline{\overline{\sigma}}) \cong -\Delta\overline{\overline{\sigma}}/3\sigma_b$, and consequently $\|\overline{\overline{\vartheta}}\| < \|\overline{\overline{\alpha}}/\sqrt{\sigma_b}\|$ for $\Delta\overline{\overline{\sigma}} > 0$. Applying the method of successive iterations to Eq. (72) does not guarantee any convergence. However, a hybrid scheme can be constructed as

$$\bar{E}_s = \bar{E}_s^{(m)} + \overline{\overline{\vartheta}}^{-1} \cdot \left[\mathbf{G_D}(\Delta\overline{\overline{\sigma}} \cdot \bar{E}_s^{(m)}) + \bar{E}_B - \bar{E}_s^{(m)}\right], \quad m = 0, 1, \ldots \tag{74}$$

For $m = 0$, one of the solutions given by Eqs. (68a)–(68e) are chosen for $\bar{E}_s^{(0)}$ on the right-hand side of Eq. (74), and for higher order ($m > 1$) $\bar{E}_s^{(m)}$ is computed from (69) ensuring convergence.

Alternatively, one can consider the total field instead of the scattered field, and obtain the corresponding counterpart of Eq. (74):

$$\bar{E} = \bar{E}^{(m)} + \overline{\overline{\vartheta}}^{-1} \cdot \left[\mathbf{G_D}(\Delta\overline{\overline{\sigma}} \cdot \bar{E}^{(m)}) + \bar{E}_b - \bar{E}^{(m)}\right], \quad m = 0, 1, \ldots \tag{75}$$

A similar scheme to that in Eq. (75) has been introduced by Gao and Torres-Verdin [17] and is denoted High-Order Generalized Born Approximation (Ho-GEBA).

If we let $D_s \rightarrow D$, then it follows from Eq. (74) (making use of Eq. (9)):

$$\bar{E}_s = \bar{E}_s^{(m)} + \overline{\overline{\Gamma}}_{QA} \cdot \left[\mathbf{G_D}(\Delta\overline{\overline{\sigma}} \cdot \bar{E}_s^{(m)}) + \bar{E}_B - \bar{E}_s^{(m)}\right], \quad m = 0, 1, \ldots \tag{76}$$

which represents another alternative scheme. But the scattering tensor $\overline{\overline{\Gamma}}_{QA}$ is not as computationally attractive as $\overline{\overline{\vartheta}}$. In the QA-limit (cf.

Eq. (6)), Eq. (76) gives us a link between iterative and non-iterative theories, e.g.,

$$\begin{aligned}\bar{E}_s &= \bar{E}_s^{(m)} + \overline{\overline{\Gamma}}_{QA} \cdot \left[\mathbf{G_D}(\Delta\overline{\overline{\sigma}} \cdot \bar{E}_s^{(m)}) + \bar{E}_B - \bar{E}_s^{(m)}\right] \\ &\cong \bar{E}_s^{(m)} + \overline{\overline{\Gamma}}_{QA} \cdot \left[\mathbf{G_D}(\Delta\overline{\overline{\sigma}}) \cdot \bar{E}_s^{(m)} + \bar{E}_B - \bar{E}_s^{(m)}\right] = \\ &\bar{E}_s^{(m)} + \overline{\overline{\Gamma}}_{QA} \cdot \left[-\overline{\overline{\Gamma}}_{QA}^{-1} \cdot \bar{E}_s^{(m)} + \bar{E}_B\right] = \overline{\overline{\Gamma}}_{QA} \cdot \bar{E}_B \end{aligned} \tag{77}$$

as expected.

5. CONCLUDING REMARKS

By introducing a common framework of analyses, we have been able to reveal similarities and links between various proposed EM scattering approximations in a simple manner. Both source-dependent and source-independent scattering theories have been investigated in a systematic manner, as well as their possible higher-order extensions. Based on new insights obtained we have been able to bridge the apparent gap between some of the approaches as well as proposing possible extensions and refinements.

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